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Analytical mechanics of finite homogeneous strains

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IT IS SHOWN that the classical methods of analytical, Hamiltonian mechanics provide a convenient framework for studying finite homogeneous strains. The geometric structure of the configuration space of a homogeneously deformed body is investigated in detail. In particular, kinematical symmetries and non-holonomic velocities related to these symmetries are studied. The phase space of deformable body is constructed and investigated. The kinetic energy of a homogeneously deformable medium is calculated and expressed explicitly in terms of the strain-parameters in two versions: 1) assuming that the physical space is endowed with some Euclidean metric; 2) in an amorphous space without any metrical concepts. Both these variants are discussed in detail. The general methods of describing interactions are presented. The general ideas concerning quantization are formulated and developed.

W pracy wykazano, że klasyczne metody hamiltonowskiej mechaniki analitycznej są dogodnym narzędziem do badania skończonych deformacji jednorodnych. Zbadano szczegółowo strukturę geometryczną przestrzeni konfiguracyjnej ciała deformowalnego jednorodnie. W szczególności zbadano symetrie kinematyczne i nieholonomiczne quasi-prędkości związane z tymi symetriami. Przedstawiono konstrukcję przestrzeni fazowej dla ośrodka deformowalnego jednorodnie i zbadano ją szczegółowo. Energia kinetyczna ośrodka została obliczona w dwu wersjach: 1) zakładając w przestrzeni fizycznej wyróżnioną metrykę euklidesową, 2) w przestrzeni amorficznej pozbawionej jakichkolwiek pojęć metrycznych. Zbadano szczegółowo obydwa warianty. Przedstawiono ogólną metodę opisu oddziaływań. Sformułowano i rozwinięto główne idee odnośnie kwantyzacji problemu.

В работе показано, что классические методы гамильтоновой аналитической механики являются хорошим аппаратом для исследования конечных однородных деформаций. Исследована подробно геометрическая структура конфигурационного пространства тела деформируемого однородным образом. В частности исследованы кинематические симметрии и неголономные квазискорости, связанные с этими симметриями. Представлено построение фазового пространства для среды деформированной однородным образом и оно исследовано подробно. Кинетическая энергия среды расчитана в двух вариантах: 1) предполагая, что в физическом пространстве задана специальная евклидовая метрика, 2) в аморфном пространстве лишенном каких-нибудь метрических понятий. Исследованы подробно оба варианта. Представлен общий метод описания взаимодействий. Сформулированы и развиты главные идеи, касающиеся квантования задачи.

1. Introduction

THE AIM of this paper is to develop analytical mechanics of a homogeneously deformable body.

We assume that the physical space in which the body is placed is endowed with an affine geometry. Displacing all material points of the medium according to a certain fixed affine transformation, we achieve precisely a certain homogeneous strain. A material point, the affine coordinates of which are x^i, then undergoes displacement to the point with coordinates $y^i = a^i{}_j x^j + b^i$, where the matrix composed of coefficients $a^i{}_j$ is nonsingular.

Let g_{ij} be the Euclidean metric tensor. Then, the finite Eulerian strain tensor e_{ij} is as follows:

$$2e_{ij} = g_{ij} - g_{kl}\bar{a}^k{}_i\bar{a}^l{}_j,$$

where $\bar{a}$ is the reciprocal matrix of a [2, 3, 6].

The constants $a^i{}_j$, b^i describe uniquely a displacement of the medium. Components of the strain tensor e_{ij} are constant, according to the coordinate-independence of $a^i{}_j$.

In this paper, we are concerned only with the simplified situation in which the translational degrees of freedom are frozen. This means that $b_i = 0$, and the body undergoes affine rotations around some fixed point (e.g. the centre of the mass). The configuration of the body is uniquely described by coefficients $a^i{}_j$. (Note, however, that in addition to pure deformations, a describes also a rigid rotation). Therefore, it is not affine, but rather the vector space which should be used as a model of the material and physical space (due to the existence of a fixed, non-moving point which will be identified with the null of the space of free vectors). Hence, the natural coordinates in the configuration space of the body are coefficients $a^i{}_j$. A homogeneously deformable body possesses a finite number of degrees of freedom. This enables us to make use of the well-known methods of analytical mechanics in both the Lagrangian and Hamiltonian forms [4, 5, 9].

The analytical mechanics of homogeneous strains is of interest at least from the purely geometrical point of view. A homogeneously deformable medium is nothing but an *affinely-rigid body*—i.e., such a one that all internal affine relations between its elements are frozen. (Let us bear in mind that in usual—i.e., metric rigid-bodies, not only affine, but also the metric relations have to be frozen). In spite of its obvious attractiveness, our treatment seems to be new.

We also feel that it will be possible to apply our approach in many physical and mechanical problems. The recent paper is considered as a starting point for developing these applications. Let us quote the most important of them:

1) oscillations of molecules;

2) finite deformations and oscillations of monocrystals, and small deformations superposed on finite deformations;

3) statistical mechanics of systems with internal degrees of freedom;

4) micromorphic solids (media composed of granules which undergo homogeneous strains);

5) large deformations of vulcanized rubber.

We develop our treatement simultaneously on both the classical and quantum levels. The quantum theory of homogeneous strains is unavoidable when studying problems 1, 2, 3, especially at low temperatures.

No approximation methods of infinitesimal elasticity are used. We are rather dealing with finite deformations only.

It is possible to take relativistic phenomena into account. We shall investigate this problem in subsequent papers.

Familiarity with elementary linear algebra [1], differential geometry [8, 11] and analytical mechanics [4, 9] is assumed.

2. Degrees of freedom and the phase space of a system. Kinematical symmetries

Let M be a physical space. We assume it to be endowed with an affine structure; the space of free vectors (translations) will be denoted as V. As was pointed out above, homogeneous deformation results from displacing all the material points of the medium, according to some fixed affine transformation of M.

In this paper, we investigate only the special case in which the body is fixed at one point. In imposing these constraints we exclude translations from our treatment. The existence of a distinguished non-moving point $p \in M$ enables us to identify the manifold M with the vector space V by means of the one-to-one mapping $t_p : M \to V$, where $t_p(q) = \overrightarrow{pq}$, and $\overrightarrow{pq}$ denotes the translation (free vector) carrying p over into q.

In this way, the finite-dimensional linear space V models our *physical space*. We assume the material points of the medium to be distinguishable and "mark" them by means of the points of auxiliary vector space U, called material space [2, 10]. We assume that the dimensions of U and V are equal. When our "marks" are initial positions at the moment t_0, then $U = V$.

When using linear bases in U, V, we shall denote them throughout this paper as $\{E_A\}$, $\{e_i\}$, respectively. Matrix elements of linear mappings $\varphi : U \to V$, $\psi : V \to U$, $A : U \to U$, $B : V \to V$ with respect to these bases will be denoted as $\varphi^i{}_A$, $\psi^A{}_i$, $A^B{}_C$, $B^i{}_j$, respectively (i.e.: $\varphi E_A = e_i \varphi^i{}_A$, $\psi e_i = E_A \psi^A{}_i$, $AE_C = E_B A^B{}_C$, $Be_j = e_i B^i{}_j$).

Arbitrary *configuration* of a body which undergoes affine rotations (homogeneous deformations without translations) is uniquely described by means of a certain linear isomorphism $\varphi : U \to V$. This should be understood in the sense that the material point "marked" by $u \in U$ occupies the position φu in the physical space V. The non-singularity of φ means that different material points occupy different physical positions (no "glueing" of the particles [2, 3, 6]).

In what follows, $L(U, V)$ denotes the space of linear mappings of U into V. Its subset, composed of isomorphisms, will be denoted as $LI(U, V)$. Obviously, $LI(U, V)$ is an open submanifold of $L(U, V)$. Finally:

The *configuration space* of our system is given by $Q = LI(U, V)$. Obviously, $\dim Q = \dim U \dim V = n^2$, where $n = \dim U = \dim V$. Throughout this paper, n is quite arbitrary. Obviously, only the special cases $n = 1, 2, 3$ (the linear, plane and space problems, respectively) possess physical meaning. In relativistic investigations, four-dimensional spaces will be used.

The *motions* of a system are described by smooth curves $\varrho : R \to Q$; obviously, $\varrho(t)$ denotes the configuration of the body at the moment $t \in R$.

In this section, no metric structures in U, V are presupposed. Hence, arbitrary $\varphi \in Q$ describes a general affine injection of the matter into physical space and it makes no sense to factorize it into a metrically-rigid and a purely-deformative part.

In the linear spaces U, V, the automorphism-groups—i.e., full linear groups $GL(U)$, $GL(V)$—act in a natural way. These actions give rise to the natural action of $GL(U)$, $GL(V)$ on the configuration space Q. The corresponding transformations of Q are defined and denoted as follows:

$$A\varphi \underset{\text{def}}{=} A \circ \varphi, \tag{2.1}$$

(2.2) $$\varphi B \underset{\text{def}}{=} \varphi \circ B,$$

for arbitrary $A \in GL(V)$, $B \in GL(U)$, $\varphi \in Q = LI(U, V)$. With actions so defined, $GL(V)$ acts on Q *on the left*, and $GL(U)$—*on the right*. This justifies the notations $A\varphi$, φB used above. Obviously, all actions of $GL(U)$ commute with all actions of $GL(V)$.

Transformations of the group $GL(V)$ possess a direct physical interpretation: they are physical mappings (symmetries) acting in the physical space V. Hence, they move the material points of the medium in V, which is in agreement with natural, operational intuitions concerning the word "deformation". In all formulas, the physical fields acting on the medium (e.g. electromagnetic field) are coupled with geometric objects of the "left" homogeneous space $(Q, GL(V))$. Roughly speaking, any transformation of the group $GL(V)$ on Q can be realized as a consequence of some external physical interaction. By contrast, the action of $GL(U)$ on Q may appear rather as a purely mathematical construction. It describes certain, rather abstract, initial deformations and rotations. However, the kinetic energy of a system becomes a much more simple expression when given in terms of geometric objects related to the "right" homogeneous space $(Q, GL(U))$.

Now, let us construct the mechanical *state-space* of our system. It is well known that the mechanical state of a system is fully described by its configuration and generalized velocity, or equivalently, by configuration and generalized momentum. As we have pointed out above, Q is an open subset of the linear space $L(U, V)$. This enables us to avoid the general mechanical formalism of the tangent and cotangent bundless, [5, 7, 8]: Generalized velocities are elements of $L(U, V)$ itself. Similarly, generalized momenta become elements of the dual space $L(U, V)^*$. Moreover, let us notice that $L(U, V)^*$ can be identified with $L(V, U)$. An element of $L(U, V)^*$ corresponding to $f \in L(V, U)$ via this identification will be denoted as $\bar{f}$, and given as follows:

(2.3) $$\langle \bar{f}, g \rangle = \mathrm{TR}(f \circ g) = f^A{}_i g^i{}_A$$

for arbitrary $g \in L(U, V)$, where $\mathrm{Tr}(f \circ g)$ denotes the trace of the linear mapping $f \circ g$: $U \to U$. Instead of $F = \bar{f}$, we shall write also $f = \bar{F}$.

This simplifies all formulas and eliminates misunderstandings.

Let us briefly sum up these preliminaries:

The *Lagrangian state-space* of our system is a differential manifold $Q \times L(U, V)$. When $\varrho: R \to Q$ describes a certain motion, then $\varrho'(t) = \dfrac{d\varrho}{dt}(t)$ is the generalized velocity of the body at time $t \in R$.

The *phase-space* of a system—i.e., its *Hamiltonian state-space*—is a manifold $P = Q \times L(V, U)$. Linear bases in U, V give rise to canonical coordinates $Q^i{}_A$, $P^B{}_j$ on P such that:

(2.4) $$Q^i{}_A(\varphi, \pi) = \varphi^i{}_A,$$

(2.5) $$P^B{}_i(\varphi, \pi) = \pi^B{}_i.$$

Poisson brackets of the coordinates above will be given by the usual formulas:

(2.6) $$\{Q^i{}_A, Q^j{}_B\} = 0 = \{P^A{}_i, P^B{}_j\},$$

(2.7) $$\{Q^i{}_A, P^B{}_j\} = -\delta^i{}_j \delta^B{}_A;$$

(Obviously, it is possible to give them also in terms of absolute symbols).

Instead of the usual, generalized velocities $\varrho'(t) \in L(U, V)$, we shall make use of what are called *quasi-velocities* (*non-holonomic velocities*), i.e., combinations of the components of $\varrho'(t)$ with coefficients depending on the configuration $\varphi \in LI(U, V)$. They will enable us to separate the rotational and deformative behaviour of the body. In $\varrho'(t)$, these two types of behaviour are non-physically mixed. Similarly, what are called quasi-momenta (non-holonomic momenta) will be used. The non-holonomic velocities and momenta we use below are strictly related to the action of kinematical symmetries $GL(U)$, $GL(V)$ on the configuration space Q, in the same way as angular velocities and angular momenta are connected with the action of the corresponding orthogonal groups on the configuration space of a rigid body. This is because this action enables us to identify $L(U, V)$ with $L(U)$, or $L(V)$ (where $L(U)$, $L(V)$ is an abbreviation of $L(U, U)$, $L(V,V)$, respectively).

The *external*, or *left quasi-velocity form* is defined as a mapping $\Omega_l: Q \times L(U, V) \to L(V)$ such that:

$$\Omega_l(\varphi, g) = g \circ \varphi^{-1}. \tag{2.8}$$

This mapping identifies the Lagrangian state-space with $Q \times L(V)$:

$$(\varphi, g) \to (\varphi, \Omega_l(\varphi, g)) = (\varphi, g \circ \varphi^{-1}).$$

When $\varrho: R \to Q$ is a motion, then $\Omega_l(t) = \Omega_l(\varrho(t), \varrho'(t))$ is the *external* (*left*) *quasi-velocity* of a system at time t.

The *internal*, or *right quasi-velocity* form is a mapping $\Omega_r: Q \times L(U, V) \to L(U)$ such that:

$$\Omega_r(\varphi, g) = \varphi^{-1} \circ g. \tag{2.9}$$

In the next section, we shall show that the Eulerian and Lagrangian velocity fields are uniquely described by the quasi-velocities (2.8) and (2.9), respectively.

Let us take the matrices $\Omega_l^{ij}(t) = \Omega_l^i{}_k(t) g^{kj}$, and $\Omega_r^{AB}(t) = \Omega_r^A{}_C(t) \eta^{CB}$, where g^{kj}, η^{AB} are the contravariant components of some fixed metric tensors g, η in V, U, respectively. Independent elements of the skew-symmetric part $\Omega_l^{[ij]}(t)$ are simply the components of the angular velocity at time t, with respect to the laboratory system of reference. Similarly, $\Omega_r^{[AB]}$ is the angular velocity referred to the co-moving frame. This example justifies the terms used above: external and internal quasi-velocities. All physical quantities referred to the physical space V (given by geometric objects in V) describe the medium in laboratory terms. Similarly, quantities referred to the material space U, describe the body in terms of the co-moving frame: they are internal characteristics of the body.

Both $\Omega_l^{[ij]}$, $\Omega_r^{[AB]}$ describe the rigid-rotational behaviour of the body. By contrast, symmetric parts $\Omega_l^{(ij)}$, $\Omega_r^{(AB)}$ describe the deformative behaviour (they are non-holonomic distorsion-velocities). Note that our separation of these two phenomena is valid for arbitrary, large deformative motions.

Analogously to non-holonomic velocities, we define the quasi-momenta:

The *external*, or *left quasi-momentum form* is defined as a mapping:
$\Sigma_l: P = Q \times L(V, U) \to L(V)$ such that:

$$\Sigma_l(\varphi, f) = \varphi \circ f. \tag{2.10}$$

Similarly:

The *internal*, or *right quasi-momentum form* $\Sigma_r: P = Q \times L(V, U) \to L(U)$ is a mapping:

$$\Sigma_r(\varphi, f) = f \circ \varphi. \tag{2.11}$$

In the definitions above, we identified $L(U)$ with $L(U)^*$ and $L(V)$ with $L(V)^*$, according to the formula:

$$\langle \bar{k}, l \rangle = \mathrm{TR}(k \circ l).$$

As already indicated, the Lie groups $GL(U)$, $GL(V)$ act on the configuration space via the formulas (2.1), (2.2). These kinematical symmetries can be "lifted" in a natural way to the state-spaces, both Lagrangian and Hamiltonian. The corresponding transformations are defined as follows:

$$A(\varphi, g) = (A \circ \varphi, A \circ g), \tag{2.12}$$

$$(\varphi, g)B = (\varphi \circ B, g \circ B), \tag{2.13}$$

$$A(\varphi, f) = (A \circ \varphi, f \circ A^{-1}), \tag{2.14}$$

$$(\varphi, f)B = (\varphi \circ B, B^{-1} \circ f), \tag{2.15}$$

for arbitrary $A \in GL(V)$, $B \in GL(U)$, $\varphi \in Q$, $g \in L(U, V)$, $f \in L(V, U)$. Obviously, (2.14) and (2.15) are *canonical mappings*; moreover, they are known as *extended point transformations* of the phase-space P [4, 9]. The infinitesimal generators of these transformations can easily be found:

To arbitrary $\alpha \in L(V)$, $\beta \in L(U)$ (linear mappings in V and U, respectively) we attach the functions $F_l[\alpha]: P \to R$, $F_r[\beta]: P \to R$, given by the following formulas:

$$F_l[\alpha](\varphi, f) = \langle \Sigma_l(\varphi, f), \alpha \rangle = \langle \varphi \circ f, \alpha \rangle = \mathrm{Tr}(\varphi \circ f \circ \alpha) = \varphi^i{}_A f^A{}_j \alpha^j{}_i, \tag{2.16}$$

$$F_r[\beta](\varphi, f) = \langle \Sigma_r(\varphi, f), \beta \rangle = \langle f \circ \varphi, \beta \rangle = \mathrm{Tr}(f \circ \varphi \circ \beta) = f^A{}_i \varphi^i{}_B \beta^B{}_A. \tag{2.17}$$

Let us assume some metric tensors g, η in V, U, respectively. When α, β are skew-symmetric with respect to these tensors, then $F_l[\alpha]$, $F_r[\beta]$ describe the corresponding components of the angular momentum of the body in the laboratory and co-moving frame, respectively.

It is easy to show that the assignment $\alpha \to F_l[\alpha]$ is a Lie algebra-representation in the sense of matrix commutators and Poisson brackets [9, 4]: $F_l[[\alpha_1, \alpha_2]] = \{F_l[\alpha_1], F_l[\alpha_2]\}$. By contrast $\beta \to F_r[\beta]$ is an antirepresentation: $F_r[[\beta_1, \beta_2]] = -\{F_r[\beta_1], F_r[\beta_2]\}$. This is because $F_l[\alpha]$, $F_r[\beta]$ generate the groups (2.14), (2.15) of canonical mappings on P: For arbitrary smooth function G on P, we have:

$$\frac{\partial}{\partial t}(G \circ h_t) = \{F_l[\alpha], G\} \circ h_t,$$

$$\frac{\partial}{\partial t}(G \circ k_t) = \{F_r[\beta], G\} \circ k_t,$$

where:

$$h_t(\varphi, \pi) = (e^{\alpha t} \circ \varphi, \pi \circ e^{-\alpha t}), \text{ and:}$$

$$(\varphi, \pi)k_t = (\varphi \circ e^{\beta t}, e^{-\beta t} \circ \pi).$$

It was mentioned above that the groups $GL(U)$, $GL(V)$ commute when acting on Q (and on P): Consequently, infinitesimal symmetries also commute:

$$\{F_l[\alpha], F_r[\beta]\} = 0. \tag{2.18}$$

3. The kinetic energy of a homogeneously deformable body

Let the medium, moving with generalized velocity $\xi \in L(U, V)$, pass the configuration $\varphi \in Q$. It can easily be shown that the Eulerian velocity field v in V is then given as:

$$v(x) = \xi \circ \varphi^{-1} x = \Omega_l(\varphi, \xi) x. \tag{3.1}$$

Carrying this vector field by means of φ^{-1}, we obtain the co-moving velocity field $\mathcal{V}$ on U:

$$\mathcal{V}(X) = \varphi^{-1} \circ \xi x = \Omega_r(\varphi, \xi) X. \tag{3.2}$$

When there is no danger of misunderstanding, we shall write simply:

$$v^i(x) = \Omega_l{}^i{}_j x^j, \tag{3.1a}$$

$$\mathcal{V}^A(X) = \Omega_r{}^A{}_B X^B. \tag{3.2a}$$

The kinetic energy of the material point is proportional to the scalar square of its velocity. Hence, to be able to introduce the notion of the kinetic energy, we should have some fixed Euclidean metric. From the purely mathematical point of view, we have, *a priori*, two opposite possibilities:

(i) The physical metric fixed. This approach consists in endowing the physical space V with some distinguished metric tensor g.

(ii) The material metric fixed. In this approach it is the material space U which is endowed with a fixed metric η. Distances in V are measured by means of the configuration-dependent Cauchy deformation tensor g_φ, where: $\eta = \varphi^* \cdot g_\varphi$ [2]. In such a way, the geometry of the physical space would be determined by configurations of the matter. This is to some extent reminiscent of the situation we encounter in general relativity.

Obviously, in the usual theory of elasticity and its applications, we have to use the *physical metric* g in V, when calculating the kinetic energy. Nevertheless, it would be advantageous and instructive to know also what results when the *material, co-moving* metric η is fixed and consequently used—i.e., when the geometry of the physical space V is given by the configuration-dependent Cauchy tensors g_φ. Obviously, the "kinetic energy" so obtained is rather a "non-physical" quantity; nevertheless, it of is interest for its elegant geometric structure and properties. For example, it possesses high dynamical symmetries—namely, it is invariant under $GL(V)$ and under certain subgroups of $GL(U)$. Moreover, when expressed by the quasi-velocities Ω_l, or Ω_r, it becomes a very simple quadratic form. This is not the case of "physical" kinetic energy. This fact is of interest in view of the profound physical interpretation of quasi-velocities via velocity fields (3.1), (3.2). Hence, the "co-moving kinetic energy", although non-physical, seems to be an appropriate (and indispensable) tool for understanding the dynamical and geometrical structure of the theory. Provided that we are interested in infinitesimal theory, both forms of the kinetic energy are asymptotically equivalent.

Let us notice that, in addition to the above "philosophical" reasons used to justify the study of the "comoving kinetic energy", there exist certain more physical ones. In fact, the theories of continuous media in amorphous spaces without any metrical notions

are mathematically possible and, perhaps, physically reasonable. Observe for example that in general relativity theory and in the theory of continuum based on it, there is no fixed metric geometry at all; rather, the components of the metric tensor are included in physical degrees of freedom and dynamically coupled with matter and its distribution.

The kinetic energy of the medium equals the sum (integral) of the kinetic energies of its infinitesimal elements. Let the measure μ describe the mass distribution of the matter in U (in the reference configuration). It can easily be shown that the physical kinetic energy in the Lagrangian state (φ, ξ) is given as:

$$T_g(\varphi, \xi) = \frac{1}{2}\langle g, (\xi \otimes \xi)J\rangle = \frac{1}{2} g_{ij}\xi^i{}_A \xi^j{}_B J^{AB}, \tag{3.3}$$

where J is the comoving quadrupole moment of the mass distribution:

$$J = \int_U X \otimes X d\mu(X), \text{ i.e.: } J^{AB} = \int_U X^A X^B d\mu(X). \tag{3.4}$$

The configuration-independent tensor J describes the inertial properties of the body. (3.3) is a configuration-independent quadratic form of generalized velocities. Unfortunately, when expressed by quasi-velocities, it becomes a configuration-dependent form.

In appropriate coordinates, (3.3) can be reduced to:

$$T_g(\varphi, \xi) = \frac{1}{2}\sum_{i,A} J^A (\xi^i{}_A)^2. \tag{3.3a}$$

Particularly interesting is the special case of the full spherical symmetry of the quadrupole tensor: $J^{AB} = \nu\delta^{AB}$. We have then:

$$T_g(\varphi, \xi) = \frac{\nu}{2}\sum_{i,A} (\xi^i{}_A)^2 = \frac{\nu}{2}\mathrm{TR}(\hat{\xi}^T\hat{\xi}), \tag{3.5}$$

(where $\hat{\xi}$ is the matrix of ξ; the lowering and raising of indices is understood in the trivial sense of the Kronecker symbol).

It is often possible to treat deviations from the exact spherical symmetry as a small perturbation of the spherical term (3.5).

To calculate the comoving kinetic energy, we replace the metric tensor g by the Cauchy deformation tensor g_φ. Then we obtain:

$$T_\eta(\varphi, \xi) = \frac{1}{2}\langle \eta, (\beta \otimes \beta)J\rangle = \frac{1}{2}\eta_{AB}\beta^A{}_C\beta^B{}_D J^{CD}, \tag{3.6}$$

where $\beta = \Omega_r(\varphi, \xi)$.

Hence, when expressed by quasi-velocities, T_η is a configuration-independent quadratic function. That is why the comoving energy, although "nonphysical", seems to be more "elegant" and geometrically interesting then the physical. In fact, it is quasi-velocities rather than generalized velocities which possess a direct physical interpretation (velocity fields and separation of the deformative and rotational behaviour).

In the special case of spherical symmetry, $J^{AB} = \nu\delta^{AB}$, we obtain:

$$T_\eta = \frac{\nu}{2}\beta^A{}_C\beta_A{}^C = \mathrm{Tr}(\hat{\beta}^T\hat{\beta})$$

(lowering and raising of indices in the sense of the Kronecker symbol). When expressed by external quasi-velocities, Ω_l, T_η becomes configuration-dependent. It is internal, geometric objects in U (material objects) which make both T_g and T_η configuration-independent. The same situation occurs in the theory of rigid body, where T_g and T_η become equal. (The theory of a rigid body results from our theory by imposing certain holonomic constraints).

4. Interactions

In what follows we assume the existence of the potential energy $\mathscr{V}: Q \to R$, which describes the static interactions. To find $\mathscr{V}$ it is necessary to postulate some interaction potential between material points of the medium, and the potential of external forces which act on such points. Obviously, summation (integration) over material points should be performed in order to express the full potential energy as a function depending on the configuration $\varphi \in Q$ only. When some non-static—i.e., velocity-dependent—interactions are present (e.g. magnetic ones), then the formulation of the problem within an analytical, Lagrangian framework is possible if, and only if, the generalized potential $\mathscr{V}: Q \times L(U, V) \to R$ does exist. Obviously, magnetic interactions satisfy this requirement.

In this paper, we do not go into details of interactions in real solids (crystals or rubbers). Rather, we restrict ourselves to presenting the general outline only. In addition, some academic, but computationally simple and instructive examples are presented.

Let us assume that the body consists of *identical material points* distributed in material space U according to the positive, regular measure (mass distribution) μ. (Both continuous and singular, in particular pointwise distributions are admissible). Then, taking into account the physical homogeneity of the space V, we can express the potential energy of internal, binary interactions, $\mathscr{V}_i^{(2)}$, as follows:

$$(4.1)\quad \mathscr{V}_i^{(2)}(\varphi) = \frac{1}{2}\int'_{U\times U} w^{(2)} \circ \varphi(X-Y)\,d\mu(X)\,d\mu(Y) = \frac{1}{2}\int'_{V\times V} w^{(2)}(x-y)\,d\mu_\varphi(x)\,d\mu_\varphi(y),$$

where the primed integration symbol $\int'_{M\times M}$ means that the integration has to be performed over the set

$$\Delta(M) = \{(m_1, m_2) : m_1 \neq m_2,\ m_1, m_2 \in M\}$$

(the "diagonal" consisting of pairs (m, m) excluded). This point is essential in the case of discrete, pointwise mass distribution μ. Obviously, when μ is absolutely continuous with respect to the Lebesgue measure, then the primed integral is equal to the usual one.

The function $w^{(2)}$ in (4.1) describes the density of potential energy per unit mass-square.

Similarly, the potential energy of internal m-body interactions $\mathscr{V}_i^{(m)}$ is given as follows:

$$(4.2)\quad \mathscr{V}_i^{(m)} = \frac{1}{m!}\int'_{\underset{m}{\times U}} w^{(m)} \circ \varphi(X-X_1, \ldots, X-X_{m-1})\,d\mu(X)\,d\mu(X_1)\ldots d\mu(X_{m-1})$$

$$= \frac{1}{m!}\int'_{\underset{m}{\times V}} w^{(m)}(x-x_1, \ldots, x-x_{m-1})\,d\mu_\varphi(x)\,d\mu_\varphi(x_1)\ldots d\mu_\varphi(x_{m-1}),$$

where the primed integration over M is performed over the subset $\Delta^{(m)}(M) \subset \underset{m}{\times} M$ such that no "coordinates" coincide:

$$\Delta^{(m)}(M) = \{(p_1, \ldots, p_m) : p_i \neq p_j,\ i, j = 1, \ldots, m\}.$$

When the body consists of non-identical material points, then it is advantageous to replace the formulas (4.1), (4.2) (although they remain valid) by others involving densities per unit "charges" of physical properties of points. For example, the energy $\mathscr{V}_i^{\mathrm{est}}$ of internal electrostatic interactions is given as follows:

$$\mathscr{V}_i^{\mathrm{est}}(\varphi) = \frac{a}{2} \int_{U \times U}' \frac{1}{|\varphi X - \varphi Y|_\eta} d\varepsilon(X) d\varepsilon(Y) = \frac{a}{2} \int_{V \times V}' \frac{1}{|x-y|_g} d\varepsilon_\varphi(x) d\varepsilon_\varphi(y), \tag{4.3}$$

where g is the physical metric tensor and $|v|_g = \sqrt{\langle g, v \otimes v \rangle}$, η is the material metric tensor; $|u|_\eta = \sqrt{\langle \eta, u \otimes u \rangle}$, ε is a regular measure which describes the distribution of electric charge in the material space U (obviously, ε need not be positive). ε_φ results from ε via φ-transport and describes the physical configuration-dependent charge distribution, a is a certain constant depending on the choice of units.

External interactions are described in a similar way. When the body consists of identical material points, then the external potential energy is given by the following expression:

$$\mathscr{V}_{\mathrm{ex}}(\varphi) = \int_U w_{\mathrm{ex}} \circ \varphi(X) d\mu(X) = \int_V w_{\mathrm{ex}}(x) d\mu_\varphi(x), \tag{4.4}$$

where $w_{\mathrm{ex}} : V \to R$ is the density of external potential energy per unit mass.

Similarly, an interaction with external electrostatic field is given by:

$$\mathscr{V}_{\mathrm{ex}}^{\mathrm{est}}(\varphi) = \int_U F \circ \varphi(X) d\varepsilon(X) = \int_V F(x) d\varepsilon_\varphi(x), \tag{4.5}$$

where the regular measure ε describes the distribution of electric charge in U, and $F : V \to R$ is the usual potential of electrostatic field (i.e., the potential energy per unit charge).

When the external magnetic field is present, then the corresponding generalized (velocity-dependent) potential energy is given by:

$$\mathscr{V}_{\mathrm{ex}}^{\mathrm{mgn}}(\varphi, \xi) = \frac{1}{c} \int_V \langle A(x), \Omega_l(\varphi, \xi) \cdot x \rangle d\varepsilon_\varphi(x) = \frac{1}{c} \int_U \langle \varphi^* A(X), \Omega_r(\varphi, \xi) \cdot X \rangle d\varepsilon(X), \tag{4.6}$$

or, in a more symbolic way:

$$\mathscr{V}_{\mathrm{ex}}^{\mathrm{mgn}}(\varphi, \xi) = \frac{1}{c} \int_V \langle A(x), dj_\varphi(x) \rangle = \frac{1}{c} \int_U \langle \varphi^* A(X), dj(X) \rangle, \tag{4.6a}$$

where the field $A : V \to V^*$ is a vector potential of external magnetic field, and the vector measures j_φ, j describe the electric current connected with the deformative motion

$$\big(dj_\varphi(x) = \Omega_l(\varphi, \xi) \cdot x \, d\varepsilon_\varphi(x),\ dj(X) = \Omega_r(\varphi, \xi) \cdot X \, d\varepsilon(X)\big).$$

In many practical problems, it is convenient to describe electromagnetic interactions in terms of multipole moments. For example, when the body is endowed with an electrostatic dipole moment, then the corresponding potential energy is given as follows:

$$\mathscr{V}_{\mathrm{ex}}^{\mathrm{dip.el.}}(\varphi) = \langle E(0), \varphi \cdot d \rangle = E(0)_i \varphi^i{}_A d^A, \tag{4.7}$$

where $E(0) \in V$ is a value of external electrostatic field at the immovable point $v \in V$, and $d \in U$ is the comoving dipole moment:

$$d = \int X d\varepsilon(X) \tag{4.8}$$

in linear coordinates:

$$d^A = \int X^A d\varepsilon(X). \tag{4.8a}$$

Obviously, d is configuration-independent.

We conclude this section with a simple example. It is non-physical, nevertheless it does help to understand how our method works. We shall omit all non-interesting details of easy proofs. The more detailed analysis will be given in subsequent papers.

Let us consider a material which consists of identical material points distributed continuously in space. The measures μ, μ_φ are absolutely continuous with respect to the Lebesgue measures in U, V, respectively; the corresponding densities will be denoted as ϱ, ϱ_φ. Moreover, we assume the density ϱ to be constant all over the body. Then, ϱ_φ is also constant. This means that the *body is completely homogeneous*. The potential energy of internal, m-body interactions $\mathscr{V}_i^{(m)}$ is given by (4.2). Assuming *short-range forces* and *disregarding the surface-energy* we can obtain some explicit results concerning $\mathscr{V}_i^{(m)}$. Obviously, *these results are valid for such configurations* $\varphi \in Q$, *only that the linear size of the body in an arbitrary direction remains much larger than the range of forces*. In particular, they are drastically wrong in the neighbourhood of "collapsed" configuration $\varphi = 0$—i.e., for a highly compressed medium. It can be shown that when our assumptions are satisfied, then:

$$\tilde{\mathscr{V}}_i^{(m)}(\varphi) = C_m/|\varphi|^{m-1}, \tag{4.9}$$

where $|\varphi|$ is the determinant of the matrix of φ with respect to some orthonormal bases in U and V. The details of the shape of functions $w^{(m)}$ are not very essential—these functions influence the constants C_m via the integral formulas:

$$C_m = \frac{M\varrho^{m-1}}{m!} \int\limits_{\underset{m-1}{\times} V} w^{(m)}(x_1, \ldots, x_{m-1}) dx^1, \ldots, dx^{m-1}, \tag{4.10}$$

where M is the mass of the body, ϱ is its undeformed (comoving) density per unit (Lebesgue) volume in U. Therefore, the full internal potential energy is given by

$$\tilde{\mathscr{V}}_i(\varphi) = \sum_{m=2}^{N} \tilde{\mathscr{V}}_i^{(m)}(\varphi) = \sum_{m=2}^{N} \frac{C_m}{|\varphi|^{m-1}}, \tag{4.11}$$

where the integer N characterizes the dynamical structure of a system (in the theory of continua, N need not be finite).

Thus, our assumptions (the large body-configurations) imply that:

$$\tilde{\mathscr{V}}_i(\varphi) = f(|\varphi|), \tag{4.12}$$

where $f: R \to R$ is some real-valued function of one variable, vanishing at infinity (dissociated body) and singular at zero (collapsed body). Obviously, the singularity mentioned is non-physical, because our approximation then becomes wrong.

The potential energy (4.12) leads to the following stress-strain relation [2, 3, 6]:

$$\tau = f'\left(\sqrt{|\mathscr{P}|}\right)\mathscr{P}, \tag{4.13}$$

where τ is the stress tensor measured per unit area of the deformed body, $\mathscr{P}$ is the so-called Pioli deformation tensor [2] (i.e., the reciprocal of the Cauchy deformation tensor), and $|\mathscr{P}|$ denotes the determinant of $\mathscr{P}$ in some orthonormal basis. Making use of the usual Eulerian strain tensor e, and denoting the reciprocal tensors by a tilda, we can write (4.12) in the following form:

$$\tau = f'\left(\sqrt{\widetilde{|g-2e|}}\right)\widetilde{(g-2e)} \tag{4.13a}$$

or, equivalently:

$$\tau^{ik}(g_{kj}-2e_{kj}) = f'\left(\frac{1}{\sqrt{|g-2e|}}\right). \tag{4.13b}$$

The quantity $|\varphi| = \sqrt{|\mathscr{P}|} = \dfrac{1}{\sqrt{|g-2e|}}$ is often denoted as $\sqrt{I_3}$ (cf. [6]). Hence:

$$\tau^{i}_{\ j}-2\tau^{ik}e_{kj} = f'\left(\sqrt{I_3}\right). \tag{4.13c}$$

Obviously, in physical applications we make use principally of the model of binary interactions. The physical properties of $\tilde{\mathscr{V}}^{(2)}_i = \dfrac{C}{|\varphi|}$ can be improved by taking the surface terms into account. In general, the surface term is rather involved. However, in some special cases it can easily be evaluated. For example, let us assume that the natural, undeformed shape of the body is given by a cube in the material space U. Then, assuming that the sizes of deformed walls are still large as compared with the range of interactions, we have:

$$\mathscr{V}^{(2)}_{i,\,\text{surface}}(\varphi) = \frac{D\varrho^2\Sigma}{|\varphi|}\sum_{A=1}^{3}\sqrt{\varphi^{-1A}{}_i\,\varphi^{-1A}{}_j g^{ij}} = \frac{D\varrho^2\Sigma}{|\varphi|}\sum_{A=1}^{3}\sqrt{\tilde{\eta}_{\varphi}{}^{AA}}, \tag{4.14}$$

where Σ is the undeformed area of a wall and the constant D depends on the shape of $w^{(2)}$ and on the radius of interactions. The full internal energy of binary interactions is a sum of (4.15) and $\tilde{\mathscr{V}}^{(2)}_i$:

$$\mathscr{V}^{(2)}_i = \tilde{\mathscr{V}}^{(2)}_i + \mathscr{V}^{(2)}_{i,\,\text{surface}}. \tag{4.15}$$

5. Legendre transformation and Hamiltonian dynamics

The full information about the dynamical structure of a system and its equations of motion is contained in its Lagrange function L: $Q\times L(U, V) \to R$. In the case we consider here, $L = T-\Phi$, where T is the kinetic energy and Φ—the potential energy of a system.

Similarly, as in previous sections, $\bar{f}\in L(V, U)$ denotes a linear mapping identified canonically with a linear function $f\in L(U, V)^*$, according to the formula:

$$\langle f, g\rangle = \mathrm{Tr}(\bar{f}\circ g). \tag{5.1}$$

The *Legendre transformation* $\mathscr{L}: Q \times L(U, V) \to P = Q \times L(V, U)$ maps the Lagrangian state-space into the phase-space P of a system. The Lagrangian L gives rise to the following Legendre transformation:

$$\mathscr{L}(\varphi, \xi) = (\varphi, \overline{D_\xi L(\varphi, \circ)}) \tag{5.2}$$

for arbitrary $(\varphi, \xi) \in Q \times L(U, V)$.

Hence, the arbitrary point (φ, ξ), linear coordinates of which are $(\varphi^i{}_A, \xi^i{}_A)$, is transformed into the point (φ, π) with coordinates $\varphi^i{}_A, \pi^A{}_i$, where:

$$\pi^A{}_i = \frac{\partial L}{\partial \xi^i{}_A}(\varphi, \xi). \tag{5.3}$$

In what follows, we restrict ourselves to the special, but super-important case in which $\mathscr{L}$ is a *diffeomorphism*. When $L = T - \Phi$ where T equals either T_η or T_g and Φ describes static or magnetic interactions, then, for arbitrary $\varphi \in Q$, $\mathscr{L}|\{\varphi\} \times L(U, V)$ is an affine mapping. Hence, $\mathscr{L}$ possesses the inverse if, and only if, the quadruple tensor $\mathscr{J} \in U \otimes U$ is algebraically non-singular (i.e., the matrix of components $\mathscr{J}^{AB}$ is non-singular). In what follows, we assume that this is the case and the corresponding reciprocal tensor will be denoted as $\tilde{\mathscr{J}} \in U^* \otimes U^*$. In linear coordinates:

$$\mathscr{J}^{AB} \tilde{\mathscr{J}}_{BC} = \delta^A{}_C. \tag{5.4}$$

Physically, this means that the matter occupies some n-dimensional region in U (where $n = \dim U = \dim V$). The special cases of singular distributions concentrated on subspaces of U call for separate treatment (for example, the generalized Dirac mechanics with irreversible Legendre transformations might be used).

The *energy* of a system is defined as a function $E: Q \times L(U, V) \to R$ such that:

$$E(\varphi, \xi) = \langle D_\xi L(\varphi, \circ), \xi \rangle - L(\varphi, \xi). \tag{5.5}$$

In linear coordinates:

$$E(\varphi, \xi) = \xi^i{}_A \frac{\partial L}{\partial \xi^i{}_A}(\varphi, \xi) - L(\varphi, \xi). \tag{5.6}$$

When $L = T - \Phi$ and Φ does not depend on generalized velocities, then:

$$E = T + \Phi. \tag{5.7}$$

The *Hamiltonian* of a system, $H: Q \times L(V, U) \to R$, is defined as follows:

$$H = E \circ \mathscr{L}^{-1}. \tag{5.8}$$

In what follows, we restrict ourselves to the *static interactions* only (Φ does not depend on generalized velocities). It is easy to show that in this case:

$$T_g \circ \mathscr{L}^{-1}(\varphi, \pi) = \frac{1}{2}\langle \tilde{\mathscr{J}}, (\pi \otimes \pi)\tilde{g} \rangle = \frac{1}{2}\langle \pi^* \tilde{\mathscr{J}}, g \rangle. \tag{5.9}$$

In linear coordinates:

$$T_g \circ \mathscr{L}^{-1}(\varphi, \pi) = \frac{1}{2} \tilde{\mathscr{J}}_{AB} \pi^A{}_i \pi^B{}_j g^{ij} \tag{5.10}$$

or, making use of functions $P^A{}_i : P \to R$ as defined in (2.5):

$$T_g \circ \mathscr{L}^{-1} = \frac{1}{2} \tilde{\mathscr{J}}_{AB} P^A{}_i P^B{}_j g^{ij}. \tag{5.11}$$

Hence, the kinetic term is a quadratic function of generalized momenta. Simple calculations show that:

$$T_\eta \circ \mathscr{L}^{-1}(\varphi, \pi) = \frac{1}{2} \langle \tilde{\mathscr{J}}, (\Sigma_r(\varphi, \pi) \otimes \Sigma_r(\varphi, \pi)) \cdot \tilde{\eta} \rangle = \frac{1}{2} \langle \Sigma_r(\varphi, \pi)^* \cdot \tilde{\mathscr{J}}, \tilde{\eta} \rangle. \tag{5.12}$$

In linear coordinates:

$$T_\eta \circ \mathscr{L}^{-1}(\varphi, \pi) = \frac{1}{2} \tilde{\mathscr{J}}_{AB} \Sigma_r(\varphi, \pi)^A{}_C \Sigma_r(\varphi, \pi)^B{}_D \eta^{CD}. \tag{5.13}$$

When we identify the phase-space P with $Q \times L(U)$, then the kinetic term above becomes $T^r : Q \times L(U) \to R$, where:

$$T^r(\varphi, \sigma) = \frac{1}{2} \langle \sigma^* \cdot \tilde{\mathscr{J}}, \eta \rangle = \frac{1}{2} \tilde{\mathscr{J}}_{AB} \sigma^A{}_C \sigma^B{}_D \eta^{CD}. \tag{5.14}$$

The function $T_\eta \circ \mathscr{L}^{-1}$ becomes a very simple expression when given in terms of generators of the group $GL(U)$ acting on P according to (2.15). In fact, let us remember that to arbitrary $\beta \in L(U)$ is attached the generator $F_r[\beta]$ such that:

$$F_r[\beta](\varphi, \pi) = \langle \Sigma_r(\varphi, \pi), \beta \rangle = \langle \pi \circ \varphi, \beta \rangle = \mathrm{Tr}(\pi \circ \varphi \circ \beta). \tag{5.15}$$

Let $\{E_A\}$ be some basis in U; $E^D{}_C$ denotes such an element of $L(U)$ that:

$$E^D{}_C \cdot E_D = E_C \qquad \text{(no summation over } D\text{!)} \tag{5.16}$$

and

$$E^D{}_C \cdot E_G = 0 \tag{5.17}$$

when $D \neq G$.

The matrix of $E^D{}_C$ with respect to the basis E_A is given as follows:

$$(E^D{}_C)^B{}_A = \delta^B{}_C \delta^D{}_A. \tag{5.18}$$

Let us denote:

$$F_r{}^D{}_C = F_r[E^D{}_C]. \tag{5.19}$$

It is seen that:

$$F_r{}^D{}_C(\varphi, \pi) = \Sigma_r(\varphi, \pi)^D{}_C. \tag{5.20}$$

Hence, finally:

$$T_\eta \circ \mathscr{L}^{-1} = \frac{1}{2} \tilde{\mathscr{J}}_{AB} F_r{}^A{}_C F_r{}^B{}_D \eta^{CD}. \tag{5.21}$$

When the potential Φ depends on the configuration only, then $T_\eta \circ \mathscr{L}^{-1}$ is a quadratic form of generators of $GL(U)$.

When remaining within the framework of Hamiltonian mechanics, the full dynamical structure of a system is described by its Hamilton function $H : P \to R$.

It gives rise to the one-parameter group $\{g_t, t \in R\}$ of canonical transformations $g_t: P \to P$. For an arbitrary differentiable function $G: P \to R$, we have:

$$\frac{d}{dt}(G \circ g_t) \underset{t}{\equiv} \{G, H\} \circ g_t. \tag{5.22}$$

In particular, we have the following *Hamiltonian equations of motion*:

$$\begin{aligned} \frac{d}{dt}(Q^i{}_A \circ g_t) &= \frac{\partial H}{\partial P^A{}_i} \circ g_t, \\ \frac{d}{dt}(P^A{}_i \circ g_t) &= -\frac{\partial H}{\partial Q^i{}_A} \circ g_t, \end{aligned} \tag{5.23}$$

where $Q^i{}_A$, $P^A{}_i$ are canonical coordinates and momenta introduced in (2.6) and (2.7).

The Eqs. (5.23) describe the deformative behaviour of the medium after imposing on it homogeneous strain-constraints. Therefore, they replace the usual Lamé equations. It should be noticed that the Lamé equations must not be used literally in the case we consider here, because they do not take into account the reactions of constraints. The constraints we refer to are described by the following homogeneity conditions:

$$\frac{\partial^2 u^i}{\partial X^A \partial X^B} = 0,$$

where u is the displacement vector.

6. General ideas of quantization

At low temperatures, quantum effects should be taken into account.

The quantization consists in replacing phase-space functions by some appropriate Hermitean operators and solving the corresponding eigenequations.

Let us start with some mathematical remarks. There exist two distinguished measures on the manifold $Q = LI(U, V)$. In fact, Q is an open subset of the finite-dimensional linear space $L(U, V)$, hence the usual Lebesgue measure [on $L(U, V)$] can be used. The corresponding integration will be denoted as follows:

$$\int F(\varphi)\, d\varphi.$$

In linear coordinates, we have:

$$\int F(\varphi)\, d\varphi = \int F(\varphi)\, d\varphi^1{}_1 \ldots d\varphi^i{}_A \ldots d\varphi^n{}_n \tag{6.1}$$

up to a constant factor. In what follows, we make use of such coordinates only where this factor equals one.

Obviously, such a measure is invariant under all affine translations:

$$\int F(\varphi + a)\, d\varphi = \int F(\varphi)\, d\varphi \tag{6.2}$$

for arbitrary $a \in L(U, V)$.

In addition to the Lebesgue measure, we have at our disposal the second distinguished measure Δ. It is uniquely (up to a constant factor) defined by the requirement of invariance under the action of groups $GL(V)$, $GL(U)$ on the manifold Q [7]:

$$\int F(A\varphi)d\Delta(\varphi) = \int F(\varphi B)d\Delta(\varphi) = \int F(\varphi)d\Delta(\varphi) \tag{6.3}$$

for arbitrary integrable function F and arbitrary $A \in GL(V)$, $B \in GL(U)$. In linear coordinates:

$$\int F(\varphi)d\Delta(\varphi) = \int F(\varphi)(\det\hat{\varphi})^{-n}d\varphi^1{}_1 \ldots d\varphi^i{}_A \ldots d\varphi^n{}_n = \int F(\varphi)(\det\hat{\varphi})^{-n}d\varphi, \tag{6.4}$$

where $\hat{\varphi}$ is a matrix of φ with respect to the chosen basis.

The measures above give rise to the scalar products in the corresponding spaces of square-integrable functions $L^2(Q, d\varphi)$, $L^2(Q, \Delta)$. These products are defined as follows:

$$(\psi_1|\psi_2) = \int \psi_1{}^*(\varphi)\psi_2(\varphi)d\varphi \tag{6.5}$$

for $\psi_1, \psi_2 \in L^2(Q, d\varphi)$ and:

$$\langle\psi_1|\psi_2\rangle = \int \psi_1{}^*(\varphi)\psi_2(\varphi)d\Delta(\varphi), \tag{6.6}$$

where $\psi_1, \psi_2 L^2(Q, \Delta)$.

Let $a \in L(U, V)$, $A \in GL(V)$, $B \in GL(U)$. We define *translation operators* T_a, L_A, R_B according to the formulas:

$$(T_a\psi)(\varphi) = \psi(\varphi - a), \tag{6.7}$$

$$(L_A\psi)(\varphi) = \psi(A^{-1}\varphi), \tag{6.8}$$

$$(R_B\psi)(\varphi) = \psi(\varphi B^{-1}). \tag{6.9}$$

The translations above are unitary operators in the corresponding L^2-spaces:

$$(T_a\psi_1|T_a\psi_2) = (\psi_1|\psi_2), \tag{6.19}$$

$$\langle L_A\psi_1|L_A\psi_2\rangle = \langle\psi_1|\psi_2\rangle, \tag{6.11}$$

$$\langle R_B\psi_1|R_B\psi_2\rangle = \langle\psi_1|\psi_2\rangle. \tag{6.12}$$

This follows easily from the invariance properties (6.2), (6.3) of measures $d\varphi$, Δ, respectively.

Now, let us sketch briefly the general outline of quantization. In principle, any of the Hilbert spaces $L^2(Q, d\varphi)$, $L^2(Q, \Delta)$ can be used to describe the set of pure quantum states. The first of them, $L^2(Q, d\varphi)$ with the scalar product (6.5), is more convenient when studying a system with the "physical" form of the kinetic energy, T_g [cf. (3.3), (3.3a), (3.5)]. By contrast, the theory of systems with "non-physical", comoving kinetic energy T_η becomes simpler and clearer when formulated in $L^2(Q, \Delta)$ using the scalar product (6.6).

Pure quantum states are described by means of *wave functions*—i.e., normed elements of the corresponding Hilbert space ($(\psi|\psi) = 1$, or, alternatively, $\langle\psi|\psi\rangle = 1$). Physical quantities are represented by *self-adjoint* operators acting on a Hilbert space.

Let us start with the usual mechanical system the kinetic energy of which is given by T_g (on the classical level, of course). Hence, we are working in $L^2(Q, d\varphi)$. *Generalized coordinates* are described by the following natural operators:

$$(\hat{Q}^i{}_A\psi)(\varphi) = \varphi^i{}_A\psi(\varphi). \tag{6.13}$$

Generalized momenta are given by the following differential operators $\hat{P}^A{}_i$:

$$\hat{P}^A{}_i = \frac{\hbar}{i}\frac{\partial}{\partial\varphi^i{}_A}. \tag{6.14}$$

Their domain consists of differentiable functions in $L^2(Q, d\varphi)$. Obviously, both $\hat{Q}^i{}_A$, $\hat{P}^A{}_i$ are *formally self-adjoint* with respect to the scalar product (6.5):

$$(\psi_1|\hat{Q}^i{}_A\psi_2) = (\hat{Q}^i{}_A\psi_1|\psi_2),$$

$$(\psi_1|\hat{P}^A{}_i\psi_2) = (\hat{P}^A{}_i\psi_1|\psi_2),$$

for arbitrary ψ_1, $\psi_2 \in C_0^\infty(Q)$. The first of the above equations follows from the reality of $\varphi^i{}_A$; the second is an obvious consequence of the unitarity of translations T_a generated infinitesimally by $\hat{P}^A{}_i$.

Replacing the classical functions $P^A{}_i$ in the formula (5.11) by the operators $\hat{P}^A{}_i$, we obtain the following *operator of the kinetic energy*:

$$\hat{T}_g = \frac{1}{2}\tilde{\mathscr{J}}_{AB}\hat{P}^A{}_i\hat{P}^B{}_j g^{ij} = -\frac{\hbar^2}{2}\tilde{\mathscr{J}}_{AB}g^{ij}\frac{\partial^2}{\partial\varphi^i{}_A\partial\varphi^j{}_B}. \tag{6.15}$$

Hence, in the case of static interactions, we have the following *Hamilton operator*:

$$\hat{H} = \hat{T}_g + \hat{\Phi}, \tag{6.16}$$

where $\hat{\Phi}$ is the potential operator the action of which consists in multiplying wave functions by the classical potential Φ:

$$\hat{\Phi}\psi = \Phi\psi.$$

The energy levels—i.e., admissible energy values E of the homogeneously deformable medium—are given by the solutions of the eigenequation for the Hamiltonian $\hat{H}$:

$$\hat{H}\psi = E\psi. \tag{6.17}$$

Some explicit forms of $\hat{H}$ and the corresponding eigenequations will be discussed in subsequent papers.

Now, let us consider the medium the kinetic energy of which is given classically by T_η. Hence, as already indicated, we shall make use of the Hilbert space $L^2(Q, \Delta)$, with the scalar product (6.6). The classical functions $F_r{}^A{}_D$ (and, similarly, $F_I{}^i{}_j$) as given in (5.20), should be replaced by the following differential operators:

$$\hat{F}_i{}^j{}_i = \frac{\hbar}{i}X_i{}^j{}_i = \frac{\hbar}{i}\hat{Q}^j{}_A\frac{\partial}{\partial\varphi^i{}_A} = \hat{Q}^j{}_A\hat{P}^A{}_i, \tag{6.18}$$

$$\hat{F}_r{}^A{}_B = \frac{\hbar}{i}X_r{}^A{}_B = \frac{\hbar}{i}\hat{Q}^i{}_B\frac{\partial}{\partial\varphi^i{}_A} = \hat{Q}^i{}_B\hat{P}^A{}_i. \tag{6.19}$$

Their domain consists of differentiable functions in $L^2(Q, \Delta)$. Operators $\hat{F}_l{}^j{}_i$, $\hat{F}_r{}^A{}_B$ are *formally selfadjoint* in $L^2(Q, \Delta)$—i.e.:

$$\langle \psi_1 | \hat{F}_l{}^j{}_i \psi_2 \rangle = \langle \hat{F}_l{}^j{}_i \psi_1 | \psi_2 \rangle,$$
$$\langle \psi_1 | \hat{F}_r{}^A{}_B \psi_2 \rangle = \langle \hat{F}_r{}^A{}_B \psi_1 | \psi_2 \rangle,$$

for arbitrary $\psi_1, \psi_2 \in C_0^\infty(Q)$.

This is implied by the unitarity of translations L_A, R_B, generated infinitesimally by $\hat{F}_l{}^j{}_i$, $\hat{F}_r{}^A{}_B$.

R e m a r k. Operators $\hat{F}_l{}^j{}_i$, $\hat{F}_r{}^A{}_B$, defined in (6.18), (6.19), are not formally selfadjoint with respect to the scalar product (6.5), although they are with respect to (6.6). To obtain the (6.5)—self-adjoint operators corresponding to the classical functions $F_l{}^j{}_i$, $F_r{}^A{}_B$ — it is necessary to replace them by the following operators:

$$\hat{\mathscr{F}}_r{}^A{}_B = F_r{}^A{}_B + \frac{\hbar n}{2i}\delta^A{}_B, \tag{6.20}$$

$$\mathscr{F}_l{}^j{}_i = \hat{F}_l{}^j{}_i + \frac{\hbar n}{2i}\delta^j{}_i. \tag{6.21}$$

The additional non-differential terms compensate the effect of the non-invariance of the measure $d\varphi$ under groups $GL(U)$, $GL(V)$ generated infinitesimally by $X_r{}^A{}_B$, $X_l{}^j{}_i$, cf. (6.18), (6.19).

Similarly as in the previous case, the position operators $Q^i{}_A$ defined by (6.13), are also formally self-adjoint:

$$\langle \psi_1 | \hat{Q}^i{}_A \psi_2 \rangle = \langle \hat{Q}^i{}_A \psi_1 | \psi_2 \rangle.$$

Replacing the functions $F_r{}^A{}_B$ in (5.21) by the operators $\hat{F}_r{}^A{}_B$, we obtain the following operator of the comoving kinetic energy:

$$\hat{T}_\eta = \frac{1}{2}\tilde{\mathscr{J}}_{AB}\hat{F}_r{}^A{}_C\hat{F}_r{}^B{}_D\eta^{CD}. \tag{6.22}$$

Similarly as in the previous, "physical" case, the quantum dynamical problem consists in the solution of the *Schrödinger equation*:

$$\hat{H}\psi = E\psi,$$

where $\hat{H} = \hat{T}_\eta + \hat{\Phi}$ (assuming no kinetic, e.g., magnetic interactions are present).

In a special case of spherically-symmetric distribution of matter, $\mathscr{J} = \mu\tilde{\eta}$, we obtain the following form of the kinetic energy:

$$\hat{T}_\eta = \frac{1}{2\varphi}\eta_{AB}\hat{F}_r{}^A{}_C\hat{F}_r{}^B{}_D\eta^{CD} \tag{6.23}$$

or, in orthonormal basis ($\eta_{AB} = \delta_{AB}$):

$$\hat{T}_\eta = \frac{1}{2\mu}\sum_{A,B}{}'\hat{F}_r{}^A{}_B\hat{F}_r{}^A{}_B. \tag{6.24}$$

Hence, $\hat{T}_\eta$ possesses very high dynamical symmetries. In fact, $\hat{T}_\eta$ commutes then with all the operators $\hat{F}_l{}^j{}_i$, and consequently, with translations L. Therefore, it is invariant under the groups $GL(V)$ [and under η-orthogonal subgroup of $GL(U)$].

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