CHAPTER VIII.

FORM $\int F(x, \sqrt{R})\,dx$, WHERE R IS QUADRATIC.

275. The integration of expressions of the type

$$\int \frac{dx}{X\sqrt{Y}}$$

can be effected in all cases for which X and Y are rational integral algebraic expressions of degree not greater than the second.

There are **four Cases**:

I. X and Y, both linear.	Put $\sqrt{Y}=y$.
II. X quadratic, Y linear.	
III. X linear, Y quadratic.	Put $X=\dfrac{1}{y}$.
IV. X quadratic, Y quadratic.	Put $\dfrac{Y}{X}=y$ or y^2.

The general substitution $\dfrac{Y}{X}=y$ or y^2 will effect the integration in all cases. But the simpler substitutions noted, viz.:

$$\sqrt{Y}=y \text{ in Cases I. and II.}$$

and $$X=\frac{1}{y} \text{ in Case III., are better.}$$

Case IV., in which we employ the substitution

$$\frac{Y}{X}=y \text{ or } y^2,$$

is much more troublesome, but *includes the previous ones*.

We shall, in all cases, assume the radical $\sqrt{Y}$ to be real.

276. Case I. **X linear, Y linear.**

Let $I=\displaystyle\int\frac{dx}{(ax+b)\sqrt{px+q}}$.

Putting $\sqrt{Y} \equiv \sqrt{px+q} = y,$

$$\frac{p\,dx}{2\sqrt{px+q}} = dy$$

and $$ax+b = \frac{a}{p}(y^2-q)+b.$$

Thus I becomes $2\int\frac{dy}{ay^2-aq+bp}$,

which being of the standard form

$$\frac{2}{a}\int\frac{dy}{y^2\pm\lambda^2}, \text{ where } \lambda^2 = \frac{bp\sim aq}{a},$$

is immediately integrable, viz.:

$$= \frac{2}{a\lambda}\tan^{-1}\frac{y}{\lambda} \quad \text{or} \quad -\frac{2}{a\lambda}\coth^{-1}\frac{y}{\lambda},$$

according as $\frac{bp-aq}{a}$ is positive or negative,

i.e. $$= \frac{2}{\sqrt{a(bp-aq)}}\tan^{-1}\frac{\sqrt{a(px+q)}}{\sqrt{bp-aq}}$$

or $$= -\frac{2}{\sqrt{a(aq-bp)}}\coth^{-1}\frac{\sqrt{a(px+q)}}{\sqrt{aq-bp}},$$

i.e. $$= \frac{2}{\sqrt{a(bp-aq)}}\sin^{-1}\sqrt{\frac{a}{p}}\sqrt{\frac{px+q}{ax+b}}$$

or $$= -\frac{2}{\sqrt{a(aq-bp)}}\cosh^{-1}\sqrt{\frac{a}{p}}\sqrt{\frac{px+q}{ax+b}},$$

with other forms, the real one to be chosen in each case.

277. Another Method.

The last form shows how the factors of the integrand are involved in the result of integration, and indicates that the substitution $\frac{px+q}{ax+b} = y^2$ mentioned above as the *general* substitution would have led directly to this result.

If we elect to proceed in this way, viz. putting $\frac{px+q}{ax+b} = y^2$, we have

$$\left(\frac{p}{px+q} - \frac{a}{ax+b}\right)dx = 2\,\frac{dy}{y};$$

$$\therefore \frac{dx}{(ax+b)(px+q)} = \frac{2}{bp-aq}\,\frac{dy}{y} \quad \text{or} \quad -\frac{2}{aq-bp}\,\frac{dy}{y}.$$

Now $x=\dfrac{by^2-q}{p-ay^2}$ and $px+q=\dfrac{(bp-aq)y^2}{p-ay^2}$ or $\dfrac{(aq-bp)y^2}{ay^2-p}$.

When $bp-aq$ is positive,

$$\int\frac{dx}{(ax+b)\sqrt{px+q}}=\frac{2}{\sqrt{bp-aq}}\int\frac{dy}{\sqrt{p-ay^2}}$$

$$=\frac{2}{\sqrt{a(bp-aq)}}\int\frac{dy}{\sqrt{\frac{p}{a}-y^2}}$$

$$=\frac{2}{\sqrt{a(bp-aq)}}\sin^{-1}\sqrt{\frac{a}{p}}\,y,$$

or other forms.

When $bp-aq$ is negative,

$$\int\frac{dx}{(ax+b)\sqrt{px+q}}=-\frac{2}{\sqrt{aq-bp}}\int\frac{dy}{\sqrt{ay^2-p}}$$

$$=-\frac{2}{\sqrt{a(aq-bp)}}\int\frac{dy}{\sqrt{y^2-\frac{p}{a}}}$$

$$=-\frac{2}{\sqrt{a(aq-bp)}}\cosh^{-1}\sqrt{\frac{a}{p}}\,y$$

or

$$=-\frac{2}{\sqrt{a(aq-bp)}}\sinh^{-1}\sqrt{-\frac{a}{p}}\,y,$$

or other forms, the real form to be chosen in each case.

278. **Illustrative Examples.**

Ex. 1. Integrate $I=\displaystyle\int\frac{dx}{(2x+3)\sqrt{4x+5}}$.

Put $\sqrt{4x+5}=y$; $\therefore\ \dfrac{2dx}{\sqrt{4x+5}}=dy$.

Also $2x+3=\dfrac{y^2+1}{2}$;

$$\therefore\ I=\int\frac{dy}{y^2+1}=\tan^{-1}\sqrt{4x+5}.$$

Again, if we put $\dfrac{4x+5}{2x+3}=z$, *i.e.* $x=\dfrac{3z-5}{2(2-z)}$ and $dx=\dfrac{dz}{2(2-z)^2}$,

$$I=\frac{1}{2}\int\frac{dy}{\sqrt{z(2-z)}}=\sin^{-1}\sqrt{\frac{z}{2}}\quad\text{(Art. 87)},$$

i.e.

$$=\sin^{-1}\frac{1}{\sqrt{2}}\sqrt{\frac{4x+5}{2x+3}},$$

which is the same as before, but exhibits the result as a function of both the factors of the integrand.

Ex. 2. Integrate $$I\equiv\int\frac{dx}{(1-x)\sqrt{2-x}}.$$

Let $\sqrt{2-x}=y$; $\therefore\ \frac{dx}{\sqrt{2-x}}=-2dy$;

$$\therefore\ I=-2\int\frac{dy}{y^2-1}=-\int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy=-\log\frac{y-1}{y+1}$$

$$=\log\frac{y+1}{y-1}=\log\frac{\sqrt{2-x}+1}{\sqrt{2-x}-1}, \text{ or other forms.}$$

279. An Extension.

The same substitution, viz. $\sqrt{Y}=y$, will suffice for the integration of

$$\int\frac{\phi(x)\,dx}{X\sqrt{Y}},$$

where X, Y are both linear and $\phi(x)$ is any rational integral algebraic function of x.

For if $Y\equiv px+q=y^2$, then $x=\frac{y^2-q}{p}$, and $p\,dx=2y\,dy$;

thus $$\int\frac{\phi(x)dx}{(ax+b)\sqrt{px+q}}=2\int\frac{\phi\left(\frac{y^2-q}{p}\right)dy}{ay^2-aq+bp};$$

and if $\phi\left(\frac{y^2-q}{p}\right)$ be expanded in descending powers of y^2 and then divided by $ay^2+(bp-aq)$ till the remainder is independent of y, we have to integrate with regard to y an expression of form

$$A_0y^{2n-2}+A_1y^{2n-4}+A_2y^{2n-6}+\dots+A_{n-1}+\frac{B}{ay^2+(bp-aq)},$$

n being the degree of $\phi(x)$ in x; and each term is at once integrable, after which operation y is to be written back as $\sqrt{px+q}$.

280. Ex. Integrate $$I\equiv\int\frac{x^4\,dx}{(x-1)\sqrt{x+2}}.$$

Writing $\sqrt{x+2}=y$, we have $\frac{dx}{\sqrt{x+2}}=2dy$ and $x=y^2-2$, so that

$$\frac{x^4}{x-1}=\frac{(y^2-2)^4}{y^2-3}=y^6-5y^4+9y^2-5+\frac{1}{y^2-3} \text{ by division.}$$

Thus

$$\frac{I}{2}=\int\left(y^6-5y^4+9y^2-5+\frac{1}{y^2-3}\right)dy$$

$$=\frac{y^7}{7}-y^5+3y^3-5y-\frac{1}{\sqrt{3}}\coth^{-1}\frac{y}{\sqrt{3}}$$

$$=\frac{(x+2)^{\frac{1}{2}}}{7}\{(x+2)^3-7(x+2)^2+21(x+2)-35\}-\frac{1}{\sqrt{3}}\coth^{-1}\sqrt{\frac{x+2}{3}},$$

i.e. $$I=\frac{2\sqrt{x+2}}{7}(x^3-x^2+5x-13)+\frac{1}{\sqrt{3}}\log\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}$$

if the logarithmic form be preferred.

281. Forms reducible to Case I.

The student should note the variety of forms reducible to the case considered, viz. X linear, Y linear, by a proper substitution. For example,

(1) $\int\frac{\sin\theta\,d\theta}{(a\cos\theta+b)\sqrt{p\cos\theta+q}}$, put $\cos\theta=x$, *i.e.* $\theta=\cos^{-1}x$.

(2) $\int\frac{\sqrt{\operatorname{cosec}\theta}\,d\theta}{(a\cos\theta+b\sin\theta)\sqrt{p\cos\theta+q\sin\theta}}$, put $\cot\theta=x$, *i.e.* $\theta=\cot^{-1}x$.

(3) $\int\frac{\cos\theta\,d\theta}{(a\cos^2\theta+b\sin^2\theta)\sqrt{p\cos^2\theta+q\sin^2\theta}}$, put $\cot^2\theta=x$, *i.e.* $\theta=\cot^{-1}\sqrt{x}$.

(4) $\int\frac{L\cos\theta+M\sin\theta\,d\theta}{(a\cos^2\theta+b\sin^2\theta)\sqrt{p\cos^2\theta+q\sin^2\theta}}$, separate into two integrals, put $\cot^2\theta=x$ in one and $\tan^2\theta=y$ in the other.

(5) $\int\frac{e^{\frac{x}{2}}\,dx}{(ae^x+be^{-x})\sqrt{pe^x+qe^{-x}}}$, put $e^{2x}=y$, *i.e.* $x=\frac{1}{2}\log y$.

(6) $\int\frac{dx}{x\log(ax^b)\sqrt{\log(cx^d)}}$, put $\log x=y$, *i.e.* $x=e^y$.

(7) $\int\frac{dx}{(ax+b)\sqrt{x(px+q)}}$, put $x=\frac{1}{y}$.

etc.

282. Ex. Integrate

$$I\equiv\int\frac{1+2\cos^4\theta}{(1+12\cos^2\theta)\sqrt{1+3\cos^2\theta}}\frac{\sin\theta}{\cos^4\theta}d\theta.$$

Put $\tan^2\theta=y$; $\therefore\ 2\tan\theta\sec^2\theta\,d\theta=dy$;

$$\therefore\ I=\int\frac{\sec^4\theta+2}{(\sec^2\theta+12)\sqrt{\sec^2\theta+3}}\tan\theta\sec^2\theta\,d\theta=\frac{1}{2}\int\frac{(y+1)^2+2}{(y+13)\sqrt{y+4}}dy.$$

Now put $\sqrt{y+4}=z$; $\therefore\ y=z^2-4,\ dy=2z\,dz$;

$$\therefore\ I=\int\frac{z^4-6z^2+11}{z^2+9}\,dz=\int\left(z^2-15+\frac{146}{z^2+9}\right)dz$$

$$=\frac{z^3}{3}-15z+\frac{146}{3}\tan^{-1}\frac{z}{3},\ \text{ where } z=\sqrt{y+4}=\sqrt{\tan^2\theta+4};$$

$$\therefore\ I=\frac{(\tan^2\theta+4)^{\frac{3}{2}}}{3}-15\,(\tan^2\theta+4)^{\frac{1}{2}}+\frac{146}{3}\tan^{-1}\frac{\sqrt{\tan^2\theta+4}}{3}$$

or $$\frac{(\sec^2\theta+3)^{\frac{3}{2}}}{3}-15\,(\sec^2\theta+3)^{\frac{1}{2}}+\frac{146}{3}\tan^{-1}\frac{\sqrt{\sec^2\theta+3}}{3}.$$

283. Case II. X quadratic, Y linear.

Let $I\equiv\int\frac{Mx+N}{(ax^2+bx+c)\sqrt{px+q}}\,dx$, M and N being constants. The terms $Mx+N$ now existent in the numerator do not introduce any difficulty and make the result more general.

The same substitution being made, viz. $\sqrt{Y}=y$, we put

$$\sqrt{px+q}=y;\ \therefore\ \frac{p}{2\sqrt{px+q}}\,dx=dy.$$

$$ax^2+bx+c \text{ reduces to the form } Ay^4+By^2+C$$

and $\quad Mx+N \quad$ reduces to the form $M'y^2+N'$

Thus I takes the form $\frac{2}{p}\int\frac{M'y^2+N'}{Ay^4+By^2+C}\,dy.$

Now $\frac{M'y^2+N'}{Ay^4+By^2+C}$ can be thrown into partial fractions of the form

$$\frac{\lambda y+\mu}{\alpha y^2+\beta y+\gamma}+\frac{\lambda' y+\mu'}{\alpha' y^2+\beta' y+\gamma'},$$

and each portion is integrable by earlier rules (Arts. 155 and 156).

284. Extension.

Further, it is evident that the same substitution will effect the integration $\int\frac{\phi(x)dx}{(ax^2+bx+c)\sqrt{px+q}}$, where $\phi(x)$ is any rational integral algebraic function of x. For when $px+q=y^2$,

i.e. $$x=\frac{y^2-q}{p}\quad\text{and}\quad\frac{dx}{\sqrt{px+q}}=\frac{2}{p}\,dy,$$

$\frac{\phi(x)}{ax^2+bx+c}$ reduces to the form

$$\frac{A_0y^{2n}+A_1y^{2n-2}+A_2y^{2n-4}+\ldots+A_n}{Ay^4+By^2+C}$$

where n is the degree of $\phi(x)$ in x; and therefore, by division and our rules for partial fractions the integrand may be expressed as

$$P_0y^{2n-4}+P_1y^{2n-6}+\dots P_{n-2}+\frac{\lambda y+\mu}{\alpha y^2+\beta y+\gamma}+\frac{\lambda' y+\mu'}{\alpha' y^2+\beta' y+\gamma'},$$

and each term is integrable.

Again, $$I\equiv\int\frac{\phi(x)}{\chi(x)}\frac{dx}{\sqrt{px+q}},$$

where ϕ and χ are any rational integral algebraic functions of x, may now be seen to be integrable by the same substitution, for it becomes

$$I=\frac{2}{p}\int\frac{\phi\left(\frac{y^2-q}{p}\right)}{\chi\left(\frac{y^2-q}{p}\right)}dy,$$

and the new integrand can be expressed by partial fraction methods (Art. 152, etc.) in the form

$$\Sigma Py^r+\sum\frac{Q}{y-\alpha}+\sum\frac{R}{(y-\beta)^s}+\sum\frac{\lambda y+\mu}{Ay^2+By+C}+\sum\frac{\lambda' y+\mu'}{(A'y^2+B'y+C')^t},$$

and integrals of the expressions of the first four terms can be obtained by the rules given before, and the integral of the last by aid of the reduction formula established in Art. 238.

285. Ex. 1. Integrate $I\equiv\int\frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx.$

Putting $\sqrt{x+1}=y$, we have $\frac{dx}{\sqrt{x+1}}=2\,dy$,

and $$I=2\int\frac{y^2+1}{y^4+y^2+1}dy=\int\left(\frac{1}{y^2+y+1}+\frac{1}{y^2-y+1}\right)dy$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\tan^{-1}\frac{2y-1}{\sqrt{3}}=-\frac{2}{\sqrt{3}}\tan^{-1}\sqrt{3}\frac{\sqrt{x+1}}{x}$$

$$=\frac{2}{\sqrt{3}}\cos^{-1}\sqrt{3}\sqrt{\frac{x+1}{x^2+3x+3}}+\text{const.}$$

Ex. 2. Integrate $$I\equiv\int\frac{x^2-5x-37}{x^2-7x-30}\frac{dx}{\sqrt{x-1}}.$$

Put $\sqrt{x-1}=y$; $\therefore\ \frac{dx}{\sqrt{x-1}}=2dy$ and $x=1+y^2$;

$$\therefore\ I=2\int\frac{(1+y^2)^2-5(1+y^2)-37}{(1+y^2)^2-7(1+y^2)-30}dy$$

$$=2\int\frac{y^4-3y^2-41}{y^4-5y^2-36}dy$$

$$=2\int\frac{y^4-3y^2-41}{(y^2+4)(y^2-9)}dy$$

$$=2\int\left(1+\frac{1}{y^2+4}+\frac{1}{y^2-9}\right)dy$$

$$=2y+\tan^{-1}\frac{y}{2}-\frac{2}{3}\coth^{-1}\frac{y}{3}$$

$$=2\sqrt{x-1}+\tan^{-1}\frac{\sqrt{x-1}}{2}-\frac{2}{3}\coth^{-1}\frac{\sqrt{x-1}}{3}.$$

286. Forms reducible to Case II.

The student should again note the variety of forms which may be brought under the foregoing rule by suitable substitutions and integrated.

Thus

(1) $\displaystyle\int\frac{\sqrt{\sin\theta}\,d\theta}{(a\cos^2\theta+b\sin\theta\cos\theta+c\sin^2\theta)\sqrt{p\cos\theta+q\sin\theta}}$, put $\theta=\cot^{-1}x$.

(2) $\displaystyle\int\frac{A\sqrt{\sin\theta}+B\sqrt{\cos\theta}}{a\cos^2\theta+b\sin\theta\cos\theta+c\sin^2\theta}\,\frac{d\theta}{\sqrt{p\cos\theta+q\sin\theta}}$, separate into two integrals. Use $\theta=\cot^{-1}x$ in the one and $\theta=\tan^{-1}y$ in the other.

(3) $\displaystyle\int\frac{A\sqrt{\sinh x}+B\sqrt{\cosh x}}{a\cosh^2x+b\sinh x\cosh x+c\sinh^2x}\,\frac{dx}{\sqrt{p\cosh x+q\sinh x}}$, similarly.

(4) $\displaystyle\int\frac{A\sqrt{\sin x}+B\sqrt{\cos x}}{\lambda+\mu\cos 2x+\nu\sin 2x}\,\frac{dx}{\sqrt{p\cos x+q\sin x}}$, from (2).

(5) $\displaystyle\int\frac{A\sqrt{\sin x}+B\sqrt{\cos x}}{a+b\cos(2x+\alpha)}\,\frac{dx}{\sqrt{\cos(x+\beta)}}$, from (4).

287. CASE III. **X linear, Y quadratic.**

The proper substitution is now $X=\dfrac{1}{y}$.

Let $$I\equiv\int\frac{dx}{(ax+b)\sqrt{px^2+qx+r}}.$$

Putting $ax+b=\dfrac{1}{y}$, we have, by logarithmic differentiation,

$$\frac{dx}{ax+b}=-\frac{1}{a}\frac{dy}{y}$$

and $$px^2+qx+r=\frac{p}{a^2}\left(\frac{1}{y}-b\right)^2+\frac{q}{a}\left(\frac{1}{y}-b\right)+r,$$

i.e. of form $$\frac{Ay^2+2By+C}{y^2}.$$

Hence the integral has been reduced to the known form

$$-\frac{1}{a}\int\frac{dy}{\sqrt{Ay^2+2By+C}},$$

which has been discussed in Art. 80.

Ex. Integrate

$$I\equiv\int\frac{dx}{(x-1)\sqrt{x^2-4x+2}}.$$

Let $x-1=\frac{1}{y}$, and therefore $\frac{dx}{x-1}=-\frac{dy}{y}$.

Hence $$I=-\int\frac{dy}{y\sqrt{\left(1+\frac{1}{y}\right)^2-4\left(1+\frac{1}{y}\right)+2}}$$

$$=-\int\frac{dy}{\sqrt{1-2y-y^2}}=-\int\frac{dy}{\sqrt{2-(y+1)^2}}=\cos^{-1}\frac{y+1}{\sqrt{2}}$$

$$=\cos^{-1}\frac{x}{(x-1)\sqrt{2}}.$$

288. **Forms reducible to Case III.**

Again we note the varieties of integrals which may be reduced to the present form by a suitable transformation, for instance:

(1) $\int\frac{\sin\theta\,d\theta}{(a\cos\theta+b)\sqrt{p\cos^2\theta+q\cos\theta+r}}$, put $\cos\theta=x$.

(2) $\int\frac{d\theta}{(a\cos\theta+b\sin\theta)\sqrt{p\cos^2\theta+q\sin\theta\cos\theta+r\sin^2\theta}}$, put $\cot\theta=x$.

(3) $\int\frac{\sin\theta\cos\theta\,d\theta}{(a\cos^2\theta+b\sin^2\theta)\sqrt{p\cos^4\theta+q\sin^2\theta\cos^2\theta+q\sin^4\theta}}$, put $\cot^2\theta=x$.

(4) $\int\frac{dx}{x(ax^n+bx^{-n})\sqrt{px^{2n}+q+rx^{-2n}}}$, put $x^{2n}=y$.

(5) $\int\frac{d\theta}{\cos(\theta+\alpha)\sqrt{a+b\cos 2(\theta+\beta)}}$ from (2)

etc.

289. Remarks.

It will now appear that any integration of the form

$$\int\frac{\phi(x)\,dx}{(ax+b)\sqrt{px^2+qx+r}}$$

can be effected, $\phi(x)$ being any rational integral algebraic function of x. For by division we can express $\frac{\phi(x)}{ax+b}$ in the form

$$A_1x^{n-1}+A_2x^{n-2}+A_3x^{n-3}+\dots+A_{n-1}x+A_n+\frac{M}{ax+b},$$

where n is the degree of $\phi(x)$ in x,

$$A_1x^{n-1}+\dots+A_{n-1}x+A_n$$

is the quotient, and M the remainder independent of x.

We have thus reduced the process to the integration of a number of terms of the class

$$\int\frac{Ex^m\,dx}{\sqrt{px^2+qx+r}}$$

and one of the class

$$\int\frac{M\,dx}{(ax+b)\sqrt{px^2+qx+r}}.$$

The latter has been discussed in Art. 287, and integrals of the former class may be obtained by the reduction formula of Art. 240, viz.

$$x^{m-1}\sqrt{px^2+qx+r}=(m-1)rI_{m-2}+\frac{2m-1}{2}qI_{m-1}+mpI_m,$$

I_m standing for $\int\frac{x^m\,dx}{\sqrt{px^2+qx+r}}$;

that is,

$$I_m=x^{m-1}\frac{\sqrt{px^2+qx+r}}{mp}-\frac{2m-1}{2m}\frac{q}{p}I_{m-1}-\frac{m-1}{m}\frac{r}{p}I_{m-2}.$$

Ex. Integrate $I\equiv\int\frac{x^5+x^2+2}{(x+1)\sqrt{x^2+1}}dx.$

By division $\frac{x^5+x^2+2}{x+1}=x^4-x^3+x^2+\frac{2}{x+1}$;

$$\therefore\ I=\int\left(\frac{x^4}{\sqrt{x^2+1}}-\frac{x^3}{\sqrt{x^2+1}}+\frac{x^2}{\sqrt{x^2+1}}+\frac{2}{(x+1)\sqrt{x^2+1}}\right)dx.$$

Let $I_n=\int\frac{x^n\,dx}{\sqrt{x^2+1}}$.

Then

$$I_0 = \sinh^{-1} x,$$

$$I_1 = \sqrt{x^2+1},$$

$$I_2 = \frac{x\sqrt{x^2+1}}{2} - \frac{1}{2}\sinh^{-1} x, \quad \text{by the reduction formula } (m=2),$$

$$I_3 = \frac{x^2\sqrt{x^2+1}}{3} - \frac{2}{3}I_1 = \frac{x^2\sqrt{x^2+1}}{3} - \frac{2}{3}\sqrt{x^2+1},$$

$$I_4 = \frac{x^3\sqrt{x^2+1}}{4} - \frac{3}{4}I_2 = \frac{x^3\sqrt{x^2+1}}{4} - \frac{3}{4}\cdot\frac{1}{2}x\sqrt{x^2+1} + \frac{3}{4}\cdot\frac{1}{2}\sinh^{-1} x;$$

and for the last part of the integral, viz. $2\int\frac{dx}{(x+1)\sqrt{x^2+1}}$, put $x+1=\frac{1}{y}$;

$$\therefore \frac{dx}{x+1} = -\frac{dy}{y};$$

$$\therefore \int\frac{dx}{(x+1)\sqrt{x^2+1}} = -\int\frac{1}{\sqrt{\left(\frac{1}{y}-1\right)^2+1}}\frac{dy}{y} = -\int\frac{dy}{\sqrt{2y^2-2y+1}}$$

$$= -\frac{1}{\sqrt{2}}\int\frac{dy}{\sqrt{(y-\frac{1}{2})^2+\frac{1}{4}}} = -\frac{1}{\sqrt{2}}\sinh^{-1}(2y-1)$$

$$= -\frac{1}{\sqrt{2}}\sinh^{-1}\frac{1-x}{1+x}.$$

Thus $$I = I_4 - I_3 + I_2 + 2\int\frac{dx}{(x+1)\sqrt{x^2+1}}$$

$$= \frac{x^3\sqrt{x^2+1}}{4} - \frac{x^2\sqrt{x^2+1}}{3} + \frac{1}{8}x\sqrt{x^2+1} + \frac{2}{3}\sqrt{x^2+1}$$

$$-\frac{1}{8}\sinh^{-1} x - \sqrt{2}\sinh^{-1}\frac{1-x}{1+x}.$$

290. Extension.

Further, we are now in a position to effect any integration of the form

$$\int\frac{\phi(x)}{\chi(x)}\frac{dx}{\sqrt{px^2+qx+r}},$$

where $\phi(x)$, $\chi(x)$ are rational integral algebraic functions of x, and all the factors of $\chi(x)$ are real and linear.

For putting $\frac{\phi(x)}{\chi(x)}$ into partial fractions, as described in Arts. 140 to 146,

$$\frac{\phi(x)}{\chi(x)} \equiv \Sigma Ax^m + \sum\frac{B}{x-b} + \sum\frac{C}{(x-c)^n}.$$

Hence the integration can be performed when we can integrate

$$\int\frac{x^m dx}{\sqrt{px^2+qx+r}},\quad \int\frac{dx}{(x-b)\sqrt{px^2+qx+r}},\quad \int\frac{dx}{(x-c)^n\sqrt{px^2+qx+r}}.$$

The first species of integral is reduced as already explained by the formula of Art. 240.

The second species was discussed in Art. 287.

The third species can be reduced as explained in Art. 244, or obtained from

$$\int\frac{dx}{(x-c)\sqrt{px^2+qx+r}}$$

by $n-1$ differentiations with regard to c, as will be explained later.

Examples.

Integrate the following expressions:

1. $\dfrac{1}{x\sqrt{x+1}},\quad \dfrac{1}{(x-1)\sqrt{x+2}},\quad \dfrac{x+1}{(x-1)\sqrt{x+2}},\quad \dfrac{x^2+x+1}{(x+2)\sqrt{x-1}}.$

2. $\dfrac{1}{(x^2+1)\sqrt{x}},\quad \dfrac{1}{(x^2+2x+2)\sqrt{x+1}},$

   $\dfrac{x}{(x^2+2x+2)\sqrt{x+1}},\quad \dfrac{x^2+1}{(x^2+2x+2)\sqrt{x+1}}.$

3. $\dfrac{1}{x\sqrt{x^2+1}},\quad \dfrac{1}{(x+1)\sqrt{x^2+1}},\quad \dfrac{x}{(x+1)\sqrt{x^2+1}},\quad \dfrac{x^2+x+1}{(x+1)\sqrt{x^2+2x+3}}.$

4. Prove that

$$\int\frac{dx}{(x+c)\sqrt{x}}=\frac{2}{\sqrt{c}}\tan^{-1}\sqrt{\frac{x}{c}} \quad\text{or}\quad \frac{1}{\sqrt{-c}}\log_e\frac{\sqrt{x}-\sqrt{-c}}{\sqrt{x}+\sqrt{-c}},$$

according as c is positive or negative. [C. S., 1904.]

5. Integrate $\dfrac{\sqrt{\sin\theta}}{\cos\theta(\cos\theta+\sin\theta)\sqrt{2\cos\theta+3\sin\theta}}.$

6. Integrate $\dfrac{\sqrt{\sin\theta}+\sqrt{\cos\theta}}{(a^2\cos^2\theta-b^2\sin^2\theta)\sqrt{\cos\left(\theta-\frac{\pi}{4}\right)}}$ $(a>b>0)$.

7. Integrate $\dfrac{\tan 2\theta}{\sqrt{\cos^6\theta+\sin^6\theta}}.$ 8. Integrate $\dfrac{(x^3+1)^2}{(x-1)\sqrt{x^2+1}}.$

9. Integrate

$$\frac{1}{\sqrt{(a-x)(x-b)}},\quad \frac{1}{(a-x)\sqrt{x-b}},\quad \frac{1}{(x-b)\sqrt{a-x}},\quad \frac{1}{(1\pm x)\sqrt{1\pm x^2}}.$$

[St. John's, 1890.]

10. Integrate $\dfrac{1}{x\sqrt{(x-a)(x-b)}}$. [Colleges, 1876.]

11. Show that

$$\int\frac{x-3}{x\sqrt{x^2-3x+2}}\,dx=\cosh^{-1}(2x-3)+\frac{3}{\sqrt{2}}\cosh^{-1}\left(\frac{4-3x}{x}\right).$$

[St. John's, 1883.]

12. Integrate $\displaystyle\int\frac{x-\alpha}{x-\beta}\frac{dx}{\sqrt{x^2+2px+q}}$ $(p^2>q)$. [α, 1887.]

13. Integrate

(i) $\displaystyle\int\frac{dx}{x\sqrt{x^2+x+1}}$, [Coll., 1879.] (ii) $\displaystyle\int\frac{dx}{x\sqrt{x^2-a^2}}$, [$\epsilon$, 1883.]

(iii) $\displaystyle\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}$. [I. C. S., 1888.]

14. Integrate

(i) $\displaystyle\int\frac{dx}{(1-x)\sqrt{1-x^2}}$ (in two ways). [Trinity, 1890.]

(ii) $\displaystyle\int\frac{dx}{(1-2x)\sqrt{1+4x}}$. [Coll., 1892.]

15. Integrate

$$\int\frac{dx}{(x-\lambda)\sqrt{x-\mu}},\quad\int\frac{x^3\,dx}{x^2+\lambda^2},\quad\int\frac{dx}{(x+1)(x+2)(x+3)}.$$

[Math. Trip., Pt. I., 1920.]

291. Case IV. X quadratic, Y quadratic.

The integral is now of the form

$$\int\frac{Mx+N}{(a_1x^2+2b_1x+c_1)\sqrt{a_2x^2+2b_2x+c_2}}\,dx,$$

where a linear factor has been inserted in the numerator, as in Case II., for the same reason. [See also Art. 1894, Vol. II.]

Before beginning the integration we make the following preliminary remarks:

292. (1). The numerator of the subject of integration is for the present linear. We shall consider later, as in previous cases, a numerator which is any rational integral algebraic function of x, viz. $\phi(x)$.

293. (2). The cases $b_1^2 \geqq a_1c_1$ and $b_2^2=a_2c_2$ are excluded. For (α) if $b_2^2=a_2c_2$, the expression $\sqrt{a_2x^2+2b_2x+c_2}$ becomes rational as regards x, and such integrations have been already considered.

(β) If $b_1^2 \geqq a_1c_1$, $\dfrac{Mx+N}{a_1x^2+2b_1x+c_1}$

would be resolvable into partial fractions either of the form

$$\frac{A}{fx+g}+\frac{B}{hx+k} \text{ (if } b_1^2>a_1c_1)$$

or $$\frac{A}{fx+g}+\frac{B}{(fx+g)^2} \text{ (if } b_1^2=a_1c_1),$$

and the forms of integral resulting have already been considered in Articles 287 to 290.

294. (3). a_1 may be regarded as positive without loss of generality, for in any case in which this is not so, we may change all the signs of the factor $a_1x^2+2b_1x+c_1$, and finally change back the sign of the result when the integration has been effected.

Hence we assume: (1) a_1 positive, (2) $a_1c_1-b_1^2$ positive.

295. (4). We shall assume the subject of integration real. If $b_2^2>a_2c_2$, the expression $a_2x^2+2b_2x+c_2$ has real factors, and may be written

$$\equiv a_2(x-\lambda_1)(x-\lambda_2), \text{ say, where } \lambda_1<\lambda_2.$$

In order that the radical should be real, we must therefore confine both the limits of integration to lie

either between $-\infty$ and λ_1 } when a_2 is positive,
or between λ_2 and $+\infty$, }

or between λ_1 and λ_2, when a_2 is negative.

If $b_2^2<a_2c_2$, the factors of $a_2x^2+2b_2x+c_2$ are unreal, and the condition a_2 *positive* is all that is necessary that the radical may be real for all values of x. The limits of integration in this case may therefore be any real quantities whatever.

296. Reduction to a Canonical Form.

(5). Lemma. Any three expressions of the forms

$$Mx+N, \quad a_1x^2+2b_1x+c_1, \quad a_2x^2+2b_2x+c_2$$

can be in general simultaneously thrown into the forms

$$P\xi_1+Q\xi_2, \quad p_1\xi_1^2+q_1\xi_2^2, \quad p_2\xi_1^2+q_2\xi_2^2,$$

where ξ_1, ξ_2 are linear expressions of forms $x-x_1$, $x-x_2$ respectively.

In order to do this it is necessary to determine the eight quantities (x_1, x_2), (P, Q), (p_1, q_1), (p_2, q_2); and we have eight linear equations to find them, viz.

$$\begin{array}{lll} p_1 + q_1 = a_1, & p_2 + q_2 = a_2, & P + Q = M, \\ p_1x_1 + q_1x_2 = -b_1, & p_2x_1 + q_2x_2 = -b_2, & Px_1 + Qx_2 = -N, \\ p_1x_1^2 + q_1x_2^2 = c_1, & p_2x_1^2 + q_2x_2^2 = c_2. & \end{array}$$

It follows that

$$\begin{vmatrix} 1, & 1, & a_1 \\ x_1, & x_2, & -b_1 \\ x_1^2, & x_2^2, & c_1 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 1, & 1, & a_2 \\ x_1, & x_2, & -b_2 \\ x_1^2, & x_2^2, & c_2 \end{vmatrix} = 0.$$

Also, as the consideration of the cases in which X or Y are perfect squares is to be rejected, we may assume x_1 not equal to x_2.

The determinants give at once on division by $x_2 - x_1$,

$$\left.\begin{array}{l} a_1x_1x_2 + b_1(x_1+x_2) + c_1 = 0, \\ a_2x_1x_2 + b_2(x_1+x_2) + c_2 = 0, \end{array}\right\} \quad \text{..................(1)}$$

i.e. $$\frac{x_1x_2}{b_1c_2 - b_2c_1} = \frac{x_1+x_2}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1};$$

whence x_1 and x_2 are determined, being the roots of

$$(a_1b_2 - a_2b_1)\rho^2 - (c_1a_2 - c_2a_1)\rho + (b_1c_2 - b_2c_1) = 0, \quad \text{......(2)}$$

i.e. $$C\rho^2 - B\rho + A = 0,$$

where A, B, C are the co-factors of a, b, c, in

$$\Delta \equiv \begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix} \equiv aA + bB + cC.$$

That is, ρ is given by $$\begin{vmatrix} 1, & -\rho, & \rho^2 \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix} = 0.$$

The remaining six quantities are found at once by solving the equations

$$\left.\begin{array}{l} p + q = a_1, \\ px_1 + qx_2 = -b_1, \end{array}\right\} \text{ or } \left.\begin{array}{r} a_2, \\ -b_2, \end{array}\right\} \text{ or } \left.\begin{array}{r} M, \\ -N, \end{array}\right\} \quad \text{........(3)}$$

which give

(p_1, q_1), $\qquad (p_2, q_2)$, $\qquad (P, Q)$ respectively.

297. It may be remarked that the equations (1) may be reproduced immediately from the functions

$$a_1x^2+2b_1x+c_1, \quad a_2x^2+2b_2x+c_2$$

by the simple rule:

"For x^2 write x_1x_2; for $2x$ write (x_1+x_2) and leave the coefficients unaltered."

In the case when $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ equation (2) has one root infinite. Now therefore the general theorem of our Lemma fails, and the case must receive separate consideration.

298. (6). In this case, viz. $\frac{a_1}{a_2}=\frac{b_1}{b_2}$, the three expressions are:

$$Mx+N, \quad a_1\left(x+\frac{b_1}{a_1}\right)^2+c_1-\frac{b_1^2}{a_1}, \quad a_2\left(x+\frac{b_2}{a_2}\right)^2+c_2-\frac{b_2^2}{a_2};$$

and putting $\qquad x+\frac{b_1}{a_1}=\xi=x+\frac{b_2}{a_2},$

they are $\quad M\left(\xi-\frac{b_1}{a_1}\right)+N, \quad a_1\xi^2+c_1-\frac{b_1^2}{a_1}, \quad a_2\xi^2+c_2-\frac{b_2^2}{a_2},$

and therefore are simultaneously reducible to the forms

$$P\xi+Q, \quad p_1\xi^2+q_1, \quad p_2\xi^2+q_2,$$

i.e. the same as if we put $\xi_2=1$ in the former transformation.

299. (7). When $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, the two quadratic functions are the same function, and the integral takes the form

$$I\equiv\int\frac{Mx+N}{(ax^2+2bx+c)^{\frac{3}{2}}}\,dx,$$

and a reduction formula may be used to connect with

$$\int\frac{M'x+N'}{(ax^2+2bx+c)^{\frac{1}{2}}}\,dx;$$

which has been considered before (Art. 85). Or the integral I may be deduced from the latter integral by differentiation with regard to c.

300. Illustrative Examples.

Ex. 1. In the case

$$\frac{3x-11}{(7x^2-46x+103)\sqrt{11x^2-70x+155}},$$

$$\left.\begin{aligned}7x_1x_2-23(x_1+x_2)+103=0,\\ 11x_1x_2-35(x_1+x_2)+155=0;\end{aligned}\right\}\quad \text{whence}\quad \begin{aligned}x_1+x_2=6,\\ x_1x_2=5,\end{aligned}$$

and therefore $x_1=1$, $x_2=5$ (the order is immaterial).

Also $\left.\begin{array}{l} p+q=7, \\ p+5q=23, \end{array}\right\}$ or $\left.\begin{array}{l} =11, \\ =35, \end{array}\right\}$ or $\left.\begin{array}{l} =3, \\ =11, \end{array}\right\}$

giving $\left.\begin{array}{l} p_1=3, \\ q_1=4, \end{array}\right\}$ or $\left.\begin{array}{l} p_2=5, \\ q_2=6, \end{array}\right\}$ or $\left.\begin{array}{l} P=1, \\ Q=2. \end{array}\right\}$

And the transformed result is therefore

$$\frac{\xi_1+2\xi_2}{(3\xi_1{}^2+4\xi_2{}^2)\sqrt{5\xi_1{}^2+6\xi_2{}^2}}.$$

Ex. 2. In the case

$$\frac{x-3}{(3x^2-30x+79)\sqrt{5x^2-50x+131}},$$

we have

$$\frac{a_1}{a_2}=\frac{b_1}{b_2},$$

$$3x^2-30x+\ 79=3(x-5)^2+4,$$

$$5x^2-50x+131=5(x-5)^2+6.$$

Putting $x-5=\xi$, the transformed result is

$$\frac{\xi+2}{(3\xi^2+4)\sqrt{5\xi^2+6}}.$$

301. Taking the general case then, we suppose for the present $\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} \neq 0,$

$$X \equiv a_1x^2+2b_1x+c_1 \equiv p_1\xi_1{}^2+q_1\xi_2{}^2,$$

$$Y \equiv a_2x^2+2b_2x+c_2 \equiv p_2\xi_1{}^2+q_2\xi_2{}^2,$$

where $\xi_1=x-x_1,\quad \xi_2=x-x_2,$

so that $\xi_2-\xi_1=x_1-x_2$ and $d\xi_1=d\xi_2=dx.$

Also, we are to use the transformation

$$y=\frac{Y}{X},\quad i.e. =\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1} \text{ or } \frac{p_2\xi_1{}^2+q_2\xi_2{}^2}{p_1\xi_1{}^2+q_1\xi_2{}^2},$$

and

$$\frac{1}{2y}\frac{dy}{dx}=\frac{p_2\xi_1+q_2\xi_2}{p_2\xi_1{}^2+q_2\xi_2{}^2}-\frac{p_1\xi_1+q_1\xi_2}{p_1\xi_1{}^2+q_1\xi_2{}^2}$$

$$=\begin{vmatrix} p_1, & p_2 \\ q_1, & q_2 \end{vmatrix}(\xi_1-\xi_2)\frac{\xi_1\xi_2}{XY};$$

$$\therefore \frac{dy}{dx}=2\begin{vmatrix} p_1, & p_2 \\ q_1, & q_2 \end{vmatrix}(x_2-x_1)\frac{\xi_1\xi_2}{X^2}=-2K\frac{\xi_1\xi_2}{X^2}, \text{ say.}$$

Now $\left.\begin{array}{l} p_1\ +q_1\ =\ a_1, \\ p_1x_1+q_1x_2=-b_1, \end{array}\right\}$ and $\left.\begin{array}{l} p_2\ +q_2\ =\ a_2, \\ p_2x_1+q_2x_2=-b_2; \end{array}\right\}$

$$\therefore\ C \equiv a_1b_2 - a_2b_1 = (p_2+q_2)(p_1x_1+q_1x_2) - (p_1+q_1)(p_2x_1+q_2x_2)$$
$$= (p_1q_2 - p_2q_1)(x_1 - x_2) = +K,$$

i.e.
$$K = +\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} \equiv C$$

when expressed in terms of the original coefficients.

The points on the graph of

$$y = \frac{a_2x^2 + 2b_2x + c_2}{a_1x^2 + 2b_1x + c_1};$$

where the ordinate has a maximum or a minimum value, *i.e.* the "turning-points," are given by

$$\frac{dy}{dx} = 0, \quad \textit{i.e.} \text{ by } \xi_1\xi_2 = 0,$$

and are therefore at $\xi_1 = 0$ and $\xi_2 = 0$, *i.e.* at $x = x_1$ and $x = x_2$; and the values of the corresponding ordinates, viz. y_1 and y_2, are plainly

$$y_1 = \frac{q_2}{q_1} \quad \text{and} \quad y_2 = \frac{p_2}{p_1}.$$

We shall suppose the graph such that $x = x_1$ gives the minimum ordinate and $x = x_2$ the maximum, and that $x_2 > x_1$.

Again, clearly $y = \frac{a_2}{a_1}$ is an asymptote, and the curve cuts the y-axis where $y = \frac{c_2}{c_1}$. It cuts the x-axis where

$$a_2x^2 + 2b_2x + c_2 = 0,$$

i.e. in real points P, Q if $b_2{}^2 > a_2c_2$,

in unreal points if $b_2{}^2 < a_2c_2$.

It cuts the asymptote where

$$\frac{a_2x^2 + 2b_2x + c_2}{a_1x^2 + 2b_1x + c_1} = \frac{a_2}{a_1},$$

i.e. where
$$x = -\frac{1}{2}\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{1}{2}\frac{B}{C},$$

i.e. at a point R at a finite distance from the y-axis, unless $a_1b_2 - a_2b_1 = 0$, a case for the present excluded.

There are three cases with which we are concerned, *i.e.* in which some portion of the graph lies on the upper side of the x-axis, otherwise $\sqrt{Y}$ would be unreal.

(1)

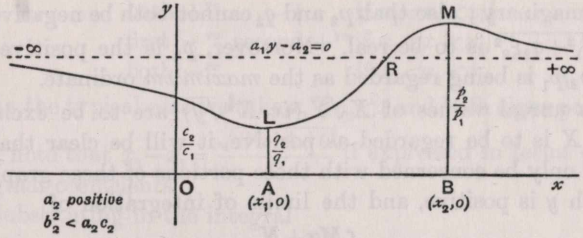

Fig. 17.

(2)

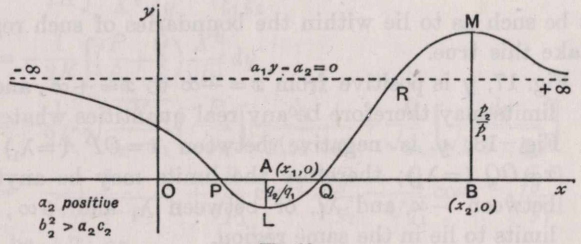

Fig. 18.

(3)

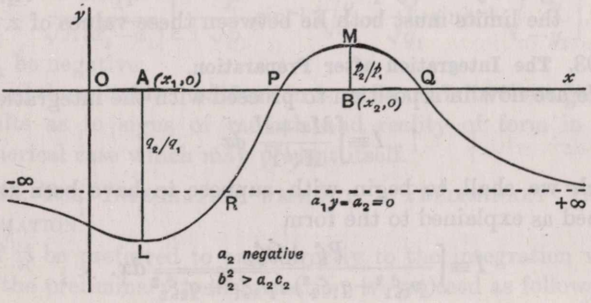

Fig. 19.

302. These are *typical* cases. It will be seen that we have taken $x_2>x_1$ and the turning-points both on the right-hand side of the y-axis, *i.e.* x_1 and x_2 both positive. The student will have no difficulty in making the necessary modifications for any particular case in which the numerical values of the several constants are given. It is to be observed that p_1 and q_1 are necessarily both positive, for a_1 has been taken positive, and the roots of

$$a_1x^2+2b_1x+c_1=0, \quad \textit{i.e. } p_1\xi_1{}^2+q_1\xi_2{}^2=0,$$

are imaginary; also that p_2 and q_2 cannot both be negative, for $\sqrt{p_2\xi_1^2+q_2\xi_2^2}$ is to be real. Moreover, p_2 is the positive one, for p_2/p_1 is being regarded as the *maximum* ordinate.

As unreal values of $X\sqrt{Y}$ (*i.e.* $X^{\frac{3}{2}}\sqrt{y}$) are to be excluded, and X is to be regarded as positive, it will be clear that we shall only be concerned with those portions of these graphs in which y is positive, and the limits of integration of

$$\int\frac{Mx+N}{X\sqrt{Y}}\,dx$$

must be such as to lie within the boundaries of such regions as make this true.

In Fig. 17, y is positive from $x=-\infty$ to $x=+\infty$, and the limits may therefore be any real quantities whatever.

In Fig. 18, y is negative between $x=OP$ $(=\lambda_1)$ and $x=OQ$ $(=\lambda_2)$; therefore the limits may be anything between $-\infty$ and λ_1, or between λ_2 and $+\infty$, both limits to lie in the same region.

In Fig. 19, y is only positive between $x=\lambda_1$ and $x=\lambda_2$, and the limits must both lie between these values of x.

303. The Integration after Preparation.

We are now in a position to proceed with the integration of

$$I\equiv\int\frac{Mx+N}{X\sqrt{Y}}\,dx,$$

which we shall, to begin with, suppose to have been transformed as explained to the form

$$I\equiv\int\frac{P\xi_1+Q\xi_2}{(p_1\xi_1^2+q_1\xi_2^2)\sqrt{p_2\xi_1^2+q_2\xi_2^2}}\,dx.$$

Putting $\dfrac{Y}{X}=y$, we have $dx=-\dfrac{1}{2K}\dfrac{X^2}{\xi_1\xi_2}\,dy$;

also, $$y_2-y=\frac{p_2}{p_1}-\frac{p_2\xi_1^2+q_2\xi_2^2}{p_1\xi_1^2+q_1\xi_2^2}=-\begin{vmatrix}p_1, & q_1\\ p_2, & q_2\end{vmatrix}\frac{\xi_2^2}{p_1X}=\frac{K}{x_2-x_1}\frac{\xi_2^2}{p_1X},$$

$$y-y_1=\frac{p_2\xi_1^2+q_2\xi_2^2}{p_1\xi_1^2+q_1\xi_2^2}-\frac{q_2}{q_1}=-\begin{vmatrix}p_1, & q_1\\ p_2, & q_2\end{vmatrix}\frac{\xi_1^2}{q_1X}=\frac{K}{x_2-x_1}\frac{\xi_1^2}{q_1X};$$

$$\therefore\quad \left.\begin{aligned}\frac{\sqrt{X}}{\xi_2}&=\pm\frac{1}{\sqrt{p_1}}\sqrt{\frac{K}{x_2-x_1}}\frac{1}{\sqrt{y_2-y}},\\ \frac{\sqrt{X}}{\xi_1}&=\pm\frac{1}{\sqrt{q_1}}\sqrt{\frac{K}{x_2-x_1}}\frac{1}{\sqrt{y-y_1}},\end{aligned}\right\}$$ the signs of the ambiguities being governed by the signs of ξ_2 and ξ_1,

i.e. both $+^{\text{ve}}$ if $x_1<x_2<x$,
first $-^{\text{ve}}$, second $+^{\text{ve}}$ if $x_1<x<x_2$,
both $-^{\text{ve}}$ if $x<x_1<x_2$.

As the typical case we take $x_1<x_2<x$ and both signs positive, and note that $x_2-x_1=\dfrac{\sqrt{B^2-4AC}}{C}$ if expressed in terms of the original coefficients.

Substituting in the integral

$$I=-\frac{1}{2K}\int\frac{P\xi_1+Q\xi_2}{X^{\frac{3}{2}}\sqrt{y}}\cdot\frac{X^2}{\xi_1\xi_2}dy$$

$$=-\frac{1}{2K}\int\left(\frac{P}{\xi_2}+\frac{Q}{\xi_1}\right)\frac{X^{\frac{1}{2}}}{\sqrt{y}}dy$$

$$=-\frac{1}{2K}\sqrt{\frac{K}{x_2-x_1}}\left[\frac{P}{\sqrt{p_1}}\int\frac{dy}{\sqrt{y(y_2-y)}}+\frac{Q}{\sqrt{q_1}}\int\frac{dy}{\sqrt{y(y-y_1)}}\right]$$

$$=\frac{1}{\sqrt{K(x_2-x_1)}}\left[+\frac{P}{\sqrt{p_1}}\cos^{-1}\sqrt{\frac{y}{y_2}}-\frac{Q}{\sqrt{q_1}}\cosh^{-1}\sqrt{\frac{y}{y_1}}\right]$$

if y_1 be $+^{\text{ve}}$; or,

$$\frac{1}{\sqrt{K(x_2-x_1)}}\left[-\frac{P}{\sqrt{p_1}}\sin^{-1}\sqrt{\frac{y}{y_2}}-\frac{Q}{\sqrt{q_1}}\sinh^{-1}\sqrt{\frac{y}{-y_1}}\right],$$

if y_1 be negative.

And the suitable modification is to be made in these general results as to signs of radicals and reality of form in each numerical case which may present itself.

304. The Integration without a Preliminary Transformation.

If it be preferred to pass directly to the integration without the preliminary transformation, we proceed as follows:

$$I\equiv\int\frac{Mx+N}{(a_1x^2+2b_1x+c_1)\sqrt{a_2x^2+2b_2x+c_2}}dx.$$

Let
$$y=\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1}.$$

Then
$$\frac{1}{2y}\frac{dy}{dx}=\frac{a_2x+b_2}{a_2x^2+2b_2x+c_2}-\frac{a_1x+b_1}{a_1x^2+2b_1x+c_1}$$

$$=\frac{(a_2x+b_2)(b_1x+c_1)-(a_1x+b_1)(b_2x+c_2)}{XY}$$

$$=-\frac{1}{4}\frac{J}{XY}=-\frac{1}{4}\frac{J}{X^2y},$$

where J is the Jacobian of the two quadratic expressions

$$a_1x^2+2b_1x+c_1, \quad a_2x^2+2b_2x+c_2,$$

viz.
$$J \equiv \begin{vmatrix} 2a_1x+2b_1, & 2b_1x+2c_1 \\ 2a_2x+2b_2, & 2b_2x+2c_2 \end{vmatrix}.$$

Hence
$$\frac{dy}{dx} = -\frac{J}{2X^2}.$$

Let x_1, x_2 be the roots of the equation $J=0$, and y_1, y_2 the corresponding values of y. Then the points (x_1, y_1), (x_2, y_2) are the "turning-points" of y, *i.e.* the points of maximum or minimum ordinates of the graph. Let y_1 be the minimum ordinate, y_2 the maximum.

The equation giving x_1, x_2, *i.e.* $J=0$, is obviously

$$(a_1b_2-a_2b_1)x^2-(c_1a_2-c_2a_1)x+(b_1c_2-b_2c_1)=0,$$

i.e.
$$Cx^2-Bx+A=0,$$

where A, B, C are the cofactors of a, b, c in the standard determinant

$$\Delta = \begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix},$$

and we may write $J \equiv +4C(x-x_1)(x-x_2)$.

Again, any straight line $y=\mu$ will cut the graph of

$$y = \frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1}$$

in two points which are coincident in the two cases $\mu=y_1$ and $\mu=y_2$.

Also
$$y-\mu \equiv \frac{Y}{X}-\mu = \frac{(a_2-a_1\mu)x^2+2(b_2-b_1\mu)x+c_2-c_1\mu}{X}.$$

Hence, when $\mu=y_1$ or y_2 the numerator must contain $(x-x_1)^2$ or $(x-x_2)^2$ as a factor, and the equation

$$(a_2-a_1\mu)x^2+2(b_2-b_1\mu)x+c_2-c_1\mu=0$$

must have in these cases equal roots.

Hence, the necessary values of μ, viz. y_1 and y_2, are the roots of the quadratic

$$(b_2-b_1\mu)^2=(a_2-a_1\mu)(c_2-c_1\mu),$$

i.e.
$$(b_1^2-a_1c_1)\mu^2+(a_1c_2+a_2c_1-2b_1b_2)\mu+(b_2^2-a_2c_2)=0,$$

and
$$y-y_1=(a_2-a_1y_1)\frac{(x-x_1)^2}{X},$$
$$y_2-y=(a_1y_2-a_2)\frac{(x-x_2)^2}{X}$$

[a_2 supposed positive, $y_1<\frac{a_2}{a_1}$, $y_2>\frac{a_2}{a_1}$, y intermediate between y_1 and y_2, Figs. 17 and 18].

Thus
$$x-x_1=\pm\frac{X^{\frac{1}{2}}}{(a_2-a_1y_1)^{\frac{1}{2}}}\sqrt{y-y_1},$$
$$x-x_2=\pm\frac{X^{\frac{1}{2}}}{(a_1y_2-a_2)^{\frac{1}{2}}}\sqrt{y_2-y},$$

the signs of the right-hand sides

being both positive, if $x_1<x_2<x$;

the first positive, the second negative, if $x_1<x<x_2$;

both negative, if $x<x_1<x_2$.

Substituting in the original integral, and taking $x_1<x_2<x$ as the standard case, we have

$$I\equiv\int\frac{Mx+N}{X\sqrt{Y}}\,dx=-\int\frac{Mx+N}{X\sqrt{Y}}\,\frac{2X^2}{J}\,dy$$
$$=-2\int\frac{Mx+N}{\sqrt{y}}\,\frac{X^{\frac{1}{2}}}{J}\,dy$$
$$=-\frac{2}{4C}\int\frac{Mx+N}{(x-x_1)(x-x_2)}\,\frac{X^{\frac{1}{2}}}{\sqrt{y}}\,dy$$
$$=-\frac{1}{2C}\int\left[\frac{Mx_1+N}{x_1-x_2}\,\frac{1}{x-x_1}\,\frac{X^{\frac{1}{2}}}{\sqrt{y}}+\frac{Mx_2+N}{x_2-x_1}\,\frac{1}{x-x_2}\,\frac{X^{\frac{1}{2}}}{\sqrt{y}}\right]dy$$
$$=-\frac{1}{2C}\,\frac{Mx_1+N}{x_1-x_2}\sqrt{a_2-a_1y_1}\int\frac{dy}{\sqrt{y(y-y_1)}}$$
$$-\frac{1}{2C}\,\frac{Mx_2+N}{x_2-x_1}\sqrt{a_1y_2-a_2}\int\frac{dy}{\sqrt{y(y_2-y)}}$$
$$=+F\cosh^{-1}\sqrt{\frac{y}{y_1}}+G\cos^{-1}\sqrt{\frac{y}{y_2}},\text{ if } y_1 \text{ be positive,}$$
or
$$=+F\sinh^{-1}\sqrt{\frac{y}{-y_1}}-G\sin^{-1}\sqrt{\frac{y}{y_2}},\text{ if } y_1 \text{ be negative,}$$

where F and G are constants, viz.

$$F=\frac{Mx_1+N}{\sqrt{B^2-4AC}}\sqrt{a_2-a_1y_1},\quad G=\frac{Mx_2+N}{\sqrt{B^2-4AC}}\sqrt{a_1y_2-a_2},$$

for it has been seen above that $C(x_2-x_1)=\sqrt{B^2-4AC}$.

The suitable modification is to be made in these general results as to sign of radicals and reality of form in each numerical case which may present itself.

305. Comparison of the Processes. Construction of Examples. Considerable arithmetical simplification accrues from the treatment shown in Art. 303, but of course at the cost of the initial reduction to the canonical form.

The method there shown indicates a method of construction of such examples, for the values of $p_1, q_1, p_2, q_2, P, Q, x_1, x_2$ are there all at choice, due care being taken that p_1, q_1 are both taken positive, and that p_2, q_2 are not both negative, as explained in Art. 302.

[See a paper by Russell, cited by Greenhill, *Chapter on the Integral Calculus.*]

306. Various Forms of the Coefficients.

The two coefficients may be thrown into various forms:

for since $$y-\mu\equiv\frac{(a_2-a_1\mu)x^2+2(b_2-b_1\mu)x+c_2-c_1\mu}{X}\quad\text{(Art. 304)}$$

is a fraction with $(x-x_1)^2$ as a factor in the numerator when $\mu=y_1$, or with $(x-x_2)^2$ in the numerator when $\mu=y_2$, we have by comparison of coefficients

$$(a_2-a_1y_1)x_1+(b_2-b_1y_1)=0$$

and $$(a_2-a_1y_2)x_2+(b_2-b_1y_2)=0,$$

so $$y_1=\frac{a_2x_1+b_2}{a_1x_1+b_1}\quad\text{and}\quad y_2=\frac{a_2x_2+b_2}{a_1x_2+b_1}$$

$$a_2-a_1y_1=-\frac{a_1b_2-a_2b_1}{a_1x_1+b_1}\quad\text{and}\quad a_1y_2-a_2=\frac{a_1b_2-a_2b_1}{a_1x_2+b_1},$$

and $$a_1b_2-a_2b_1=K=C\quad\text{(Art. 301).}$$

Also $$\left.\begin{array}{l}p_1+q_1\ \ =\ \ a_1,\\ p_1x_1+q_1x_2=-\ b_1\end{array}\right\}\ \therefore\ p_1=\frac{a_1x_2+b_1}{x_2-x_1},\quad q_1=-\frac{a_1x_1+b_1}{x_2-x_1};$$

$$\left.\begin{array}{l}p_2+q_2\ \ =\ \ a_2,\\ p_2x_1+q_2x_2=-b_2;\end{array}\right\}\ \therefore\ p_2=\frac{a_2x_2+b_2}{x_2-x_1},\quad q_2=-\frac{a_2x_1+b_2}{x_2-x_1};$$

whence we have the following modifications of the coefficients in Art. 303, viz.:

$$\frac{P}{\sqrt{Kp_1(x_2-x_1)}}=\frac{P}{\sqrt{K(a_1x_2+b_1)}}=\frac{P}{\sqrt{p_1}\sqrt[4]{B^2-4AC}}=\frac{P}{K}\sqrt{a_1y_2-a_2}$$

$$=\frac{Mx_2+N}{(x_2-x_1)\sqrt{K(a_1x_2+b_1)}}=\frac{Mx_2+N}{\sqrt{a_1x_2+b_1}}\sqrt{\frac{C}{B^2-4AC}}$$

$$=\frac{Mx_2+N}{\sqrt{B^2-4AC}}\sqrt{a_1y_2-a_2}, \text{ etc.}$$

And similarly for the coefficient involving Q.

307. **Convenient General Form of the Result.**

It appears then that if y_1 and y_2 be respectively the minimum and maximum ordinates of

$$y=\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1}\left(\equiv\frac{Y}{X}\right),$$

and x_1, x_2 the corresponding abscissae, and if $Mx+N$ be written in the form $P(x-x_1)+Q(x-x_2)$, then the integral

$$I\equiv\int\frac{Mx+N}{X\sqrt{Y}}\,dx$$

can be written, amongst many other ways, in the convenient form

$$PP_1\cos^{-1}\sqrt{\frac{y}{y_2}}-QQ_1\cosh^{-1}\sqrt{\frac{y}{y_1}}$$

or

$$-PP_1\sin^{-1}\sqrt{\frac{y}{y_2}}-QQ_1\sinh^{-1}\sqrt{\frac{y}{-y_1}},$$

according as y_1 is $+^{\text{ve}}$ or $-^{\text{ve}}$,

where $$P_1=\frac{\sqrt{a_1y_2-a_2}}{a_1b_2-a_2b_1} \quad\text{and}\quad Q_1=\frac{\sqrt{a_2-a_1y_1}}{a_1b_2-a_2b_1},$$

provided $a_1b_2-a_2b_1\neq 0$.

308. **Remark.**

It is further to be noticed that the two quadratics involved in this discussion, viz.

$$(b_1{}^2-a_1c_1)y^2+(a_1c_2+a_2c_1-2b_1b_2)y+(b_2{}^2-a_2c_2)=0,$$
$$(a_1b_2-a_2b_1)x^2-(c_1a_2-c_2a_1)x+(b_1c_2-b_2c_1)=0,$$

are transformable the one into the other by the homographic transformation

$$y=\frac{a_2x+b_2}{a_1x+b_1}.$$

The one gives the ordinates (y_1, y_2), the other the abscissae (x_1, x_2) of the turning points. [See Salmon, *Higher Algebra*, p. 173.]

309. A Special Case.

It remains to discuss the case we have so far excluded,

$$\text{viz. when } \frac{a_1}{a_2}=\frac{b_1}{b_2}.$$

In this case the asymptote of the graph of

$$y=\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1}\left(\equiv\frac{Y}{X}\right),\ \text{viz. } y=\frac{a_2}{a_1},$$

does not meet the graph at a finite distance from the y-axis, and one of the two turning points has disappeared. It has been seen that the expression can, however, be written

$$y=\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1},\quad \text{where}\quad \begin{aligned} p_1&=a_1 && =+^{\text{ve}} \text{ by Art. 294,}\\ p_2&=a_2, &&\\ \xi&=x-x_1 && x_1=-\frac{b_2}{a_2}=-\frac{b_1}{a_1},\\ q_1&=c_1-\frac{b_1{}^2}{a_1} && =+^{\text{ve}} \text{ by Art. 294,}\\ q_2&=c_2-\frac{b_2{}^2}{a_2}. && \end{aligned}$$

$$\text{Now } \frac{1}{2y}\frac{dy}{dx}=\frac{p_2\xi}{p_2\xi^2+q_2}-\frac{p_1\xi}{p_1\xi^2+q_1}=-\ (p_1q_2-p_2q_1)\frac{\xi}{X^2y}$$

$$\text{and}\qquad \frac{dy}{dx}=-2(p_1q_2-p_2q_1)\frac{\xi}{X^2}.$$

Also $\xi=0$ gives the turning point, viz. $x=x_1$, $y=y_1$; and y_1 obviously is $=\frac{q_2}{q_1}$. The only forms of the graph with which we are concerned are the four following. Cases, in which the graph lies entirely below the x-axis, give rise to entirely unreal values of $\sqrt{Y}$. Note the symmetry in all cases of the graph about an ordinate through the turning point.

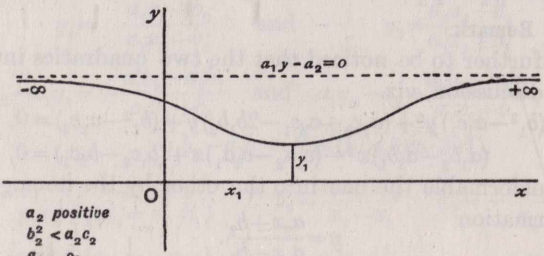

Fig. 20.

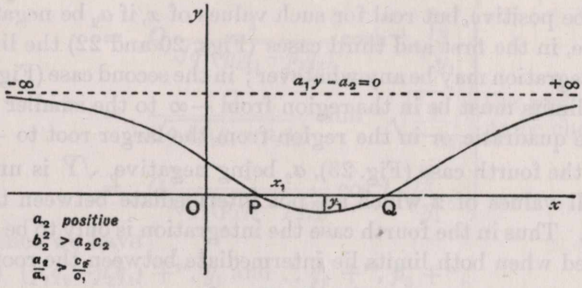

Fig. 21.

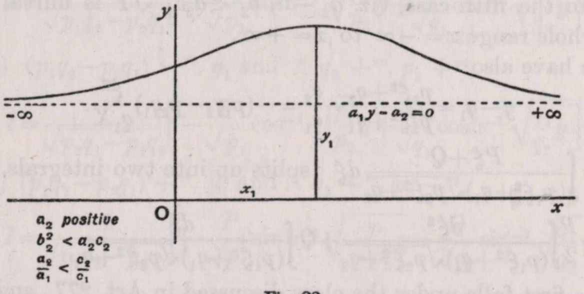

Fig. 22.

with corresponding forms if a_2 be negative, viz.

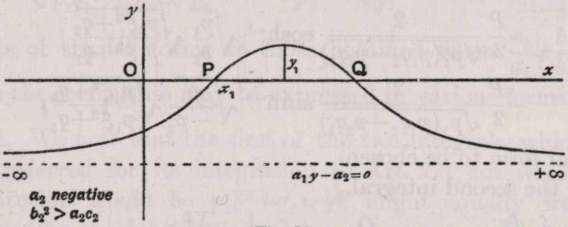

Fig. 23.

In the case a_2 negative, $b_2{}^2<a_2c_2$, the graph is entirely below the x-axis.

310. When the graph cuts the x-axis, as in Figs. 21 and 23, at points P, Q, $\sqrt{Y}$ is unreal for the value of x intermediate between P and Q, *i.e.* intermediate between the roots of

$$a_2x^2+2b_2x+c_2=0,$$

if a_2 be positive, but real for such values of x, if a_2 be negative. Hence, in the first and third cases (Figs. 20 and 22) the limits of integration may be any whatever; in the second case (Fig. 21) both limits must be in the region from $-\infty$ to the smaller root of the quadratic, or in the region from the larger root to $+\infty$; or in the fourth case (Fig. 23), a_2 being negative, $\sqrt{Y}$ is unreal for all values of x which are not intermediate between these roots. Thus in the fourth case the integration is only to be considered when both limits lie intermediate between the roots of

$$a_2x^2+2b_2x+c_2=0.$$

And in the fifth case, viz. a_2 $-^{\text{ve}}$, $b_2{}^2<a_2c_2$, $\sqrt{Y}$ is unreal for the whole range $x=-\infty$ to $x=+\infty$.

We have also

$$y-y_1=\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}-\frac{q_2}{q_1}=-(p_1q_2-p_2q_1)\frac{\xi^2}{q_1X}.$$

$I\equiv\int\frac{P\xi+Q}{(p_1\xi^2+q_1)\sqrt{p_2\xi^2+q_2}}d\xi$ splits up into two integrals, viz.

$$=\frac{P}{2}\int\frac{d\xi^2}{(p_1\xi^2+q_1)\sqrt{p_2\xi^2+q_2}}+Q\int\frac{d\xi}{(p_1\xi^2+q_1)\sqrt{p_2\xi^2+q_2}}.$$

The first falls under the class discussed in Art. 277, and

$$=\frac{P}{2}\frac{2}{\sqrt{p_1(p_2q_1-p_1q_2)}}\sin^{-1}\sqrt{\frac{p_1}{p_2}}\sqrt{\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}}$$

or $$-\frac{P}{2}\frac{2}{\sqrt{p_1(p_1q_2-p_2q_1)}}\cosh^{-1}\sqrt{\frac{p_1}{p_2}}\sqrt{\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}}$$

or $$-\frac{P}{2}\frac{2}{\sqrt{p_1(p_1q_2-p_2q_1)}}\sinh^{-1}\sqrt{\frac{p_1}{-p_2}}\sqrt{\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}},$$

the real form to be chosen.

For the second integral,

$$Q\int\frac{dx}{X\sqrt{Y}}=\frac{Q}{2(p_1q_2-p_2q_1)}\int\frac{1}{X^{\frac{3}{2}}\sqrt{y}}\frac{X^2}{\xi}dy$$

$$=-\frac{Q}{2(p_1q_2-p_2q_1)}\int\frac{1}{\sqrt{y}}\frac{\sqrt{X}}{\xi}dy$$

$$=\frac{Q}{2}\frac{1}{\sqrt{q_1(p_2q_1-p_1q_2)}}\int\frac{dy}{\sqrt{y(y-y_1)}}$$

or $$=-\frac{Q}{2}\frac{1}{\sqrt{q_1(p_1q_2-p_2q_1)}}\int\frac{dy}{\sqrt{y(y_1-y)}},$$

according as $y>$ or $<y_1$,

$$\left.\begin{aligned} \textit{i.e.} \quad &= Q\frac{1}{\sqrt{q_1(p_2q_1-p_1q_2)}}\cosh^{-1}\sqrt{\frac{y}{y_1}} \\ \text{or} \quad &= \frac{Q}{\sqrt{q_1(p_2q_1-p_1q_2)}}\sinh^{-1}\sqrt{\frac{y}{-y_1}} \\ \text{or} \quad &= Q\frac{1}{\sqrt{q_1(p_1q_2-p_2q_1)}}\cos^{-1}\sqrt{\frac{y}{y_1}}, \end{aligned}\right\}\text{the real form to be chosen.}$$

Hence we have

(1) $(p_1q_2-p_2q_1)+^{ve}$, y_1 and $\therefore q_2 +^{ve}$, $p_2 +^{ve}$,

$$I=\frac{1}{\sqrt{p_1q_2-p_2q_1}}\left[-\frac{P}{\sqrt{p_1}}\cosh^{-1}\sqrt{\frac{p_1}{p_2}y}+\frac{Q}{\sqrt{q_1}}\cos^{-1}\sqrt{\frac{q_1}{q_2}y}\right];$$

(2) $(p_1q_2-p_2q_1)-^{ve}$, y_1 and $\therefore q_2 +^{ve}$, $p_2 +^{ve}$,

$$I=\frac{1}{\sqrt{p_2q_1-p_1q_2}}\left[-\frac{P}{\sqrt{p_1}}\cos^{-1}\sqrt{\frac{p_1}{p_2}y}+\frac{Q}{\sqrt{q_1}}\cosh^{-1}\sqrt{\frac{q_1}{q_2}y}\right];$$

(3) $(p_1q_2-p_2q_1)+^{ve}$, y_1 and $\therefore q_2 +^{ve}$, $p_2 -^{ve}$,

$$I=\frac{1}{\sqrt{p_1q_2-p_2q_1}}\left[-\frac{P}{\sqrt{p_1}}\sinh^{-1}\sqrt{\frac{p}{-p_2}y}-\frac{Q}{\sqrt{q_1}}\sin^{-1}\sqrt{\frac{q_1}{q_2}y}\right];$$

(4) $(p_1q_2-p_2q_1)-^{ve}$, y_1 and $\therefore q_2 -^{ve}$, $p_2 +^{ve}$,

$$I=\frac{1}{\sqrt{p_2q_1-p_1q_2}}\left[\frac{P}{\sqrt{p_1}}\sin^{-1}\sqrt{\frac{p_1}{p_2}y}+\frac{Q}{\sqrt{q_1}}\sinh^{-1}\sqrt{\frac{q_1}{-q_2}y}\right];$$

results of similar forms to those obtained when $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$, and again the coefficients may be expressed in various forms.

311. We note that the first of the two integrals, which has been referred for its integration to Art. 277, for which the substitution would be $p_2\xi^2+q_2=y^2$, might equally well be obtained by the substitution

$$\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}=y,$$

i.e. the same as used in the second integral. Upon this substitution being made in the integral I, we get a result of form

$$A\int\frac{\sqrt{X}}{\sqrt{y}}\,dy+B\int\frac{\sqrt{X}}{\xi}\frac{dy}{\sqrt{y}}$$

In the first of these we substitute for $\sqrt{X}$ its value in terms of y, viz. $\sqrt{\dfrac{p_1q_2-p_2q_1}{p_1y-p_2}}$.

In the second we form $y_1-y=+\dfrac{(p_1q_2-p_2q_1)}{q_1}\dfrac{\xi^2}{X}$,

and substitute for $\dfrac{\sqrt{X}}{\xi}$ its value, viz. $\sqrt{\dfrac{p_1q_2-p_2q_1}{q_1}}\dfrac{1}{\sqrt{y_1-y}}$, as shown.

312. Illustrative Examples.

Ex. 1. Consider the integral

$$I\equiv\int\frac{3x-1}{(3x^2-2x+1)\sqrt{2x^2-2x+1}}dx,$$

(*a*) without reduction to the canonical form, (*b*) first reducing it as in Art. 296.

(*a*) Putting
$$y=\frac{2x^2-2x+1}{3x^2-2x+1}\left(\equiv\frac{Y}{X}\right),$$

$$\frac{1}{2y}\frac{dy}{dx}=\frac{2x-1}{2x^2-2x+1}-\frac{3x-1}{3x^2-2x+1}$$

$$=\frac{x(x-1)}{XY}=\frac{x(x-1)}{X^2y}.$$

The turning points are given by $x=0$ and $x=1$. If $x=0$, $y=1$; if $x=1$, $y=\frac{1}{2}$.

$$1-y=1-\frac{2x^2-2x+1}{3x^2-2x+1}=\frac{x^2}{X},$$

$$y-\frac{1}{2}=\frac{2x^2-2x+1}{3x^2-2x+1}-\frac{1}{2}=\frac{(x-1)^2}{2X};$$

$$\therefore\ \frac{\sqrt{X}}{x}=\frac{1}{\sqrt{1-y}},\quad \frac{\sqrt{X}}{x-1}=\frac{1}{\sqrt{2}\sqrt{y-\frac{1}{2}}},\quad \text{assuming } x>1;$$

$$\therefore\ I=\int\frac{3x-1}{X\sqrt{Xy}}\cdot\frac{X^2dy}{2x(x-1)}$$

$$=\frac{1}{2}\int\left(\frac{1}{x}+\frac{2}{x-1}\right)\frac{\sqrt{X}\,dy}{\sqrt{y}}$$

$$=\frac{1}{2}\int\left(\frac{1}{\sqrt{y(1-y)}}+\frac{\sqrt{2}}{\sqrt{y(y-\frac{1}{2})}}\right)dy$$

$$=-\cos^{-1}\sqrt{y}+\sqrt{2}\cosh^{-1}\sqrt{2y}$$

$$=-\cos^{-1}\sqrt{\frac{2x^2-2x+1}{3x^2-2x+1}}+\sqrt{2}\cosh^{-1}\sqrt{2\frac{2x^2-2x+1}{3x^2-2x+1}}.$$

The graph of the transformation formula in this case is shown in Fig. 24.

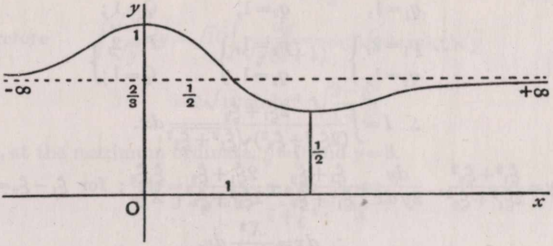

Fig. 24.

And the signs selected refer to values of $x > 1$, and we have

$$I_1 = \Big[-\cos^{-1}\sqrt{y} + \sqrt{2}\cosh^{-1}\sqrt{2y} \Big]_{\lambda_1}^{\lambda_2}$$

if λ_1, λ_2 be the limits and both > 1.

If x lie between 0 and 1, we have

$$\frac{\sqrt{X}}{x} = \frac{1}{\sqrt{1-y}} \quad \text{and} \quad \frac{\sqrt{X}}{x-1} = -\frac{1}{\sqrt{2}\sqrt{y-\frac{1}{2}}},$$

and we shall have

$$I_2 = \Big[-\cos^{-1}\sqrt{y} - \sqrt{2}\cosh^{-1}\sqrt{2y} \Big]_{\lambda_1}^{\lambda_2}$$

if λ_1, λ_2 both lie between 0 and 1.

If x lie between $-\infty$ and 0,

$$\frac{\sqrt{X}}{x} = -\frac{1}{\sqrt{1-y}} \quad \text{and} \quad \frac{\sqrt{X}}{x-1} = -\frac{1}{\sqrt{2}\sqrt{y-\frac{1}{2}}},$$

and we shall have

$$I_3 = \Big[+\cos^{-1}\sqrt{y} - \sqrt{2}\cosh^{-1}\sqrt{2y} \Big]_{\lambda_1}^{\lambda_2}$$

if λ_1, λ_2 be both negative.

If one limit, λ_1, falls on one side of a turning point, say $x=1$, and $0 < \lambda_1 < 1$, and the other, λ_2, on the opposite side, *i.e.* $\lambda_2 > 1$, the integration should be conducted from the lower limit to the turning point with the corresponding result, say I_2, and from the turning point to the upper limit with the result I_1.

(*b*) Next let us transform to the canonical form

$$\int \frac{P\xi_1 + Q\xi_2}{(p_1\xi_1^2 + q_1\xi_2^2)\sqrt{p_2\xi_1^2 + q_2\xi_2^2)}}\,dx$$

before integration.

Here, by the rule of Art. 297,

$$\left.\begin{array}{l} 3x_1x_2 - (x_1 + x_2) + 1 = 0, \\ 2x_1x_2 - (x_1 + x_2) + 1 = 0; \end{array}\right\}; \quad \therefore \left.\begin{array}{ll} x_1x_2 = 0, & x_1 + x_2 = 1, \\ x_1 = 0, & x_2 = 1; \end{array}\right\}$$

and $p_1+q_1=3, \quad p_2+q_2=2, \quad P+Q=3,$

$$q_1=1, \quad q_2=1, \quad Q=1;$$

$$\therefore \left.\begin{array}{l} p_1=2, \\ q_1=1, \end{array}\right\} \quad \left.\begin{array}{l} p_2=1, \\ q_2=1, \end{array}\right\} \quad \left.\begin{array}{l} P=2, \\ Q=1; \end{array}\right\}$$

$$\therefore I=\int \frac{2\xi_1+\xi_2}{(2\xi_1^2+\xi_2^2)\sqrt{\xi_1^2+\xi_2^2}}dx.$$

Let $y=\dfrac{\xi_1^2+\xi_2^2}{2\xi_1^2+\xi_2^2}$, $\dfrac{dy}{2y\,dx}=\dfrac{\xi_1+\xi_2}{\xi_1^2+\xi_2^2}-\dfrac{2\xi_1+\xi_2}{2\xi_1^2+\xi_2^2}=\dfrac{\xi_1\xi_2}{X^2y}$; for $\xi_1-\xi_2=1$;

$$\therefore dx=\frac{X^2}{2\xi_1\xi_2}dy.$$

$$\xi_1=0 \text{ gives } y=1,$$

$$\xi_2=0 \text{ gives } y=\tfrac{1}{2}.$$

$$1-y=\frac{\xi_1^2}{X},$$

$$y-\tfrac{1}{2}=\frac{\xi_2^2}{2X};$$

$$\therefore I=\frac{1}{2}\int\frac{2\xi_1+\xi_2}{\xi_1\xi_2}\frac{\sqrt{X}}{\sqrt{y}}dy$$

$$=\frac{1}{2}\int\left(\frac{2}{\xi_2}+\frac{1}{\xi_1}\right)\frac{\sqrt{X}}{\sqrt{y}}dy$$

$$=\frac{1}{2}\int\frac{dy}{\sqrt{y(1-y)}}+\frac{1}{\sqrt{2}}\int\frac{dy}{\sqrt{y(y-\frac{1}{2})}}$$

$$=-\cos^{-1}\sqrt{y}+\sqrt{2}\cosh^{-1}\sqrt{2y},$$

as before.

Ex. 2. As a case where $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$, consider the integration of

$$I=\int\frac{5x-9}{(x^2-6x+10)\sqrt{6x-x^2}}dx.$$

Writing $x-3=\xi$, $\quad dx=d\xi$,

$$I=\int\frac{5\xi+6}{(\xi^2+1)\sqrt{9-\xi^2}}d\xi=\int\frac{5\xi+6}{X\sqrt{Y}}d\xi, \text{ say.}$$

Let $$y=\frac{9-\xi^2}{\xi^2+1};$$

$$\therefore \frac{dy}{2y\,d\xi}=\frac{-\xi}{9-\xi^2}-\frac{\xi}{\xi^2+1}=\frac{-10\xi}{X^2y} \text{ and } d\xi=-\frac{1}{20}\frac{X^2}{\xi}dy;$$

$$\therefore I=-\frac{1}{20}\int\frac{5\xi+6}{\sqrt{y}}\frac{\sqrt{X}}{\xi}dy$$

$$=-\frac{1}{4}\int\frac{\sqrt{X}}{\sqrt{y}}dy-\frac{3}{10}\int\frac{\sqrt{X}}{\xi}\frac{dy}{\sqrt{y}}.$$

Now $(\xi^2+1)y=9-\xi^2$; $\therefore\ \xi^2=\dfrac{9-y}{1+y}$ and $X=\xi^2+1=\dfrac{10}{1+y}$.

Therefore
$$\int\frac{\sqrt{X}}{\sqrt{y}}\,dy=\sqrt{10}\int\frac{dy}{\sqrt{y(y+1)}}=2\sqrt{10}\sinh^{-1}\sqrt{y}$$
$$=2\sqrt{10}\sinh^{-1}\sqrt{\frac{9-\xi^2}{1+\xi^2}}.$$

Also, at the maximum ordinate, $\xi=0$ and $y=9$.

And
$$9-y=9-\frac{9-\xi^2}{1+\xi^2}=\frac{10\xi^2}{X};$$
$$\therefore\ \frac{\sqrt{X}}{\xi}=\frac{\sqrt{10}}{\sqrt{9-y}},$$
taking $x>3$, *i.e.* ξ as $+^{ve}$.

Therefore, in the second integral,
$$\int\frac{\sqrt{X}}{\xi}\frac{dy}{\sqrt{y}}=\sqrt{10}\int\frac{dy}{\sqrt{y(9-y)}}=2\sqrt{10}\sin^{-1}\frac{\sqrt{y}}{3};$$
$$\therefore\ I=-\frac{1}{2}\sqrt{10}\sinh^{-1}\sqrt{\frac{9-\xi^2}{1+\xi^2}}-\frac{6}{\sqrt{10}}\sin^{-1}\frac{1}{3}\sqrt{\frac{9-\xi^2}{1+\xi^2}},$$
i.e.
$$=-\frac{\sqrt{10}}{2}\left(\sinh^{-1}\sqrt{\frac{6x-x^2}{x^2-6x+10}}+\frac{6}{5}\sin^{-1}\frac{1}{3}\sqrt{\frac{6x-x^2}{x^2-6x+10}}\right).$$

The graph of the substitution formula,
$$y=\frac{6x-x^2}{x^2-6x+10},$$
is shown in Fig. 25,

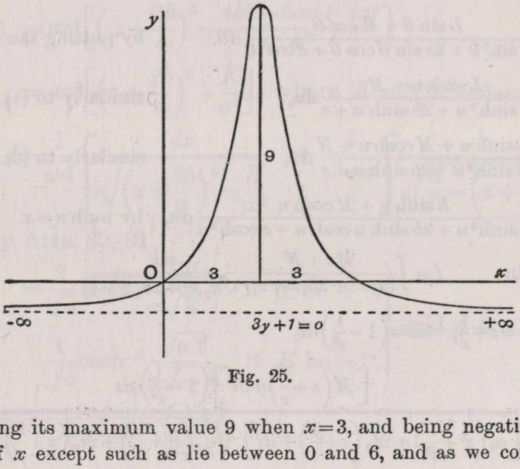

Fig. 25.

y attaining its maximum value 9 when $x=3$, and being negative for all values of x except such as lie between 0 and 6, and as we confine the integration to real values of $\sqrt{Y}$ the limits of integration are to be such that both lie within the region from 0 to 6. Also the sign of $\frac{\sqrt{X}}{\xi}$ changes as x increases through the value 3. Hence the signs adopted above when we take $\frac{\sqrt{X}}{\xi} = +\frac{\sqrt{10}}{\sqrt{9-y}}$ apply to values of x between 3 and 6. For values between 0 and 3 we must use $\frac{\sqrt{X}}{\xi} = -\frac{\sqrt{10}}{\sqrt{9-y}}$ and make the corresponding change in the sign of the part of the result dependent thereon.

313. Forms reducible to Case IV.

As in previous cases, Arts. 281, 286, 288, attention is called to the varieties of form of integrals deducible from the case just considered, viz. X quadratic, Y quadratic.

(1) Thus $\int \frac{L\sin\theta + M}{a\sin^2\theta + 2b\sin\theta + c}\,d\theta$ reduces to

$$\int \frac{Lx+M}{ax^2+2bx+c}\,\frac{dx}{\sqrt{1-x^2}}, \quad \text{if } \sin\theta = x.$$

(2) $\int \frac{L\sin\theta + M\cos\theta + N}{a\sin^2\theta + 2b\sin\theta + c}\,d\theta$ reduces to

$$\int \frac{Lx+N}{ax^2+2bx+c}\,\frac{dx}{\sqrt{1-x^2}} + \int \frac{M\,dx}{ax^2+2bx+c}.$$

(3) $\int \frac{L\sin\theta + M\cos\theta + N}{a\cos^2\theta + 2b\cos\theta + c}\,d\theta$, similarly.

(4) $\int \frac{L\sin\theta + M\cos\theta}{a\sin^2\theta + 2b\sin\theta\cos\theta + c\cos^2\theta}\,d\theta$, by putting $\tan\theta = x$.

(5) $\int \frac{L\sinh u + M}{a\sinh^2 u + 2b\sinh u + c}\,du$, similarly to (1).

(6) $\int \frac{L\sinh u + M\cosh u + N}{a\sinh^2 u + 2b\sinh u + c}\,du$, similarly to (2).

(7) $\int \frac{L\sinh u + M\cosh u}{a\sinh^2 u + 2b\sinh u\cosh u + c\cosh^2 u}\,du$, by $\tanh u = x$.

(8) If in $\qquad I \equiv \int \frac{Mx+N}{(a_1x^2+2b_1x+c_1)}\,\frac{dx}{\sqrt{a_2x^2+2b_2x+c_2}}$

we put $x = z + \frac{1}{z}$, $dx = \left(1-\frac{1}{z^2}\right)dz$,

$$I \equiv \int \frac{\left[M\left(z+\frac{1}{z}\right)+N\right]\left(1-\frac{1}{z^2}\right)dz}{\left[a_1\left(z^2+\frac{1}{z^2}\right)+2b_1\left(z+\frac{1}{z}\right)+d_1\right]\sqrt{a_2\left(z^2+\frac{1}{z^2}\right)+2b_2\left(z+\frac{1}{z}\right)+d_2}},$$

where d_1, d_2 are written for c_1+2a_1, c_2+2a_2 respectively ; so that

$$I=\int\frac{(Mz^2+Nz+M)(z^2-1)\,dz}{(a_1z^4+2b_1z^3+d_1z^2+2b_1z+a_1)\sqrt{a_2z^4+2b_2z^3+d_2z^2+2b_2z+a_2}}.$$

Hence, if F be a "reciprocal" quadratic function of z, and X, Y reciprocal quartic expressions in z, we can integrate

$$I=\int\frac{F}{X\sqrt{Y}}(z^2-1)\,dz \quad \text{by the substitution } z+\frac{1}{z}=x.$$

(9) Similarly

$$\int\frac{(Mz^2+Nz-M)(z^2+1)\,dz}{(a_1z^4+2b_1z^3+d_1z^2-2b_1z+a_1)\sqrt{a_2z^4+2b_2z^3+d_2z^2-2b_2z+a_2}}$$

integrates by the substitution $z-\frac{1}{z}=x$.

314. The Case of $Y\equiv$ a Reciprocal Quartic.

Let Y be any reciprocal binary quartic expression

$$\equiv ax^4+4bx^3+6cx^2+4bx+a.$$

Then $I=\int\frac{x^2-1}{x}\frac{dx}{\sqrt{Y}}$ reduces at once to the form

$$\int\frac{dz}{\sqrt{\text{Quadratic}}},$$

by the substitution $x+\frac{1}{x}=z$, whence $\left(1-\frac{1}{x^2}\right)dx=dz$.

For $Y=x^2\left[a\left(x^2+\frac{1}{x^2}\right)+4b\left(x+\frac{1}{x}\right)+6c\right]$

$$=x^2[az^2+4bz+6c-2a]$$

$$=ax^2\left[\left(z+\frac{2b}{a}\right)^2-\frac{4b^2-6ac+2a^2}{a^2}\right]$$

$$=ax^2\left[\left(z+\frac{2b}{a}\right)^2-\frac{K}{a^2}\right], \text{ where } K=2(2b^2-3ac+a^2);$$

$$\therefore\ I=\frac{1}{\sqrt{a}}\int\frac{dz}{\sqrt{\left(z+\frac{2b}{a}\right)^2-\frac{K}{a^2}}} \text{ or } \frac{1}{\sqrt{-a}}\int\frac{dz}{\sqrt{\frac{K}{a^2}-\left(z+\frac{2b}{a}\right)^2}},$$

which, by Arts. 80, 81,

$$=\frac{1}{\sqrt{a}}\sinh^{-1}\frac{\sqrt{aY}}{x\sqrt{K}}, \quad \text{if } K \text{ be } +^{\text{ve}},$$

or

$$\frac{1}{\sqrt{a}}\cosh^{-1}\frac{\sqrt{aY}}{x\sqrt{-K}}, \quad \text{if } K \text{ be } -^{\text{ve}},$$

and $a+^{\text{ve}}$,

or

$$\frac{1}{\sqrt{-a}}\cos^{-1}\frac{\sqrt{-aY}}{x\sqrt{K}}, \text{ if } a \text{ be } -^{\text{ve}}.$$

Note that if K be positive, the factors of Y as expressed in terms of z are real;
if K be negative, unreal;
and that $aY+Kx^2$ is a perfect square.

315. A Similar Case.

In the same way, if

$$Y_1 \equiv ax^4+4bx^3+6cx^2-4bx+a,$$

the integration of

$I_1=\int\frac{x^2+1}{x}\frac{dx}{\sqrt{Y_1}}$ can be effected by the substitution $x-\frac{1}{x}=z$.

$$\text{For } Y_1=x^2\left[a\left(x^2+\frac{1}{x^2}\right)+4b\left(x-\frac{1}{x}\right)+6c\right]$$

$$=x^2[az^2+4bz+6c+2a]$$

$$=ax^2\left[\left(z+\frac{2b}{a}\right)^2-\frac{4b^2-6ac-2a^2}{a^2}\right]$$

$$=ax^2\left[\left(z+\frac{2b}{a}\right)^2-\frac{K_1}{a^2}\right], \text{ where } K_1=2(2b^2-3ac-a^2);$$

$$\therefore\ I_1=\frac{1}{\sqrt{a}}\int\frac{dz}{\sqrt{\left(z+\frac{2b}{a}\right)^2-\frac{K_1}{a^2}}} \text{ or } \frac{1}{\sqrt{-a}}\int\frac{dz}{\sqrt{\frac{K_1}{a^2}-\left(z+\frac{2b}{a}\right)^2}}$$

$$\left.\begin{aligned} &=\frac{1}{\sqrt{a}}\sinh^{-1}\frac{\sqrt{aY_1}}{x\sqrt{K_1}}, \quad \text{if } K_1 \text{ be } +^{\text{ve}}, \\ \text{or} \quad &\frac{1}{\sqrt{a}}\cosh^{-1}\frac{\sqrt{aY_1}}{x\sqrt{-K_1}}, \text{ if } K_1 \text{ be } -^{\text{ve}}, \end{aligned}\right\} \text{and } a\ +^{\text{ve}},$$

$$\text{or} \quad \frac{1}{\sqrt{-a}}\cos^{-1}\frac{\sqrt{-aY_1}}{x\sqrt{K_1}}, \text{ if } a \text{ be } -^{\text{ve}};$$

also, Y_1 expressed in terms of z has real or unreal factors, as K_1 is $+^{\text{ve}}$ or $-^{\text{ve}}$, and $aY_1+K_1x^2$ is a perfect square.

In the integrations of these two articles, since the final form exhibited is arrived at by the conversion of a function of z into a function of Y, or of Y_1, in which process a square root is extracted

$$\left(e.g.\ \sin^{-1}\frac{az+2b}{\sqrt{K}}=\cos^{-1}\sqrt{1-\frac{(az+2b)^2}{K}}=\text{etc.}\right),$$

it is desirable to check by direct differentiation the sign of all numerical results obtained.

316. Other Forms.

The substitutions

$$x+\frac{1}{x}=\frac{1}{z}, \quad x-\frac{1}{x}=\frac{1}{z}$$

respectively reduce

$$\int\frac{x^2-1}{x^2+1}\frac{dx}{\sqrt{Y}} \quad \text{and} \quad \int\frac{x^2+1}{x^2-1}\frac{dx}{\sqrt{Y_1}},$$

Y and Y_1 denoting the same quartic functions as before. [See Greenhill's *Chapter on the Integral Calculus*, p. 41.]

For taking $x+\frac{1}{x}=\frac{1}{z}$ we have, differentiating logarithmically,

$$\frac{1-\frac{1}{x^2}}{x+\frac{1}{x}}dx=-\frac{dz}{z}, \quad \textit{i.e.} \ \frac{x^2-1}{x^2+1}dx=-\frac{x\,dz}{z},$$

and $$\sqrt{Y}=x\,[az^{-2}+4bz^{-1}+6c-2a]^{\frac{1}{2}};$$

$$\therefore \quad \int\frac{x^2-1}{x^2+1}\frac{dx}{\sqrt{Y}}=-\int\frac{dz}{\sqrt{a+4bz+(6c-2a)z^2}},$$

whose integral can be written down by Art. 80.

And similarly, if $x-\frac{1}{x}=\frac{1}{z}$,

$$\frac{x^2+1}{x^2-1}dx=-\frac{x\,dz}{z}$$

and $$\int\frac{x^2+1}{x^2-1}\frac{dx}{\sqrt{Y_1}}=-\int\frac{dz}{\sqrt{a+4bz+(6c+2a)z^2}},$$

whose integral can be written down as before.

The integrals

$$\int\frac{(x^2-1)}{a_1x^2+b_1x+a_1}\frac{dx}{\sqrt{Y}}, \qquad \int\frac{x^2+1}{a_1x^2+b_1x-a_1}\frac{dx}{\sqrt{Y_1}},$$

$$\int\frac{x^2-1}{x^2+1}\frac{x}{a_1x^2+b_1x+a_1}\frac{dx}{\sqrt{Y}}, \quad \int\frac{x^2+1}{x^2-1}\frac{x}{a_1x^2+b_1x-a_1}\frac{dx}{\sqrt{Y_1}},$$

are reduced to forms already considered by the same substitutions, and are therefore integrable.

Similarly, if $Y \equiv ap^2x^4+4bpx^3+6cx^2+4bqx+aq^2$,
$Y_1 \equiv ap^2x^4+4bpx^3+6cx^2-4bqx+aq^2$,
the integrations of

$$\left.\begin{array}{l}\int \frac{px^2-q}{x}\cdot\frac{dx}{\sqrt{Y}},\\ \int\frac{px^2-q}{px^2+q}\cdot\frac{dx}{\sqrt{Y}},\\ \int\frac{px^2+q}{x}\cdot\frac{dx}{\sqrt{Y_1}},\\ \int\frac{px^2+q}{px^2-q}\cdot\frac{dx}{\sqrt{Y_1}},\end{array}\right\}\text{can be effected by the respective substitutions}\left\{\begin{array}{l}px+\frac{q}{x}=z,\\ px+\frac{q}{x}=\frac{1}{z},\\ px-\frac{q}{x}=z,\\ px-\frac{q}{x}=\frac{1}{z}.\end{array}\right.$$

Ex. Consider the integral

$$I\equiv\int\frac{(x^4-1)\,dx}{(x^4+6x^2+1)\sqrt{x^4+x^2+1}}.$$

Here
$$I=\int\frac{x^3\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)dx}{x^3\left\{\left(x+\frac{1}{x}\right)^2+4\right\}\sqrt{\left(x+\frac{1}{x}\right)^2-1}};$$

and putting $x+\frac{1}{x}=z$, $\quad I=\int\frac{z\,dz}{(z^2+4)\sqrt{z^2-1}}$.

Put $z^2-1=w^2$, $z\,dz=w\,dw$;

$$\therefore\ I=\int\frac{dw}{5+w^2}=\frac{1}{\sqrt{5}}\tan^{-1}\frac{w}{\sqrt{5}}$$
$$=\frac{1}{\sqrt{5}}\sin^{-1}\frac{w}{\sqrt{w^2+5}}=\frac{1}{\sqrt{5}}\sin^{-1}\sqrt{\frac{z^2-1}{z^2+4}}$$
$$=\frac{1}{\sqrt{5}}\sin^{-1}\sqrt{\frac{x^4+x^2+1}{x^4+6x^2+1}}.$$

317. Summing up.

It will now be clear that any integration of the form

$$\int\frac{\phi(x)}{\psi(x)}\frac{dx}{\sqrt{ax^2+bx+c}}$$

can be effected, where ϕ and ψ are rational integral algebraic functions of x.

For if $\frac{\phi(x)}{\psi(x)}$ be put into the form

$$\sum A_nx^n+\sum\frac{\lambda}{x-a}+\sum\frac{\mu}{(x-\beta)^r}+\sum\frac{\lambda'x+\mu'}{Ax^2+Bx+C}$$
$$+\sum\frac{\lambda''x+\mu''}{(A'x^2+B'x+C')^s},$$

as explained in the chapter on Partial Fractions, then of the resulting integrals

(1) $\int \frac{x^n\,dx}{\sqrt{ax^2+bx+c}}$ is reducible to a lower order by Art. 240, and integrable.

(2) $\int \frac{dx}{(x-\alpha)\sqrt{ax^2+bx+c}}$ has been considered in Art. 287.

(3) $\int \frac{dx}{(x-\beta)^r\sqrt{ax^2+bx+c}}$ reduces by the method of Art. 290.

(4) $\int \frac{(\lambda' x+\mu')dx}{(Ax^2+Bx+C)\sqrt{ax^2+bx+c}}$ has been considered in Art. 291.

and (5) $\int \frac{(\lambda'' x+\mu'')dx}{(Ax^2+Bx+C)^s\sqrt{ax^2+bx+c}}$

is best got by differentiation with regard to C of the result for the case where $s=1$, as will be explained later. This method may also be adopted in (3).

318. General Consideration of the Position.

We have therefore now completed the integration of the most general function of x of form

$$\frac{A+B\sqrt{R}}{C+D\sqrt{R}},$$

where A, B, C, D are rational integral algebraic functions of x of any degree, and R is a rational integral algebraic function of x of degree 1 or 2.

For rationalizing the denominator,

$$\frac{A+B\sqrt{R}}{C+D\sqrt{R}}=\frac{(A+B\sqrt{R})(C-D\sqrt{R})}{C^2-D^2R}$$

$$=\frac{AC-BDR}{C^2-D^2R}+\frac{(BC-AD)R}{C^2-D^2R}\cdot\frac{1}{\sqrt{R}}$$

$$=\frac{P}{Q}+\frac{M}{N}\cdot\frac{1}{\sqrt{R}}, \text{ say,}$$

where P, Q, M, N are rational integral algebraic functions of x.

Now $\int \frac{P}{Q}dx$ is integrable by the methods of partial fractions; and if $\frac{M}{N}$ be put into partial fractions, $\int \frac{M}{N}\frac{1}{\sqrt{R}}\,dx$ can, as has been explained, be expressed as the sum of a finite number of such terms as have been discussed in the present chapter, and each term may then be integrated.

Hence the theory of the integration of

$$\int \frac{A+B\sqrt{R}}{C+D\sqrt{R}}\,dx$$

is now complete, where R is linear or quadratic. And it will be noted that the integration has been in all cases effected in terms of *the known* algebraic, logarithmic, inverse circular or inverse hyperbolic functions.

When R is of higher degree than the second, it has been seen that in *some special cases* the integration can still be effected in terms of the elementary functions, but for the *general* discussion of the cases where R is cubic or quartic, we shall require the elliptic functions, and in general for forms of R of higher degree than the fourth, we should require the functions known as hyperelliptic.

GENERAL EXAMPLES.

1. Obtain the following integrals:

(i) $\int (1+x)^{-1}x^{-\frac{1}{2}}\,dx.$ (ii) $\int (1+x)^{-1}(1+2x)^{-\frac{1}{2}}\,dx.$

(iii) $\int x^{-1}(2-3x+x^2)^{-\frac{1}{2}}\,dx.$ (iv) $\int (1+x)^{-1}(1+x+x^2)^{-\frac{1}{2}}\,dx.$

(v) $\int \frac{\sqrt{1+x+x^2}}{1+x}\,dx.$ (vi) $\int \frac{x^2+x-1}{(x+1)\sqrt{x^2-1}}\,dx.$

(vii) $\int \frac{dx}{x\sqrt{a^n+x^n}}.$ (viii) $\int \frac{1-x}{1+x}\,\frac{1}{\sqrt{x+x^2+x^3}}\,dx.$

2. Integrate (i) $\int \frac{dx}{(x+2)\sqrt{x^2-1}}.$ (ii) $\int \frac{\sqrt{x^2-1}}{x+2}\,dx.$

[Barnes Schol., 1887.]

3. Show that

$$\int \frac{dx}{(x-p)\sqrt{a+2bx+cx^2}} = \frac{1}{\{-(a+2bp+cp^2)\}^{\frac{1}{2}}}\sin^{-1}\left\{\frac{(a+bx)+p(b+cx)}{(x-p)\sqrt{b^2-ac}}\right\}$$

where p lies between the roots of $a+2bx+cx^2=0$, supposed real.

[Trinity, 1886 and 1891.]

4. Show that

$$\int\frac{dx}{(x^2+a^2)\sqrt{x^2+b^2}}=\frac{1}{a\sqrt{b^2-a^2}}\cos^{-1}\frac{a}{b}\sqrt{\frac{x^2+b^2}{x^2+a^2}},\quad \text{if } a<b,$$

$$\text{and } =\frac{1}{a\sqrt{a^2-b^2}}\cosh^{-1}\frac{a}{b}\sqrt{\frac{x^2+b^2}{x^2+a^2}},\quad \text{if } a>b.$$

5. Prove that

$$\int\frac{(x+1)\,dx}{(2x^2-2x+1)\sqrt{3x^2-2x+1}}=\cosh^{-1}\sqrt{\frac{3x^2-2x+1}{2x^2-2x+1}}+2\cos^{-1}\frac{1}{\sqrt{2}}\sqrt{\frac{3x^2-2x+1}{2x^2-2x+1}}.$$

6. Integrate

$$\text{(i)}\int\frac{(3x+4)\,dx}{(5x^2+8x)\sqrt{4x^2-2x+1}},\quad \text{(ii)}\int\frac{dx}{(x^2+2ax+b^2)\sqrt{x^2+2ax+c^2}},$$

where $a<b<c$.

7. Integrate (i) $\displaystyle\int\frac{x\,dx}{(a^2+b^2-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}}$. [ST. JOHN'S, 1888.]

(ii) $\displaystyle\int\frac{(x+b)\,dx}{(x^2+a^2)\sqrt{x^2+c^2}}$ $(a>c)$. [ST. JOHN'S, 1889.]

(iii) $\displaystyle\int\frac{d\theta}{\sin\theta\sqrt{a\cos^2\theta+b\sin^2\theta+c}}$. [TRINITY, 1888.]

8. Find the values of

(i) $\displaystyle\int\frac{\sin x\,dx}{(\cos x+\cos\alpha)\sqrt{(\cos x+\cos\beta)(\cos x+\cos\gamma)}}$. [$\gamma$, 1890.]

(ii) $\displaystyle\int\frac{dx}{\cos(x+\alpha)\sqrt{\cos(x+\beta)\cos(x+\gamma)}}$. [$\gamma$, 1890.]

9. Integrate $\displaystyle\int\frac{a^2-x^2}{(a^2-ax+x^2)(a^4+a^2x^2+x^4)^{\frac{1}{2}}}\,dx$,

transforming by the substitution

$$x^2+ax+a^2=y^2(x^2-ax+a^2).$$ [α, 1884.]

10. Integrate (i) $\displaystyle\int\frac{(8x-13)\,dx}{(3x^2-10x+9)\sqrt{-x^2+10x-13}}$.

(ii) $\displaystyle\int\frac{dx}{(x-1)(x-2)\sqrt{(x-3)(x-4)}}$. [COLL., 1892.]

(iii) $\displaystyle\int\frac{(x+9)\,dx}{(x^2-5x+4)\sqrt{x^2-2x+2}}$.

(iv) $\displaystyle\int\frac{(x-a)\,dx}{(x-b)(x-c)(x-d)\sqrt{x-e}}$.

(v) $\displaystyle\int\frac{(x+3)\,dx}{(x^2+x+1)\sqrt{x^2+x+2}}$.

11. Integrate (i) $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx$; [COLL., 1901.]

(ii) $\int \frac{x^{-1}(x^2-1)}{\sqrt{x^4+x^2+1}}\,dx$.

12. Show that

$$\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx = \cos\alpha\cos^{-1}\left(\frac{\cos x}{\cos\alpha}\right) - \sin\alpha\cosh^{-1}\left(\frac{\sin x}{\sin\alpha}\right).$$

[COLL., 1901.]

13. Integrate (i) $\int \frac{1-2x^2}{1+2x^2}\sqrt{\frac{1+x^2}{1-x^2}}\,x\,dx$. [R. P.]

(ii) $\int \frac{dx}{(1+x^4)\sqrt{(\sqrt{1+x^4}-x^2)}}$. [R. P.; EULER, *C.I.*, vol. iv.]

(iii) $\int \frac{dx}{(a^2-ax-x^2)\sqrt{ax+x^2}}$. [OXF. I. P., 1900.]

14. Evaluate the integral

$$\int \frac{dx}{(a^2-\tan^2 x)(b^2-\tan^2 x)^{\frac{1}{2}}}.$$ [MATH. TRIP., 1886.]

15. Prove that

$$\int \frac{dx}{\sin x\sin^{\frac{1}{2}}(2x+\alpha)} = -\frac{1}{\sqrt{\sin\alpha}}\cosh^{-1}\frac{\sin(x+\alpha)}{\sin x}.$$ [COLL., 1892.]

16. Show how to integrate

$$\int (fx+g)(ax^2+bx+c)^{\frac{n}{2}}\,dx,$$

where n is any positive or negative integer. [α, 1890.]

17. Show that $\int \frac{(a+bx)\,dx}{(a+2bx+cx^2)^{\frac{3}{2}}} = \frac{x}{(a+2bx+cx^2)^{\frac{1}{2}}}$. [TRINITY, 1889.]

18. Prove by the substitution

$$y^2 = (ax^2+2bx+c)/(Ax^2+2Bx+C),$$

where A and $AC-B^2$ are positive, that the integral

$$\int \frac{(Mx+N)\,dx}{(Ax^2+2Bx+C)\sqrt{ax^2+2bx+c}}$$

becomes of the form

$$P_1\int \frac{dy}{\sqrt{\lambda_1-y^2}} + P_2\int \frac{dy}{\sqrt{y^2-\lambda_2}},$$

where λ_1 and λ_2 are the roots of the quadratic

$$(a-\lambda A)(c-\lambda C)-(b-\lambda B)^2=0,$$

and P_1, P_2 are definite constants.

Integrate completely the function

$$\frac{x+3}{(2x^2-10x+17)\sqrt{4x^2-26x+49}}.$$

[MATH. TRIP., 1891.]

19. Prove $\displaystyle\int_0^a \sqrt{\tanh^2 a - \tanh^2 x}\, dx = \frac{\pi}{2}(1 - \text{sech}\, a).$ [β, 1889.]

20. Integrate $\displaystyle\int (x^2+b^2)^{-1}(x^2+a^2+b^2)^{-\frac{1}{2}}\, dx.$ [OX. II. P., 1902.]

21. Integrate $\displaystyle\int \frac{\sin^3 x\, dx}{(1+\cos^2 x)\sqrt{1+\cos^2 x+\cos^4 x}},$ [ST. JOHN'S, 1882.]

and evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{(a^2+c^2x^2)\sqrt{1-x^2}}.$

22. Show that $\displaystyle\int \frac{dx}{(x-a)\sqrt{Ax^2+2Bx+C}}$

is transcendental unless $Aa^2+2Ba+C=0$. [J. M. SCH. OX., 1904.]

Establish the results

$$\text{(i)} \int \frac{dx}{(x-1)\sqrt{x^2-4x+5}} = \frac{1}{\sqrt{2}}\cosh^{-1}\frac{\sqrt{2(x^2-4x+5)}}{x-1}$$

and $$\text{(ii)} \int \frac{dx}{(x-2)\sqrt{(3+2x-2x^2)}} = \sin^{-1}\frac{\sqrt{3+2x-2x^2}}{\sqrt{7}(x-2)}.$$

[COLL. a, 1890.]

23. Show that

$$\int_{-1}^{+1} \frac{(1-ax)(1-bx)}{(1-2ax+a^2)(1-2bx+b^2)}\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\frac{2-ab}{1-ab}.$$ [δ, 1884.]

24. Describe the steps whereby the integral of a rational function of a single variable, x, can be obtained.

Prove that if the sign of summation refer to the suffixes 1, 2, 3 in cyclical order, the integral

$$\int dx\sqrt{(x-a)(x-b)}\Sigma(c_2-c_3)\left[\frac{2(c_1-a)(c_1-b)}{(x-c_1)^2} - \frac{2c_1-a-b}{x-c_1}\right]$$

is a certain constant multiple of

$$(x-a)^{\frac{3}{2}}(x-b)^{\frac{3}{2}}(x-c_1)^{-1}(x-c_2)^{-1}(x-c_3)^{-1}.$$

[MATH. TRIP., 1896.]

25. Determine the degenerate form of the elliptic integral

$$\int \frac{ds}{\sqrt{4(s-s_1)(s-s_2)(s-s_3)}}, \quad s_1 > s_2 > s_3,$$

when s_2 is made to coincide with s_1 or with s_3. [INT. ARTS, LONDON.]

26. Prove, by means of the substitution $\frac{x-\beta}{x-a}=y^2$, that

$$\int\frac{dx}{(x-\gamma)\sqrt{(x-a)(x-\beta)}}=\frac{-2}{\sqrt{(a-\gamma)(\gamma-\beta)}}\tan^{-1}\sqrt{\frac{a-\gamma}{\gamma-\beta}\cdot\frac{x-\beta}{x-a}}$$

$$\text{or }=\frac{2}{\sqrt{(a-\gamma)(\beta-\gamma)}}\,\text{th}^{-1}\sqrt{\frac{a-\gamma}{\beta-\gamma}\cdot\frac{x-\beta}{x-a}}.$$

[INT. ARTS, LONDON.]

27. Prove that

$$\int_0^1\frac{dx}{(1+x)(2+x)\sqrt{x(1-x)}}=\pi\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}}\right).$$

[MATH. TRIP. I., 1912.]

28. Show that the integral

$$\int\frac{dx}{x\sqrt{3x^2+2x-1}}$$

is rationalized by the assumption $x=(1+y^2)/(3-y^2)$, and hence, or otherwise, find its value.

Prove that if m be a positive proper fraction, the value of the above integral when taken between limits $\frac{1}{3}$ and $\frac{1}{2+m}$ is the same as when taken between limits $\frac{1}{2+m}$ and $\frac{1}{m(2+m)}$.

[MATH. TRIP. I., 1910.]

29. Prove by means of the substitution

$$\frac{a-x}{x-b}=\frac{a-d}{b-c}\,\frac{c-y}{y-d},$$

that, if m be any positive quantity, and $a>b>c>d$,

$$\int_b^a\frac{\{(a-x)(x-c)\}^{m-1}}{\left\{\frac{(a-x)(x-d)}{a-d}+\frac{(x-b)(x-c)}{b-c}\right\}^m}dx$$

$$=\int_d^c\frac{\{(a-x)(c-x)\}^{m-1}}{\left\{\frac{(a-x)(x-d)}{a-d}+\frac{(b-x)(c-x)}{b-c}\right\}^m}dx.$$

[MATH. TRIP., 1878.]

[See Wolstenholme's *Mathematical Problems*, Numbers 1900-1903 for a group of similar examples.]

30. By the transformation $px+\frac{q}{x}=\sqrt{2pq}/z$, integrate

$$\int\frac{px^2-q}{px^2+q}\,\frac{dx}{\sqrt{p^2x^4+q^2}}.$$

[Cf. EULER, *C.I.*, iv., p. 22.]

31. Apply the transformation $x^2+\frac{1}{x^2}=2/z^2$ to integrate

$$\text{(i)} \int \frac{\sqrt{1+x^4}}{1-x^4}\,dx, \quad \text{(ii)} \int \frac{x^2\,dx}{(1-x^4)\sqrt{1+x^4}}.$$

[EULER, *C.I.*, iv.]

32. Show that

$$\int \frac{2\,d\theta}{\sin\theta\sqrt[4]{\cos 2\theta}} = \tanh^{-1}\frac{\cos\theta}{\sqrt[4]{\cos 2\theta}} + \tan^{-1}\frac{\cos\theta}{\sqrt[4]{\cos 2\theta}}.$$

33. Show that the transformation

$$\{(a+bx^n)^\lambda - b^\lambda x^{\lambda n}\} = \left(\frac{x}{u}\right)^{\lambda n}$$

will reduce the integration

$$\int \frac{x^{m-1}\,dx}{(a+bx^n)\{(a+bx^n)^\lambda - b^\lambda x^{\lambda n}\}^{\frac{m}{\lambda n}}} \quad \text{to the form} \quad \frac{1}{a}\int \frac{u^{m-1}du}{1+b^\lambda u^{\lambda n}}.$$

[EULER, *C.I.*, iv., 53 and 56; PEACOCK, p. 305.]

34. (i) Show that

$$\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}\,dx = e^x\sqrt{\frac{1+x}{1-x}},$$

[PEACOCK, p. 309.]

and (ii) integrate $\int e^x \frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\,dx.$

35. Integrate

$$\text{(i)} \int \frac{2-3x}{2+3x}\sqrt{\frac{1+x}{1-x}}\,dx. \qquad \text{(ii)} \int \frac{\sin^5\theta + 2\cos^5\theta}{\cos\theta\sin 4\theta}\,d\theta.$$

[ST. JOHN'S, 1881.]

36. Show that

$$\text{(i)} \int \sqrt{\frac{a^2-c^2x^2}{a^2-x^2}}\,\frac{dx}{x} = \frac{1}{2}\log\left\{\frac{y-1}{y+1}\,\frac{(y+c)^c}{(y-c)^c}\right\},$$

where $y=\sqrt{\frac{a^2-c^2x^2}{a^2-x^2}}.$

$$\text{(ii)} \int \frac{x^2+1}{x^2-1}\,\frac{dx}{\sqrt{1-ax^2+x^4}} = \frac{1}{\sqrt{a-2}}\cos^{-1}\frac{x\sqrt{a-2}}{x^2-1}.$$

[HALL, *I.C.*, p. 325.]

37. If $F(x, y)$ be a rational algebraic function of x and y, show that

$$\int F(x, \sqrt{1+x^2})(x+\sqrt{1+x^2})^{\frac{p}{q}}\,dx$$

may be integrated by the transformation $x=\sinh(q\log z)$.

38. Show that

(i) $\int (\cos 2\theta)^{\frac{3}{2}} \cos\theta\, d\theta$

$$= \frac{1}{8}\sin\theta(3+2\cos 2\theta)\sqrt{\cos 2\theta} + \frac{3}{8\sqrt{2}}\sin^{-1}(\sqrt{2}\sin\theta).$$

(ii) $\int_a^\theta \frac{\sin\theta}{\sqrt{\sin^2 a - \sin^2\theta}}\, d\theta = \log\left(\frac{\cos a}{\cos\theta + \sqrt{\sin^2 a - \sin^2\theta}}\right).$

(iii) $\int_0^\pi e^{2x\cos\theta}\, d\theta = \pi\left[1 + \frac{x^2}{(1!)^2} + \frac{x^4}{(2!)^2} + \frac{x^6}{(3!)^2} + \dots\right].$

39. Show that

(i) $\int_0^1 x^{a \pm cx}\, dx = \frac{1}{a+1} \mp \frac{c}{(a+2)^2} + \frac{c^2}{(a+3)^3} \mp \frac{c^3}{(a+4)^4} + \dots.$

(ii) $\int_0^1 x^{\pm x^2}\, dx = 1 \pm \frac{1}{3^2} + \frac{1}{5^3} \mp \frac{1}{7^4} + \dots.$ [ANGLIN.]

40. If $\phi(x) = a_0 + \frac{1}{2}a_2 x^2 + \frac{1.3}{2.4}a_4 x^4 + \dots$, show that

(i) $\frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos^2\theta\,\phi(\sin\theta)\, d\theta = 1^2.a_0 + \frac{1}{2}\left(\frac{1}{2}\right)^2 a_2 + \frac{1}{3}\left(\frac{1.3}{2.4}\right)^2 a_4 + \dots.$

(ii) $\frac{20}{9\pi} = \frac{1}{2}.1^2 + \frac{1}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{1.3}{2.4}\right)^2 + \frac{1}{5}\left(\frac{1.3.5}{2.4.6}\right)^2 + \dots.$

(iii) $\frac{4}{\pi} - 1 = \frac{1}{1^2}\left(\frac{1}{2}\right)^2 + \frac{1}{3^2}\left(\frac{1.3}{2.4}\right)^2 + \frac{1}{5^2}\left(\frac{1.3.5}{2.4.6}\right)^2 + \dots$

[ANGLIN.]

41. Integrate

(i) $\int \frac{\sin 2\theta\, d\theta}{\sqrt{\sin^4\theta + 4\sin^2\theta\cos^2\theta + 2\cos^4\theta}}.$

(ii) $\int \frac{(z^2-1)\, dz}{(z^2+z+1)\sqrt{z^4+4z^3+4z^2+4z+1}}.$

42. If J be the Jacobian of two quadratic functions of x, viz.

$$u_1 \equiv a_1x^2 + 2b_1x + c_1, \quad u_2 \equiv a_2x^2 + 2b_2x + c_2,$$

$$J \equiv \begin{vmatrix} 2(a_1x+b_1), & 2(b_1x+c_1) \\ 2(a_2x+b_2), & 2(b_2x+c_2) \end{vmatrix},$$

show that if $u_1 = 0$, $u_2 = 0$ have no positive roots, then

$$\int_0^\infty \frac{J}{u_1u_2}\, dx = 2\log\frac{a_1c_2}{a_2c_1}.$$

43. By means of the identity

$$\int_0^{\frac{\pi}{2}} (a+\sin^2 x)^n \cos x\,dx = \int_0^{\frac{\pi}{2}} (1+a-\sin^2 x)^n \sin x\,dx,$$

prove that

$$a^n + {}^nC_1 \frac{a^{n-1}}{3} + {}^nC_2 \frac{a^{n-2}}{5} + {}^nC_3 \frac{a^{n-3}}{7} + \dots$$

$$= (1+a)^n - 2\frac{n}{3}(1+a)^{n-1} + 2^2 \frac{n(n-1)}{3.5}(1+a)^{n-2}$$

$$- 2^3 \frac{n(n-1)(n-2)}{3.5.7}(1+a)^{n-3} + \dots.$$

[WOLSTENHOLME, *Problems*, No. 1929; WIGGINS, *E. Times*, No. 13323.]

44. Show that

(i) $$a^n + {}^nC_1 \frac{1}{p+2} a^{n-1} + {}^nC_2 \frac{1.3}{(p+2)(p+4)} a^{n-2} + {}^nC_3 \frac{1.3.5}{(p+2)(p+4)(p+6)} a^{n-3} + \dots$$

$$= (1+a)^n - {}^nC_1 \frac{p+1}{p+2}(1+a)^{n-1} + {}^nC_2 \frac{(p+1)(p+3)}{(p+2)(p+4)}(1+a)^{n-2} - {}^nC_3 \frac{(p+1)(p+3)(p+5)}{(p+2)(p+4)(p+6)}(1+a)^{n-3} + \dots.$$

(ii) $$a^n + {}^nC_1 \frac{1}{p+2} a^{n-1}b + {}^nC_2 \frac{1.3}{(p+2)(p+4)} a^{n-2}b^2 + {}^nC_3 \frac{1.3.5}{(p+2)(p+4)(p+6)} a^{n-3}b^3 + \dots$$

$$= (a+b)^n - {}^nC_1 \frac{p+1}{p+2}(a+b)^{n-1}b + {}^nC_2 \frac{(p+1)(p+3)}{(p+2)(p+4)}(a+b)^{n-2}b^2 - {}^nC_3 \frac{(p+1)(p+3)(p+5)}{(p+2)(p+4)(p+6)}(a+b)^{n-3}b^3 + \dots.$$

45. (i) Integrate

$$\int \frac{2x^3-1}{x^6+2x^3-x^2+1}\,dx.$$

[Ox. I. P., 1903.]

(ii) Integrate

$$\int \frac{(3x^4-1)\,dx}{x^8+2x^4-16x^2+1}.$$

(iii) Prove that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1-\sin\theta\cos\theta+\sin^2\theta)\cos^2\theta\,d\theta}{(1+\sin\theta\cos\theta)^3} = \frac{10\pi}{9}\sqrt{3}.$$

46. (i) Show that

$$\int_0^a x^{n-1}(1-a+x)^{-n-1}\,dx = \frac{1}{n}\frac{a^n}{1-a}.$$

(ii) Evaluate

$$\int_a^{a+h} \frac{(a+h-x)^{n-1}}{(n-1)!}\frac{d^n}{dx^n}f(x)\,dx.$$

How could your result be applied to the summation of series?

[α, 1886.]

47. Discuss the integration of

$$\int f\left[x,\ \left(\frac{a+bx}{a'+b'x}\right)^{\frac{m}{n}},\ \left(\frac{a+bx}{a'+b'x}\right)^{\frac{p}{q}},\ \left(\frac{a+bx}{a'+b'x}\right)^{\frac{r}{s}}\right]dx,$$

where f denotes a rational integral algebraic function of the quantities indicated. [Lacroix, *C.I.*, ii., p. 35.]

48. If $F(x)$ be a rational integral algebraic function of x, show that $\int_{-1}^{+1}\frac{F(x)}{\sqrt{1-x^2}} = \pi k$, where k is the coefficient of $\frac{1}{a}$ in the product

$$F(a)\left[\frac{1}{a}+\frac{1}{2}\cdot\frac{1}{a^3}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{a^5}+\ldots\right];$$ [St. John's, 1891.]

or where k is the constant term in the expansion of $\frac{x\,F(x)}{(x^2-1)^{\frac{1}{2}}}$.

[Coll., 1892.]

49. If $f(x)$ be an arbitrary algebraic polynomial of degree $n-1$, and

$$P_n(x) \equiv A\frac{d^n}{dx^n}(x-a)^n(x-b)^n,$$

where A is a constant, then

$$\int_a^b f(x)\,P_n(x)\,dx = 0.$$ [Lond. Univ.]

50. Prove that

$$\int_0^\alpha \frac{x\,dx}{\cos x\cos(\alpha-x)} = \frac{\alpha}{\sin\alpha}\log\sec\alpha.$$ [Coll., 1896.]

51. Show that if α be less than unity,

$$\int_0^\pi \frac{x\sin x\,dx}{1+\alpha^2\cos^2 x} = \pi\frac{\tan^{-1}\alpha}{\alpha}.$$ [α, 1891.]

52. Integrate (i) $\int \frac{b^4x+x^5}{(b^4-x^4)^2}\sin^{-1}\frac{x^2}{b^2}\,dx.$ [δ, 1881.]

(ii) $\int \frac{d\phi}{\cos\phi}\sqrt{1-\tfrac{1}{2}\sin^2\phi}.$ [ST. JOHN'S, 1885.]

53. From the definition of a Bessel's function, viz.

$$J_n(x)=\frac{x^n}{2^n\,\Gamma(n+1)}\left[1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2\,.\,4(2n+2)(2n+4)}-\dots\right],$$

derive the results

$$\frac{\sin x}{x}=\int_0^{\frac{\pi}{2}} J_0(x\sin\theta)\sin\theta\,d\theta,$$

$$\frac{1-\cos x}{x}=\int_0^{\frac{\pi}{2}} J_1(x\sin\theta)\,d\theta.$$ [COLL., 1896.]

54. Integrate (i) $\frac{x^3}{(x+1)^3(x^3-1)}.$

(ii) $\frac{1}{\sqrt{1+3\sin x\cos x+2\sin^2x\cos^2x}}.$

(iii) $\frac{1}{\sqrt{(1+\sin x)(2+\sin x)}}.$ [MATH. TRIP., 1897.]

55. Show that

$$\int \sin^m x\sin nx\,dx=\sin^2 nx\,\frac{d}{dx}\left\{\frac{\phi(\sin x)}{\sin nx}\right\},$$

where the form of the function ϕ is defined by the relation

$$\phi(z)=\frac{z^m}{n^2-m^2}-\frac{m(m-1)}{(n^2-m^2)\{n^2-(m-2)^2\}}z^{m-2}$$
$$+\frac{m(m-1)(m-2)(m-3)}{(n^2-m^2)\{n^2-(m-2)^2\}\{n^2-(m-4)^2\}}z^{m-4}-\dots,$$

m being a positive integer and n not being of the form $\pm(m-2r)$, where r is a positive integer not greater than $\frac{m}{2}$.

[MATH. TRIP., 1897.]

56. Draw graphs of the transformation formula

$$(a_2x^2+2b_2x+c_2)y^2=a_1x^2+2b_1x+c_1$$

corresponding to those of Arts. 301 and 309 for

$$(a_2x^2+2b_2x+c_2)y=a_1x^2+2b_1x+c_1.$$