CHAPTER VI.

INTEGRALS OF FORMS $\int \frac{dx}{(a+b\cos x+c\sin x)^n}$, etc.

170. Integration of forms

$$\int \frac{dx}{a+b\cos x},\quad \int \frac{dx}{a+b\sin x},\quad \int \frac{dx}{a+b\cos x+c\sin x},\ \text{etc.}$$

To integrate $\int \frac{dx}{a+b\cos x}$, we may write $a+b\cos x$ as

$$a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right),$$

$$\text{i.e. } (a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2},$$

or
$$(a-b)\cos^2\frac{x}{2}\left[\frac{a+b}{a-b}+\tan^2\frac{x}{2}\right].$$

Thus
$$\int \frac{dx}{a+b\cos x}=\frac{2}{a-b}\int \frac{\frac{1}{2}\sec^2\frac{x}{2}\,dx}{\frac{a+b}{a-b}+\tan^2\frac{x}{2}}. \quad\ldots\ldots\ldots\ldots\ldots(1)$$

171. Case I. If $a^2>b^2$, this becomes

$$\frac{2}{a-b}\,\frac{1}{\sqrt{\frac{a+b}{a-b}}}\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{\frac{a+b}{a-b}}}$$

or
$$\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2},$$

$$\text{i.e. } \frac{2}{a\sin\alpha}\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{x}{2}\right),\ \text{where } b=a\cos\alpha.$$

This may be written in other forms:

e.g. since $$2\tan^{-1} z = \cos^{-1}\frac{1-z^2}{1+z^2},$$

we may write the result as

$$\frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{1-\dfrac{a-b}{a+b}\tan^2\dfrac{x}{2}}{1+\dfrac{a-b}{a+b}\tan^2\dfrac{x}{2}},$$

or $$\frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b+a\cos x}{a+b\cos x} = \frac{1}{a\sin\alpha}\cos^{-1}\frac{\cos\alpha+\cos x}{1+\cos\alpha\cos x}.$$

Further forms are:

$$\frac{2}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a-b}\sin\dfrac{x}{2}}{\sqrt{a+b\cos x}} \quad \text{or} \quad \frac{2}{\sqrt{a^2-b^2}}\cos^{-1}\frac{\sqrt{a+b}\cos\dfrac{x}{2}}{\sqrt{a+b\cos x}},$$

or $$\frac{1}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}\sin x}{a+b\cos x}, \text{ etc.}$$

172. Case II. If $a^2 < b^2$, writing the integral in the form

$$\frac{2}{b-a}\int\frac{d\tan\dfrac{x}{2}}{\dfrac{b+a}{b-a}-\tan^2\dfrac{x}{2}}$$

in place of the form (1), we have by Art. 127,

$$\int\frac{dx}{a+b\cos x} = \frac{2}{b-a}\,\frac{1}{2\sqrt{\dfrac{b+a}{b-a}}}\log\frac{\sqrt{\dfrac{b+a}{b-a}}+\tan\dfrac{x}{2}}{\sqrt{\dfrac{b+a}{b-a}}-\tan\dfrac{x}{2}}$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b+a}+\sqrt{b-a}\tan\dfrac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\dfrac{x}{2}}$$

$$= \frac{1}{a\tan\alpha}\log\frac{\cos\dfrac{x-\alpha}{2}}{\cos\dfrac{x+\alpha}{2}},$$

where $b = a\sec\alpha$.

By Art. 64, this may also be written as

$$\frac{2}{\sqrt{b^2-a^2}}\tanh^{-1}\sqrt{\frac{b-a}{b+a}}\tan\frac{x}{2} \quad\text{or}\quad \frac{2}{a\tan\alpha}\tanh^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{x}{2}\right)$$

or, since $\qquad 2\tanh^{-1}z=\cosh^{-1}\dfrac{1+z^2}{1-z^2},$

we may still further exhibit the result as

$$\frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{1+\dfrac{b-a}{b+a}\tan^2\dfrac{x}{2}}{1-\dfrac{b-a}{b+a}\tan^2\dfrac{x}{2}}$$

or

$$\frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\cos x}{a+b\cos x}, \quad \textit{i.e.}\ \frac{1}{a\tan\alpha}\cosh^{-1}\frac{1+\cos\alpha\cos x}{\cos\alpha+\cos x},$$

and in other but equivalent forms as in Case I.

173. We therefore have

$$\int\frac{dx}{a+b\cos x}=\left\{\begin{array}{l}\left.\begin{array}{l}\dfrac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2},\\ \textit{i.e.}\ \dfrac{1}{\sqrt{a^2-b^2}}\cos^{-1}\dfrac{b+a\cos x}{a+b\cos x},\end{array}\right\}a^2>b^2,\\ \qquad\qquad\text{or}\\ \left.\begin{array}{l}\dfrac{2}{\sqrt{b^2-a^2}}\tanh^{-1}\sqrt{\dfrac{b-a}{b+a}}\tan\dfrac{x}{2},\\ \textit{i.e.}\ \dfrac{1}{\sqrt{b^2-a^2}}\log\dfrac{\sqrt{b+a}+\sqrt{b-a}\tan\dfrac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\dfrac{x}{2}},\\ \text{or}\ \dfrac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\dfrac{b+a\cos x}{a+b\cos x},\end{array}\right\}a^2<b^2,\end{array}\right.$$

with many other forms.

174. In the cases $b=\pm a$, the integral is at once obtainable, for

$$\int\frac{dx}{a+a\cos x}=\frac{1}{2a}\int\sec^2\frac{x}{2}\,dx=\frac{1}{a}\tan\frac{x}{2}$$

and $$\int\frac{dx}{a-a\cos x}=\frac{1}{2a}\int\operatorname{cosec}^2\frac{x}{2}\,dx=-\frac{1}{a}\cot\frac{x}{2}.$$

175. The integration of $\int \frac{dx}{a+b\sin x}$ is reduced to the foregoing forms by the substitution $x=\frac{\pi}{2}+y$, when we have

$$\int \frac{dx}{a+b\sin x}=\int \frac{dy}{a+b\cos y}$$

$$\left.\begin{aligned}
&=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\\
\text{or}\quad &=\frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b+a\sin x}{a+b\sin x}\\
&=\frac{1}{a\cos\alpha}\cos^{-1}\frac{\sin\alpha+\sin x}{1+\sin\alpha\sin x},
\end{aligned}\right\} a^2>b^2,$$

where $b=a\sin\alpha$,

$$\left.\begin{aligned}
\text{or}\quad &\frac{2}{\sqrt{b^2-a^2}}\tanh^{-1}\sqrt{\frac{b-a}{b+a}}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\\
\text{or}\quad &\frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}{\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}\\
\text{or}\quad &\frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\sin x}{a+b\sin x}\\
&=\frac{1}{a\cot\alpha}\cosh^{-1}\frac{1+\sin\alpha\sin x}{\sin\alpha+\sin\alpha},
\end{aligned}\right\} a^2<b^2,$$

where $b=a\operatorname{cosec}\alpha$, with many other forms.

176. We might also treat $\int \frac{dx}{a+b\sin x}$ independently.

Proceeding in the same way as for $\int \frac{dx}{a+b\cos x}$, we write

$$a+b\sin x=a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+2b\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=a\cos^2\frac{x}{2}\left[\left(\tan\frac{x}{2}+\frac{b}{a}\right)^2+\frac{a^2-b^2}{a^2}\right].$$

Thus,

$$\int \frac{dx}{a+b\sin x}=\frac{2}{a}\int \frac{d\left(\tan\frac{x}{2}\right)}{\left(\tan\frac{x}{2}+\frac{b}{a}\right)^2+\frac{a^2-b^2}{a^2}},$$

and two cases arise as before, viz. $a \gtrless b$, when we apply Art. 127;

$$\therefore \int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}, \quad a^2 > b^2,$$

or
$$-\frac{1}{\sqrt{b^2-a^2}} \coth^{-1} \frac{a\tan\frac{x}{2}+b}{\sqrt{b^2-a^2}}, \quad a^2 < b^2,$$

showing the result in different forms from those already given, but of course differing from them only by quantities independent of x. The student should consider this statement and reconcile the results, as it is a matter of some little ingenuity.

177. **Extension.** Again, since $b\cos x + c\sin x$ may be written as $R\cos(x-\gamma)$, where $R=\sqrt{b^2+c^2}$ and $\tan\gamma = \frac{c}{b}$, we may deduce $\int \frac{dx}{a+b\cos x + c\sin x}$ from $\int \frac{dx}{a+b\cos x}$, or we may proceed independently, at our pleasure. Adopting the former course, we have

$$\int \frac{dx}{a+b\cos x+c\sin x}$$

$$= \int \frac{d(x-\gamma)}{a+R\cos(x-\gamma)}$$

$$\left.\begin{aligned} &= \frac{2}{\sqrt{a^2-R^2}} \tan^{-1} \sqrt{\frac{a-R}{a+R}} \tan\frac{x-\gamma}{2} \\ \text{or} \quad &= \frac{1}{\sqrt{a^2-R^2}} \cos^{-1} \frac{R+a\cos\overline{x-\gamma}}{a+R\cos\overline{x-\gamma}} \end{aligned}\right\} \text{if } a^2 > R^2$$

$$\left.\begin{aligned} \text{or} \quad & \frac{2}{\sqrt{R^2-a^2}} \tanh^{-1} \sqrt{\frac{R-a}{R+a}} \tan\frac{x-\gamma}{2}, \\ \textit{i.e.} \quad & \frac{1}{\sqrt{R^2-a^2}} \log \frac{\sqrt{R+a}+\sqrt{R-a}\tan\frac{x-\gamma}{2}}{\sqrt{R+a}-\sqrt{R-a}\tan\frac{x-\gamma}{2}} \\ \text{or} \quad & \frac{1}{\sqrt{R^2-a^2}} \cosh^{-1} \frac{R+a\cos\overline{x-\gamma}}{a+R\cos\overline{x-\gamma}}, \end{aligned}\right\} \text{if } a^2 < R^2$$

with other forms.

And these of course include the forms of Arts. 171 to 176 as particular cases, viz. when $c=0$ or $b=0$.

178. The reduction to the form

$$\int \frac{dx}{a+b\cos x}$$

has the advantage of making the integral depend upon the integration of

$$\int \frac{dx}{x^2 \pm k^2},$$

whilst the independent treatment throws the integration upon the form

$$\int \frac{dx}{ax^2+2bx+c},$$

and involves the completion of the square in the denominator.

179. **Illustrative Examples.**

Ex. 1.

$$\int \frac{dx}{3+5\cos x} = \int \frac{dx}{3\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+5\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{8\cos^2\frac{x}{2}-2\sin^2\frac{x}{2}}$$

$$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{4-\tan^2\frac{x}{2}}dx$$

$$= \frac{1}{4}\int\left(\frac{1}{2-\tan\frac{x}{2}}+\frac{1}{2+\tan\frac{x}{2}}\right)d\tan\frac{x}{2}$$

$$= \frac{1}{4}\log\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}} = \frac{1}{2}\tanh^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right) = \frac{1}{4}\cosh^{-1}\frac{5+3\cos x}{3+5\cos x}.$$

Ex. 2.

$$\int \frac{dx}{3-5\cos x} = \int \frac{dy}{3+5\cos y}, \text{ where } x=\pi+y$$

$$= \tfrac{1}{4}\cosh^{-1}\frac{5+3\cos y}{3+5\cos y} = \tfrac{1}{4}\cosh^{-1}\frac{5-3\cos x}{3-5\cos x}.$$

Ex. 3.
$$\int\frac{dx}{5+3\cos x}=\int\frac{dx}{5\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+3\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$
$$=\int\frac{dx}{8\cos^2\frac{x}{2}+2\sin^2\frac{x}{2}}=\frac{1}{2}\int\frac{\sec^2\frac{x}{2}dx}{4+\tan^2\frac{x}{2}}$$
$$=\tfrac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)=\tfrac{1}{4}\cos^{-1}\frac{3+5\cos x}{5+3\cos x}.$$

Ex 4. $\displaystyle\int\frac{dx}{5+3\sin x}=\int\frac{dy}{5+3\cos y}$, where $x=\dfrac{\pi}{2}+y$,
$$=\tfrac{1}{4}\cos^{-1}\frac{3+5\cos y}{5+3\cos y}=\tfrac{1}{4}\cos^{-1}\frac{3+5\sin x}{5+3\sin x}.$$

Ex. 5. $\displaystyle\int\frac{dx}{13+3\cos x+4\sin x}=\int\frac{dx}{13+5\cos(x-a)}$, where $\tan a=\frac{4}{3}$,
$$=\frac{1}{12}\cos^{-1}\frac{5+13\cos(x-a)}{13+5\cos(x-a)}=\frac{1}{6}\tan^{-1}\left(\frac{2}{3}\tan\frac{x-a}{2}\right).$$

180. The integrals
$$\int\frac{dx}{a+b\cosh x},\quad \int\frac{dx}{a+b\sinh x},\quad \int\frac{dx}{a+b\cosh x+c\sinh x}$$
may be treated similarly.

Thus,
$$\int\frac{dx}{a+b\cosh x}=\int\frac{dx}{a\left(\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}\right)+b\left(\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}\right)}$$
$$=\frac{2}{b-a}\int\frac{d\tanh\frac{x}{2}}{\frac{b+a}{b-a}+\tanh^2\frac{x}{2}}$$
or
$$\frac{2}{a-b}\int\frac{d\tanh\frac{x}{2}}{\frac{a+b}{a-b}-\tanh^2\frac{x}{2}}.$$

Hence, if $a^2<b^2$, we have the forms
$$\frac{2}{\sqrt{b^2-a^2}}\tan^{-1}\sqrt{\frac{b-a}{b+a}}\tanh\frac{x}{2}\quad\text{or}\quad\frac{1}{\sqrt{b^2-a^2}}\cos^{-1}\frac{b+a\cosh x}{a+b\cosh x};$$
and if $a^2>b^2$,
$$\frac{2}{\sqrt{a^2-b^2}}\tanh^{-1}\sqrt{\frac{a-b}{a+b}}\tanh\frac{x}{2}\quad\text{or}\quad\frac{1}{\sqrt{a^2-b^2}}\cosh^{-1}\frac{b+a\cosh x}{a+b\cosh x}.$$

Again,

$$\int \frac{dx}{a+b\sinh x} = \int \frac{dx}{a\left(\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}\right)+2b\sinh\frac{x}{2}\cosh\frac{x}{2}}$$

$$=\frac{2}{a}\int \frac{d\tanh\frac{x}{2}}{\frac{a^2+b^2}{a^2}-\left(\tanh\frac{x}{2}-\frac{b}{a}\right)^2}$$

$$=\frac{2}{\sqrt{a^2+b^2}}\tanh^{-1}\left(\frac{a\tanh\frac{x}{2}-b}{\sqrt{a^2+b^2}}\right);$$

and other forms will be exhibited later.

Similarly, in the general case,

$$\int \frac{dx}{a+b\cosh x+c\sinh x}$$

$$=\int \frac{dx}{a\left(\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}\right)+b\left(\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}\right)+2c\sinh\frac{x}{2}\cosh\frac{x}{2}}$$

$$=\int \frac{\operatorname{sech}^2\frac{x}{2}\,dx}{a+b+2c\tanh\frac{x}{2}-(a-b)\tanh^2\frac{x}{2}}$$

$$=\frac{2}{a-b}\int \frac{d\tanh\frac{x}{2}}{\left\{\frac{a+b}{a-b}+\frac{c^2}{(a-b)^2}\right\}-\left(\tanh\frac{x}{2}-\frac{c}{a-b}\right)^2}$$

or
$$\frac{2}{b-a}\int \frac{d\tanh\frac{x}{2}}{\left\{\frac{a+b}{b-a}-\frac{c^2}{(b-a)^2}\right\}+\left(\tanh\frac{x}{2}+\frac{c}{b-a}\right)^2}$$

$$=\frac{2}{\sqrt{a^2-b^2+c^2}}\tanh^{-1}\frac{(a-b)\tanh\frac{x}{2}-c}{\sqrt{a^2-b^2+c^2}}, \quad a^2+c^2>b^2,$$

or
$$\frac{2}{\sqrt{b^2-a^2-c^2}}\tan^{-1}\frac{(b-a)\tanh\frac{x}{2}+c}{\sqrt{b^2-a^2-c^2}}, \quad a^2+c^2<b^2.$$

But we notice also that just as $a+b\cos\theta+c\sin\theta$ may be written
$$a+R\cos\overline{\theta-\alpha}$$
by putting $b=R\cos\alpha$ and $c=R\sin\alpha$, where $R=\sqrt{b^2+c^2}$ and $\tan\alpha=\frac{c}{b}$, we may write
$$a+b\cosh x+c\sinh x \quad \text{as} \quad a+R\cosh\overline{x+\gamma}$$
by putting $b=R\cosh\gamma$ and $c=R\sinh\gamma$ if $b^2>c^2$, where $R=\sqrt{b^2-c^2}$ and $\tanh\gamma=\frac{c}{b}$, or as
$$a+R\sinh\overline{x+\gamma}$$
by putting $b=R\sinh\gamma$, $c=R\cosh\gamma$, where $R=\sqrt{c^2-b^2}$ and $\tanh\gamma=\frac{b}{c}$ when $b^2<c^2$, and therefore the case may be regarded as one of the previous ones or *vice versâ*.

181. **Another Method.** A further method of treatment will be obvious if we remember that these hyperbolic functions are merely functions of a real exponential.

Taking the general integral in this way, we have
$$\int\frac{dx}{a+b\cosh x+c\sinh x}=\int\frac{2dx}{2a+b(e^x+e^{-x})+c(e^x-e^{-x})}$$
$$=\int\frac{2e^x\,dx}{(b+c)e^{2x}+2ae^x+b-c}$$
$$=\frac{2}{b+c}\int\frac{de^x}{\left(e^x+\frac{a}{b+c}\right)^2+\frac{b^2-c^2-a^2}{(b+c)^2}}$$
or
$$\frac{2}{b+c}\int\frac{de^x}{\left(e^x+\frac{a}{b+c}\right)^2-\frac{a^2+c^2-b^2}{(b+c)^2}},$$
giving the forms
$$\frac{2}{\sqrt{b^2-c^2-a^2}}\tan^{-1}\frac{(b+c)\,e^x+a}{\sqrt{b^2-c^2-a^2}} \quad \text{if } b^2>a^2+c^2$$
or
$$-\frac{2}{\sqrt{a^2+c^2-b^2}}\coth^{-1}\frac{(b+c)e^x+a}{\sqrt{a^2+c^2-b^2}} \quad \text{if } b^2<a^2+c^2.$$

Comparing with the results of Art. 180, it will be remarked that the integrals of such expressions differ much in appearance according to the method adopted in integration. Integrals of the same expression, however, can only differ by a quantity (real or unreal) which does not contain x, and it will be a useful exercise to deduce one form from another; and, as has been said previously, this will sometimes require some ingenuity.

182. **The Integration expressed in terms of the Integrand.** Far more symmetry, however, will be obtained in the results if we attempt to express the integration in terms of the integrand, as we now proceed to show.

These integrals may be deduced from the form

$$\int \frac{dx}{\sqrt{Ax^2+2Bx+C}},$$

which is

$$\frac{1}{\sqrt{A}}\cosh^{-1}\frac{Ax+B}{\sqrt{B^2-AC}}, \quad A>0,\ B^2>AC \text{ (Arts. 80 and 81)},$$

or $$\frac{1}{\sqrt{-A}}\cos^{-1}\frac{Ax+B}{\sqrt{B^2-AC}}, \quad A<0,\ B^2>AC,$$

or $$\frac{1}{\sqrt{A}}\sinh^{-1}\frac{Ax+B}{\sqrt{AC-B^2}}, \quad A>0,\ B^2<AC,$$

the case $A<0$, $B^2<AC$ being omitted because the radical in the integrand becomes unreal in that case.

The rule is to *substitute y for the integrand in all cases and integrate in terms of y.* This method leads to remarkable symmetry of form, and expresses the result in terms of the integrand itself, and yields new forms for the integration.

Thus, considering the general case, and writing

$$\int \frac{d\theta}{a+b\cos\theta+c\sin\theta}=\int y\,d\theta,$$

where $$\frac{1}{a+b\cos\theta+c\sin\theta}=y,$$

we have $$b\cos\theta+c\sin\theta=\frac{1}{y}-a;$$

and therefore $$b\sin\theta-c\cos\theta=\frac{1}{y^2}\frac{dy}{d\theta}.$$

Squaring and adding,

$$b^2+c^2-a^2=\frac{1}{y^2}-\frac{2a}{y}+\frac{1}{y^4}\left(\frac{dy}{d\theta}\right)^2.$$

Hence

$$\int y\,d\theta=\pm\int\frac{dy}{\sqrt{(b^2+c^2-a^2)\,y^2+2ay-1}}$$

$$=\pm\frac{1}{\sqrt{b^2+c^2-a^2}}\cosh^{-1}\frac{(b^2+c^2-a^2)\,y+a}{\sqrt{b^2+c^2}}\quad\text{if } b^2+c^2>a^2$$

or

$$=\pm\frac{1}{\sqrt{a^2-b^2-c^2}}\cos^{-1}\frac{(b^2+c^2-a^2)\,y+a}{\sqrt{b^2+c^2}}\quad\text{if } b^2+c^2<a^2,$$

where $y^{-1}=a+b\cos\theta+c\sin\theta$.

The sign is to be determined by examining whether y increases or decreases with θ.

If y and θ increase together, $\frac{dy}{d\theta}$ is $+$; *e.g.* in $\int\frac{d\theta}{a+b\cos\theta}$, provided it be a case where b is $+^{\text{ve}}$ and in which $0<\theta<\frac{\pi}{2}$ throughout the integration, we use a $+$, for in the first quadrant as θ increases $\cos\theta$ diminishes; $\therefore\ \frac{1}{a+b\cos\theta}$ increases, that is, y increases.

In $\int\frac{d\theta}{a+b\sin\theta}$, supposing θ to lie in the first quadrant throughout the integration, we should use the $-$ sign.

183. In the same way, to integrate

$$\int\frac{dx}{a+b\cosh x+c\sinh x}\quad\text{or}\quad\int y\,dx,\ \text{say},$$

where

$$\frac{1}{a+b\cosh x+c\sinh x}=y,$$

we have

$$b\cosh x+c\sinh x=\frac{1}{y}-a,$$

$$b\sinh x+c\cosh x=-\frac{1}{y^2}\frac{dy}{dx}.$$

Squaring and subtracting,

$$b^2-c^2-a^2=\frac{1}{y^2}-\frac{2a}{y}-\frac{1}{y^4}\left(\frac{dy}{dx}\right)^2,$$

and taking the case b and c both positive, y decreases as x increases;

$$\therefore \int y\,dx = -\int \frac{dy}{\sqrt{(a^2+c^2-b^2)y^2-2ay+1}}$$

$$= -\frac{1}{\sqrt{a^2+c^2-b^2}}\cosh^{-1}\frac{(a^2+c^2-b^2)y-a}{\sqrt{b^2-c^2}}$$

$$= \frac{1}{\sqrt{a^2+c^2-b^2}}\cosh^{-1}\frac{a-(a^2+c^2-b^2)y}{\sqrt{b^2-c^2}}+\text{const.}$$

if $b^2>c^2$ and $a^2+c^2>b^2$

or
$$= -\frac{1}{\sqrt{-a^2-c^2+b^2}}\cos^{-1}\frac{(a^2+c^2-b^2)y-a}{\sqrt{b^2-c^2}}$$

$$= \frac{1}{\sqrt{-a^2-c^2+b^2}}\cos^{-1}\frac{a-(a^2+c^2-b^2)y}{\sqrt{b^2-c^2}}+\text{const.}$$

if $a^2+c^2<b^2$

or
$$= -\frac{1}{\sqrt{a^2+c^2-b^2}}\sinh^{-1}\frac{(a^2+c^2-b^2)y-a}{\sqrt{c^2-b^2}}$$

$$= \frac{1}{\sqrt{a^2+c^2-b^2}}\sinh^{-1}\frac{a-(a^2+c^2-b^2)y}{\sqrt{c^2-b^2}} \quad \text{if } b^2<c^2,$$

where $y^{-1}\equiv a+b\cosh x+c\sinh x$.

184. Hence we get the following **particular results** by putting b or $c=0$ in the general results of Arts. 182, 183,

$$\int\frac{d\theta}{a+b\cos\theta} = \frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\cos\theta}{a+b\cos\theta} \quad (b^2>a^2)$$

$$\text{or } = \frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b+a\cos\theta}{a+b\cos\theta} \quad (b^2<a^2),$$

$$\int\frac{d\theta}{a+b\sin\theta} = -\frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\sin\theta}{a+b\sin\theta} \quad (b^2>a^2)$$

$$\text{or } = \frac{1}{\sqrt{a^2-b^2}}\sin^{-1}\frac{b+a\sin\theta}{a+b\sin\theta} \quad (b^2<a^2),$$

$$\int\frac{dx}{a+b\cosh x} = \frac{1}{\sqrt{b^2-a^2}}\cos^{-1}\frac{b+a\cosh x}{a+b\cosh x} \quad (b^2>a^2)$$

$$\text{or } = \frac{1}{\sqrt{a^2-b^2}}\cosh^{-1}\frac{b+a\cosh x}{a+b\cosh x} \quad (b^2<a^2),$$

$$\int\frac{dx}{a+b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\sinh^{-1}\frac{-b+a\sinh x}{a+b\sinh x}.$$

The symmetrical form of the several results was given (without proof) by Greenhill in his *Chapter on the Integral Calculus*, p. 34.

When $a=0$ we arrive at results obtained earlier in other forms, viz.

$$\int\frac{d\theta}{\cos\theta}=\cosh^{-1}(\sec\theta)\ \text{(compare Art. 74)}$$

$$\int\frac{d\theta}{\sin\theta}=-\cosh^{-1}(\operatorname{cosec}\theta),$$

$$\int\frac{dx}{\cosh x}=\cos^{-1}(\operatorname{sech} x),$$

$$\int\frac{dx}{\sinh x}=\sinh^{-1}(-\operatorname{cosech} x)=-\sinh^{-1}(\operatorname{cosech} x);$$

and from the general results

$$\int\frac{d\theta}{b\cos\theta+c\sin\theta}=\frac{1}{\sqrt{b^2+c^2}}\operatorname{sech}^{-1}\frac{b\cos\theta+c\sin\theta}{\sqrt{b^2+c^2}},$$

$$\int\frac{d\theta}{b\cosh x+c\sinh x}=\frac{1}{\sqrt{b^2-c^2}}\sec^{-1}\frac{b\cosh x+c\sinh x}{\sqrt{b^2-c^2}}\quad\text{if } b^2>c^2$$

$$=-\frac{1}{\sqrt{c^2-b^2}}\operatorname{cosech}^{-1}\frac{b\cosh x+c\sinh x}{\sqrt{c^2-b^2}}\quad\text{if } b^2<c^2$$

or again, $$=\frac{2}{\sqrt{b^2-c^2}}\tan^{-1}\sqrt{\frac{b+c}{b-c}}e^x\quad\text{if } b^2>c^2,$$

or $$-\frac{2}{\sqrt{c^2-b^2}}\coth^{-1}\sqrt{\frac{c+b}{c-b}}e^x\quad\text{if } b^2<c^2,$$

forms which the student should compare with those previously obtained.

185. **Reduction formulae for integrals of form $I_n\equiv\int\frac{dx}{X^n}$, where**

$$X=a+b\begin{matrix}\cos\\ \sin\end{matrix}x.$$

Let us consider the case

$$I_2\equiv\int\frac{dx}{(a+b\cos x)^2}.$$

We shall connect the integral with another, viz.

$$I_1\equiv\int\frac{dx}{a+b\cos x}.$$

Put $$P\equiv\frac{\sin x}{a+b\cos x}.$$

[*Note.*—That is, to form P, $\sin x$ is introduced into the numerator of the integrand of I_2, and the index of the denominator is lowered by unity.]

Thus $$\frac{dP}{dx}=\frac{\cos x(a+b\cos x)+b(1-\cos^2 x)}{(a+b\cos x)^2}$$

$$=\frac{b+a\cos x}{(a+b\cos x)^2}=\frac{b-\dfrac{a^2}{b}+\dfrac{a}{b}(a+b\cos x)}{(a+b\cos x)^2}$$

$$=\frac{a}{b}\frac{1}{a+b\cos x}-\frac{a^2-b^2}{b}\frac{1}{(a+b\cos x)^2}.$$

Therefore integrating,

$$\frac{\sin x}{a+b\cos x}=\frac{a}{b}I_1-\frac{a^2-b^2}{b}I_2.$$

Hence $$I_2=-\frac{b}{a^2-b^2}\frac{\sin x}{a+b\cos x}+\frac{a}{a^2-b^2}I_1,$$

and I_1 has been given in various forms in Art. 173, *e.g.*

$$\frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b+a\cos x}{a+b\cos x} \quad\text{or}\quad \frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\cos x}{a+b\cos x},$$

according as a^2 is greater or less than b^2.

$$\therefore\ I_2=-\frac{b}{a^2-b^2}\frac{\sin x}{a+b\cos x}+\frac{a}{(a^2-b^2)^{\frac{3}{2}}}\cos^{-1}\frac{b+a\cos x}{a+b\cos x}\ (a^2>b^2),$$

or $$=-\frac{b}{a^2-b^2}\frac{\sin x}{a+b\cos x}-\frac{a}{(b^2-a^2)^{\frac{3}{2}}}\cosh^{-1}\frac{b+a\cos x}{a+b\cos x}\ (a^2<b^2).$$

186. Again, in the general case, if

$$I_n\equiv\int\frac{dx}{(a+b\cos x)^n},$$

put $$P\equiv\frac{\sin x}{(a+b\cos x)^{n-1}}.$$

Then

$$\frac{dP}{dx}=\frac{\cos x(a+b\cos x)+(n-1)\,b(1-\cos^2 x)}{(a+b\cos x)^n}$$

$$=\frac{A+B(a+b\cos x)+C(a+b\cos x)^2}{(a+b\cos x)^n},\ \text{say,}$$

where $$A+Ba+Ca^2=(n-1)b,$$

$$Bb+2Cab=a,$$

$$Cb^2=(2-n)b$$

giving $C=-\frac{(n-2)}{b},\quad B=\frac{a}{b}+2\frac{a}{b}(n-2)=(2n-3)\frac{a}{b},$

$$A=\ (n-1)b-(2n-3)\frac{a^2}{b}+(n-2)\frac{a^2}{b}$$
$$=-(n-1)\frac{a^2-b^2}{b}.$$

Hence, substituting these values and integrating,

$$\frac{\sin x}{(a+b\cos x)^{n-1}}=-(n-1)\frac{a^2-b^2}{b}I_n+(2n-3)\frac{a}{b}I_{n-1}-\frac{n-2}{b}I_{n-2}.$$

The reduction formula is then

$$I_n=-\frac{b}{(n-1)(a^2-b^2)}\frac{\sin x}{(a+b\cos x)^{n-1}}+\frac{2n-3}{n-1}\frac{a}{a^2-b^2}I_{n-1}$$
$$-\frac{n-2}{n-1}\frac{1}{a^2-b^2}I_{n-2}.$$

Thus, as I_1 and I_2 have already been found in finite terms, we can successively deduce the values of I_3, I_4, etc.

It will be noted that I_n is in this case shown to be dependent upon *two* integrals of lower order, viz. I_{n-1} and I_{n-2}, except when $n=2$.

Also, the result of Art. 185 could have been obtained by putting $n=2$ in the present result.

187. **Generalization of above method.**

As $\int\frac{dx}{(a+b\sin x)^n}$ reduces to $\int\frac{dy}{(a+b\cos y)^n}$ on substituting $\frac{\pi}{2}+y$ for x, and

$$\int\frac{dx}{(a+b\cos x+c\sin x)^n}$$

may be written as $\int\frac{d(x-\gamma)}{[a+R\cos(x-\gamma)]^n}$, where $R=\sqrt{b^2+c^2}$ and $\gamma=\tan^{-1}\frac{c}{b}$, it is usual to refer these integrals to the case considered in Art. 186. We may, however, establish a reduction formula independently for each case.

Taking $\qquad I_n\equiv\int\frac{dx}{(a+b\cos x+c\sin x)^n}.$

Let $\qquad P\equiv\frac{-b\sin x+c\cos x}{(a+b\cos x+c\sin x)^{n-1}}$

$$\left[\textit{i.e.}\text{ if } D\equiv a+b\cos x+c\sin x,\ P=\frac{D'}{D^{n-1}}\right].$$

Then

$$\frac{dP}{dx}=\frac{-b\cos x-c\sin x}{(a+b\cos x+c\sin x)^{n-1}}-(n-1)\frac{(-b\sin x+c\cos x)^2}{(a+b\cos x+c\sin x)^n}$$

$$=\frac{-a(b\cos x+c\sin x)-(b\cos x+c\sin x)^2-(n-1)[b^2+c^2-(b\cos x+c\sin x)^2]}{(a+b\cos x+c\sin x)^n}$$

$$=\frac{A+B(a+b\cos x+c\sin x)+C(a+b\cos x+c\sin x)^2}{(a+b\cos x+c\sin x)^n}, \text{ say,}$$

where A, B, C are constants to be determined so that

$$\left.\begin{aligned} A+Ba+Ca^2&=-(n-1)(b^2+c^2),\\ B+2aC&=-a,\\ C&=n-2,\end{aligned}\right\}$$

whence $A=(n-1)(a^2-b^2-c^2)$, $B=-(2n-3)a$, $C=n-2$.

Therefore the proper reduction formula for I_n is

$$\frac{-b\sin x+c\cos x}{(a+b\cos x+c\sin x)^{n-1}}$$
$$=(n-1)(a^2-b^2-c^2)I_n-(2n-3)aI_{n-1}+(n-2)I_{n-2}.$$

We note that when $n=2$, the last term disappears, and

$$(a^2-b^2-c^2)I_2=aI_1+\frac{-b\sin x+c\cos x}{(a+b\cos x+c\sin x)},$$

i.e. $$(a^2-b^2-c^2)\int\frac{dx}{(a+b\cos x+c\sin x)^2}=\frac{-b\sin x+c\cos x}{a+b\cos x+c\sin x}+aI_1,$$

the real form of I_1 being selected from the various forms in Art. 177.

Also I_1 and I_2 now having been found, we can proceed to deduce I_3, I_4, etc., successively by aid of the reduction formula established.

188. **Corresponding formulae for the case of Hyperbolic Functions.**

In like manner reduction formulae for

$$\int\frac{dx}{(a+b\cosh x)^n},\quad \int\frac{dx}{(a+b\sinh x)^n},\quad \int\frac{dx}{(a+b\cosh x+c\sinh x)^n}$$

may be constructed.

As the last includes the first two as particular cases, we consider that one in particular, and proceed as before.

Put $$P \equiv \frac{b \sinh x + c \cosh x}{(a+b\cosh x + c \sinh x)^{n-1}}.$$

Then

$$\frac{dP}{dx} = \frac{(b\cosh x + c\sinh x)(a+b\cosh x+c\sinh x) - (n-1)(b\sinh x + c\cosh x)^2}{(a+b\cosh x+c\sinh x)^n}$$

$$= \frac{a(b\cosh x+c\sinh x)+(b\cosh x+c\sinh x)^2 - (n-1)[(b\cosh x+c\sinh x)^2-(b^2-c^2)]}{(a+b\cosh x+c\sinh x)^n}$$

$$= \frac{A+B(a+b\cosh x+c\sinh x)+C(a+b\cosh x+c\sinh x)^2}{(a+b\cosh x+c\sinh x)^n},$$

say,

where $$\left.\begin{aligned} A+Ba+Ca^2 &= (n-1)(b^2-c^2),\\ B+2Ca &= a,\\ C &= -(n-2),\end{aligned}\right\}$$

whence $A=(n-1)(-a^2+b^2-c^2)$, $B=(2n-3)a$, $C=-(n-2)$.

And the proper reduction formula is

$$\frac{b\sinh x+c\cosh x}{(a+b\cosh x+c\sinh x)^{n-1}} = (n-1)(-a^2+b^2-c^2)I_n+(2n-3)aI_{n-1}-(n-2)I_{n-2}.$$

As before, the last term disappears in the case $n=2$.

Hence

$$(-a^2+b^2-c^2)I_2 = \frac{b\sinh x+c\cosh x}{a+b\cosh x+c\sinh x} - aI_1,$$

the real form of I_1 being selected from the various forms shown in Art. 180.

I_1 and I_2 being now known, we can proceed as before to deduce successively I_3, I_4, etc., by aid of the reduction formula.

189. **Special Cases.**

We notice also that, putting $a=0$, $b=0$ or $c=0$, or two of them, in these reduction formulae, we have a mode of reduction for such expressions as

$$\int \operatorname{sech}^n x\,dx,\quad \int \operatorname{cosech}^n x\,dx,\quad \int\frac{dx}{(b\cosh x+c\sinh x)^n},$$
$$\int\frac{dx}{(b\cos x+c\sin x)^n},\quad \int\frac{dx}{(a+b\sinh x)^n},\quad \text{etc.}$$

190. Fractions of form

$$\frac{a+b\cos x+c\sin x}{a_1+b_1\cos x+c_1\sin x}.$$

The numerator of this fraction can be thrown into the form

$$A(a_1+b_1\cos x+c_1\sin x)+B(-b_1\sin x+c_1\cos x)+C$$

i.e. $\quad A(\text{denr.})+B(\text{diff. co. of denr.})+C,$

by taking $\quad Aa_1+C=a, \quad Ab_1+Bc_1=b, \quad Ac_1-Bb_1=c,$

which determine A, B and C.

The fraction then takes the form

$$A+B\frac{-b_1\sin x+c_1\cos x}{a_1+b_1\cos x+c_1\sin x}+\frac{C}{a_1+b_1\cos x+c_1\sin x},$$

and the integral is

$$Ax+B\log(a_1+b_1\cos x+c_1\sin x)+C\int\frac{dx}{a_1+b_1\cos x+c_1\sin x},$$

and the last integral has been evaluated.

191. Extension of above Method.

In the same way $\dfrac{a+b\cos x+c\sin x}{(a_1+b_1\cos x+c_1\sin x)^n}$ may be arranged as

$$\frac{A}{(a_1+b_1\cos x+b_1\sin x)^{n-1}}+B\frac{-b_1\sin x+c_1\cos x}{(a_1+b_1\cos x+c_1\sin x)^n}$$
$$+\frac{C}{(a_1+b_1\cos x+c_1\sin x)^n}.$$

The integrals of the first and last fractions may be deduced by the reduction formula of Art. 187, and that of the second fraction is

$$-\frac{B}{n-1}\frac{1}{(a_1+b_1\cos x+c_1\sin x)^{n-1}} \quad (n>1).$$

192. Case of Hyperbolic Functions.

Exactly in the same way fractions of the forms

$$\frac{a+b\cosh x+c\sinh x}{a_1+b_1\cosh x+c_1\sinh x}, \quad \frac{a+b\cosh x+c\sinh x}{(a_1+b_1\cosh x+c_1\sinh x)^n}$$

may be integrated.

193. Further Generalization.

If $\quad \prod_{r=1}^{r=n}(a_r+b_r\cos\theta+c_r\sin\theta)$

stands for the product of n factors, some of which may be repeated, and of which the one exhibited is a type, and if $\phi(x, y)$ be any rational integral algebraic function of x and y, the integral of

$$\int \frac{\phi(\cos\theta, \sin\theta)}{\prod_{r=1}^{r=n}(a_r + b_r\cos\theta + c_r\sin\theta)}\,d\theta$$

can now be found. For expressing $\cos\theta$ and $\sin\theta$ in terms of the tangent of the half angle, and writing $t = \tan\frac{\theta}{2}$,

$$\phi(\cos\theta, \sin\theta) = \phi\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) = \frac{\chi(t)}{(1+t^2)^p},$$

where p is the degree of $\phi(x, y)$ in x and y, not necessarily homogeneous, and $\chi(t)$ is a rational and integral algebraic function of t of degree $2p$ at most.

Also $\quad a_r + b_r\cos\theta + c_r\sin\theta = a_r + b_r\frac{1-t^2}{1+t^2} + c_r\frac{2t}{1+t^2},$

whence

$$\prod_{r=1}^{r=n}(a_r + b_r\cos\theta + c_r\sin\theta) = \frac{\prod_{r=1}^{r=n}[a_r + b_r + 2c_r t + (a_r - b_r)t^2]}{(1+t^2)^n}.$$

also $$d\theta = \frac{2dt}{1+t^2}.$$

Hence

$$\frac{\phi(\cos\theta, \sin\theta)\,d\theta}{\prod_{r=1}(a_r + b_r\cos\theta + c_r\sin\theta)}$$

$$= \frac{2\chi(t)\,dt}{(1+t^2)^{-n+p+1}\prod_{r=1}^{r=n}(a_r + b_r + 2c_r t + \overline{a_r - b_r}\,t^2)},$$

and supposing $a_r \neq b_r$ for any of the values of r, the degree of $\chi(t)$ in t, *i.e.* $2p$, is lower than that of the denominator, which is $2(p+1-n)+2n$, *i.e.* $2p+2$.

This expression may then be put into partial fractions, some of type $\frac{A+Bt}{(1+t^2)^\lambda}$, others of type $\frac{C+Dt}{(F+Gt+Ht^2)^\mu}$.

The proper reduction formulae for such cases will be found in the next chapter. The integration can now be effected.

The reader may consider for himself the effect of $a_r = b_r$ for any value or values of r.

194. A different Method.

To obtain integrals of form

$$\int\frac{d\theta}{(a+b\cos\theta+c\sin\theta)^n} \text{ or } \int\frac{dx}{(a+b\cosh x+c\sinh x)^n}$$

and their particular cases, **we may avoid the reduction formulae** referred to, and proceed as follows, using a reduction of different nature.

Consider the first of these.

Case $b^2+c^2>a^2$.

Taking

$$\int\frac{d\theta}{a+b\cos\theta+c\sin\theta}$$

$$=\pm\frac{1}{\sqrt{b^2+c^2-a^2}}\cosh^{-1}\frac{(b^2+c^2-a^2)y+a}{\sqrt{b^2+c^2}}=\frac{u}{\sqrt{b^2+c^2-a^2}}, \text{ say,}$$

where $y^{-1}=a+b\cos\theta+c\sin\theta$ and $b^2+c^2>a^2$ (Art. 182),

$$y\,d\theta=\frac{du}{\sqrt{b^2+c^2-a^2}} \text{ and } (b^2+c^2-a^2)\,y=\sqrt{b^2+c^2}\cosh u-a;$$

$$\therefore\ y^n\,d\theta=\frac{(\sqrt{b^2+c^2}\cosh u-a)^{n-1}}{(b^2+c^2-a^2)^{\frac{2n-1}{2}}}du,$$

$$\int y^n\,d\theta=\frac{1}{(b^2+c^2-a^2)^{\frac{2n-1}{2}}}\int(\sqrt{b^2+c^2}\cosh u-a)^{n-1}\,du,$$

i.e. $$\int\frac{d\theta}{(a+b\cos\theta+c\sin\theta)^n}$$

$$=\frac{1}{(b^2+c^2-a^2)^{\frac{2n-1}{2}}}\int(\sqrt{b^2+c^2}\cosh u-a)^{n-1}\,du.$$

We may then expand $(\sqrt{b^2+c^2}\cosh u-a)^{n-1}$ and integrate each term, finally substituting back for u its value

$$\pm\cosh^{-1}\frac{(b^2+c^2-a^2)\,y+a}{\sqrt{b^2+c^2}},$$

$$\textit{i.e. } \pm\cosh^{-1}\frac{1}{\sqrt{b^2+c^2}}\left[\frac{b^2+c^2-a^2}{a+b\cos\theta+c\sin\theta}+a\right],$$

the proper sign having been selected as indicated in Art. 182.

Case $b^2+c^2<a^2$.

$$\int\frac{d\theta}{a+b\cos\theta+c\sin\theta}$$

$$=\pm\frac{1}{\sqrt{a^2-b^2-c^2}}\cos^{-1}\frac{(b^2+c^2-a^2)\,y+a}{\sqrt{b^2+c^2}}=\frac{u}{\sqrt{a^2-b^2-c^2}},\text{ say}$$

i.e. $$y\,d\theta=\frac{du}{\sqrt{a^2-b^2-c^2}}\quad\text{and}\quad(a^2-b^2-c^2)\,y=a-\sqrt{b^2+c^2}\cos u;$$

$$\therefore\ y^n\,d\theta=\frac{(a-\sqrt{b^2+c^2}\cos u)^{n-1}}{(a^2-b^2-c^2)^{\frac{2n-1}{2}}}\,du,$$

i.e. $$\int\frac{d\theta}{(a+b\cos\theta+c\sin\theta)^n}$$

$$=\frac{1}{(a^2-b^2-c^2)^{\frac{2n-1}{2}}}\int(a-\sqrt{b^2+c^2}\cos u)^{n-1}\,du.$$

195. In exactly the same way, from the three forms

(where $y^{-1}=a+b\cosh x+c\sinh x$)

$$\int\frac{dx}{a+b\cosh x+c\sinh x}$$

$$=\frac{1}{\sqrt{-a^2-c^2+b^2}}\cos^{-1}\frac{a-(a^2+c^2-b^2)\,y}{\sqrt{b^2-c^2}}=\frac{u}{\sqrt{-a^2-c^2+b^2}},$$

where $b^2>a^2+c^2$;

or $$=\frac{1}{\sqrt{a^2+c^2-b^2}}\cosh^{-1}\frac{a-(a^2+c^2-b^2)\,y}{\sqrt{b^2-c^2}}=\frac{u}{\sqrt{a^2+c^2-b^2}},$$

where $a^2+c^2>b^2>c^2$;

or $$=\frac{1}{\sqrt{a^2+c^2-b^2}}\sinh^{-1}\frac{a-(a^2+c^2-b^2)\,y}{\sqrt{c^2-b^2}}=\frac{u}{\sqrt{a^2+c^2-b^2}},$$

where $b^2<c^2$.

we obtain respectively,

Case $b^2>a^2+c^2$,

$$\int\frac{dx}{(a+b\cosh x+c\sinh x)^n}$$

$$=\frac{1}{(-a^2-c^2+b^2)^{\frac{2n-1}{2}}}\int(\sqrt{b^2-c^2}\cos u-a)^{n-1}\,du,$$

where $$a-\frac{a^2+c^2-b^2}{a+b\cosh x+c\sinh x}=\sqrt{b^2-c^2}\cos u.$$

Case $a^2+c^2>b^2>c^2$,

$$\int\frac{dx}{(a+b\cosh x+c\sinh x)^n}$$

$$=\frac{1}{(a^2+c^2-b^2)^{\frac{2n-1}{2}}}\int(a-\sqrt{b^2-c^2}\cosh u)^{n-1}\,du,$$

$$\text{where } a-\frac{a^2+c^2-b^2}{a+b\cosh x+c\sinh x}=\sqrt{b^2-c^2}\cosh u.$$

Case $c^2>b^2$,

$$\int\frac{dx}{(a+b\cosh x+c\sinh x)^n}$$

$$=\frac{1}{(a^2+c^2-b^2)^{\frac{2n-1}{2}}}\int(a-\sqrt{c^2-b^2}\sinh u)^{n-1}\,du,$$

$$\text{where } a-\frac{a^2+c^2-b^2}{a+b\cosh x+c\sinh x}=\sqrt{c^2-b^2}\sinh u.$$

196. Important Particular Cases.

The particular cases (according as b or $c=0$ in the general formulae, and which should be worked *ab initio* by the student) are

$$\int\frac{d\theta}{(a+b\cos\theta)^n}=\frac{1}{(b^2-a^2)^{\frac{2n-1}{2}}}\int(b\cosh u-a)^{n-1}\,du,$$

$$b^2>a^2,\ \left(\frac{b+a\cos\theta}{a+b\cos\theta}=\cosh u\right),$$

$$=\frac{1}{(a^2-b^2)^{\frac{2n-1}{2}}}\int(a-b\cos u)^{n-1}\,du,$$

$$b^2<a^2,\ \left(\frac{b+a\cos\theta}{a+b\cos\theta}=\cos u\right).$$

$$\int\frac{d\theta}{(a+b\sin\theta)^n}=\frac{1}{(b^2-a^2)^{\frac{2n-1}{2}}}\int(b\cosh u-a)^{n-1}\,du,$$

$$b^2>a^2,\ \left(\frac{b+a\sin\theta}{a+b\sin\theta}=\cosh u\right),$$

$$=\frac{1}{(a^2-b^2)^{\frac{2n-1}{2}}}\int(a-b\sin u)^{n-1}\,du,$$

$$b^2<a^2,\ \left(\frac{b+a\sin\theta}{a+b\sin\theta}=\sin u\right).$$

$$\int\frac{dx}{(a+b\cosh x)^n}=\frac{1}{(b^2-a^2)^{\frac{2n-1}{2}}}\int(b\cos u-a)^{n-1}\,du,$$

$$b^2>a^2,\ \left(\frac{b+a\cosh x}{a+b\cosh x}=\cos u\right),$$

$$=\frac{1}{(a^2-b^2)^{\frac{2n-1}{2}}}\int(a-b\cosh u)^{n-1}\,du,$$

$$b^2<a^2,\ \left(\frac{b+a\cosh x}{a+b\cosh x}=\cosh u\right).$$

$$\int\frac{du}{(a+b\sinh x)^n}=\frac{1}{(a^2+b^2)^{\frac{2n-1}{2}}}\int(a-b\sinh u)^{n-1}\,du,$$

$$\left(\frac{-b+a\sinh x}{a+b\sinh x}=\sinh u\right)$$

197. We have the further results, from putting $a=0$ and $b=1$ in the above, viz.

$$\int\sec^n\theta\,d\theta=\int\cosh^{n-1}u\,du,\quad\text{where }\theta=\sec^{-1}\cosh u;$$

$$\int\operatorname{cosec}^n\theta\,d\theta=\int\cosh^{n-1}u\,du,\quad\text{where }\theta=\operatorname{cosec}^{-1}\cosh u.$$

Hence either integral may be expressed in the form

$$\frac{1}{2^{n-1}}\int(e^u+e^{-u})^{n-1}\,du=\frac{1}{2^{n-2}}\int\left[\cosh(n-1)u+{}^{n-1}C_1\cosh(n-3)u+{}^{n-1}C_2\cosh(n-5)u+\text{etc.}\right]du$$

$$=\frac{1}{2^{n-2}}\left[\frac{\sinh(n-1)u}{n-1}+{}^{n-1}C_1\frac{\sinh(n-3)u}{n-3}+{}^{n-1}C_2\frac{\sinh(n-5)u}{n-5}+\dots+\tfrac{1}{2}\,{}^{n-1}C_{\frac{n-1}{2}}u\quad\text{or}\quad+{}^{n-1}C_{\frac{n}{2}}\sinh u\right].$$

(Compare the forms in Art. 122.)

198. Further, if in the results of Art. 196 we write $n-1=-m$, we have

$$\int\frac{du}{(b\cosh u-a)^m}=\frac{1}{(b^2-a^2)^{\frac{2m-1}{2}}}\int(a+b\cos\theta)^{m-1}\,d\theta,\quad b^2>a^2,$$

$$\int\frac{du}{(a-b\cos u)^m}=\frac{1}{(a^2-b^2)^{\frac{2m-1}{2}}}\int(a+b\cos\theta)^{m-1}\,d\theta,\quad b^2<a^2,$$

etc.

Several of these results are given in Greenhill's *Chapter on the Integral Calculus.* The geometrical significance of some of these transformations will appear later.

199. Cases required for the time in an Elliptic Orbit.

The cases of $\int \frac{d\theta}{(a+b\cos\theta)^n}$, where $a=1$, $b=e$, $n=2$, are required in the theory of Planetary Motion in finding the time in an assigned portion of an elliptic (or hyperbolic) orbit. We may either quote the results from Art. 185, or proceed independently as follows.

If $e<1$, by Art. 171,

$$\int \frac{d\theta}{1+e\cos\theta}=\frac{1}{\sqrt{1-e^2}}\cos^{-1}\frac{e+\cos\theta}{1+e\cos\theta}=\frac{u}{\sqrt{1-e^2}}, \text{ say;}$$

and if $e>1$, $\quad =\frac{1}{\sqrt{e^2-1}}\cosh^{-1}\frac{e+\cos\theta}{1+e\cos\theta}=\frac{v}{\sqrt{e^2-1}}$, say.

Taking $e<1$,

$$\frac{d\theta}{1+e\cos\theta}=\frac{du}{\sqrt{1-e^2}} \quad\text{and}\quad \frac{1}{1+e\cos\theta}=\frac{1-e\cos u}{1-e^2};$$

$$\therefore \int \frac{d\theta}{(1+e\cos\theta)^2}=\frac{1}{(1-e^2)^{\frac{3}{2}}}\int (1-e\cos u)\,du, \text{ (or by Art. 196),}$$

$$=\frac{1}{(1-e^2)^{\frac{3}{2}}}(u-e\sin u)$$

$$=\frac{1}{(1-e^2)^{\frac{3}{2}}}\left[\cos^{-1}\frac{e+\cos\theta}{1+e\cos\theta}-e\frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right].$$

The time T for a planet measured from passing Perihelion is expressed by this integral as

$$nT=(1-e^2)^{\frac{3}{2}}\int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2},$$

where n is a certain constant (see E. J. Routh, or Tait & Steele, *Dynamics of a Particle*). It follows that $nT=u-e\sin u$.

If $e>1$,

$$\frac{d\theta}{1+e\cos\theta}=\frac{dv}{\sqrt{e^2-1}} \quad\text{and}\quad \frac{1}{1+e\cos\theta}=\frac{e\cosh v-1}{e^2-1};$$

$$\therefore \int \frac{d\theta}{(1+e\cos\theta)^2}=\frac{1}{(e^2-1)^{\frac{3}{2}}}\int (e\cosh v-1)\,dv, \text{ (or by Art. 196),}$$

$$=\frac{1}{(e^2-1)^{\frac{3}{2}}}(e\sinh v-v)$$

$$=\frac{1}{(e^2-1)^{\frac{3}{2}}}\left[e\frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta}-\cosh^{-1}\frac{e+\cos\theta}{1+e\cos\theta}\right].$$

200. In practice, each example should be worked *ab initio.*

For example, suppose we require

$$\int_0^\pi \frac{dx}{(5+3\cos x)^4}.$$

Putting $5+3\cos x=\frac{1}{v}$, $\quad 3\sin x=\frac{1}{y^2}\frac{dy}{dx}$;

$$\therefore\ 9-\left(\frac{1}{y}-5\right)^2=\frac{1}{y^4}\left(\frac{dy}{dx}\right)^2;$$

$$\therefore\ \int y\,dx=+\int\frac{dy}{y\sqrt{9-\left(\frac{1}{y}-5\right)^2}}.$$

We take the + sign, because, as x increases in the first quadrant, $5+3\cos x$ decreases and y increases.

Thus,
$$\int\frac{dx}{5+3\cos x}=\int\frac{dy}{\sqrt{-1+10y-16y^2}}$$

$$=\frac{1}{4}\int\frac{dy}{\sqrt{\frac{9}{16^2}-\left(y-\frac{5}{16}\right)^2}}$$

$$=\frac{1}{4}\sin^{-1}\left(\frac{16y-5}{3}\right)+\text{const.}$$

$$=-\frac{1}{4}\sin^{-1}\frac{3+5\cos x}{5+3\cos x}+\text{const.}$$

$$=\frac{1}{4}\cos^{-1}\frac{3+5\cos x}{5+3\cos x}+\text{const.};$$

call this
$$=\frac{1}{4}u+\text{const.}$$

Then
$$\frac{dx}{5+3\cos x}=\frac{1}{4}du,\ \text{where}\ \frac{3+5\cos x}{5+3\cos x}=\cos u,$$

and
$$\therefore\ \frac{1}{5+3\cos x}=\frac{5-3\cos u}{16};$$

$$\therefore\ \frac{1}{(5+3\cos x)^3}=\frac{(5-3\cos u)^3}{16^3};$$

$$\therefore\ \frac{dx}{(5+3\cos x)^4}=\frac{(5-3\cos u)^3\,du}{2^{14}}.$$

$$\int_0^\pi\frac{dx}{(5+3\cos x)^4}=\frac{1}{2^{14}}\int_0^\pi[5^3-3\,.\,5^2\,.\,3\cos u+3\,.\,5\,.\,3^2\cos^2 u-3^3\cos^3 u\ du$$

[for when $x=0$, $\cos u=1$, and when $x=\pi$, $\cos u=-1$],

$$=\frac{1}{2^{14}}\,.\,2\int_0^{\frac{\pi}{2}}(5^3+3\,.\,5\,.\,3^2\cos^2 u)\,du$$

$$=\frac{1}{2^{14}}\,.\,2\left[5^3\frac{\pi}{2}+3^3\,.\,5\frac{1}{2}\frac{\pi}{2}\right]$$

$$=\frac{\pi}{2^{14}}\,.\,5\,.\left(25+\frac{27}{2}\right)=\frac{5\pi}{2^{15}}\times 77$$

$$=\frac{385\pi}{2^{15}}.$$

201. **The integrals**

$$I_1 \equiv \int \frac{\sin^m x}{a+b\cos x}\,dx \quad \text{and} \quad I_2 \equiv \int \frac{\sin^m x}{(a+b\cos x)^2}\,dx$$

can both be integrated in finite terms when m is a positive integer.

Consider the first, viz. $\int \frac{\sin^m x}{a+b\cos x}\,dx$.

The case $m=1$ obviously gives $-\frac{1}{b}\log(a+b\cos x)$.

If m be odd, $=2k+1$, say, put $a+b\cos x=z$, and therefore $b\sin x\,dx=-dz$.

Thus, $$\int \frac{\sin^{2k+1} x}{a+b\cos x}\,dx = -\frac{1}{b}\int \frac{\left[1-\left(\frac{z-a}{b}\right)^2\right]^k dz}{z},$$

every term of which is integrable when expanded in powers of z.

If m be even, $=2k$, say,

$$\int \frac{\sin^{2k} x}{a+b\cos x}\,dx = \int \frac{(1-\cos^2 x)^k}{a+b\cos x}\,dx,$$

and if the numerator be expanded in descending powers of $\cos x$, and then divided by $b\cos x+a$, we arrive at an expression of form

$$\int \left(\lambda_1 \cos^{2k-1} x + \lambda_2 \cos^{2k-2} x + \ldots + \lambda_{2k} + \frac{\lambda_{2k+1}}{a+b\cos x}\right)dx,$$

where the λ's are numerical coefficients.

Hence, in all cases, $\int \frac{\sin^m x}{a+b\cos x}\,dx$ can be integrated in finite terms.

The same argument applies to $\int \frac{\sin^m x}{(a+b\cos x)^2}\,dx$.

202. If $I_n \equiv \int \frac{\sin^m x}{(a+b\cos x)^n}\,dx$, **there is a reduction formula connecting I_n with I_{n-1} and I_{n-2}.** Hence all such integrations can be effected in finite terms.

To obtain this reduction formula, put

$$P = \frac{\sin^{m+1} x}{(a+b\cos x)^{n-1}}$$

[*i.e. increase the index of the numerator by unity and decrease that of the denominator by unity*].

Then

$$\frac{dP}{dx}=\frac{(m+1)\sin^m x\cos x\,(a+b\cos x)+(n-1)\,b\sin^m x\,(1-\cos^2 x)}{(a+b\cos x)^n}$$

$$=\frac{\sin^m x}{(a+b\cos x)^n}\left[A+B(a+b\cos x)+C(a+b\cos x)^2\right],$$

where
$$A+Ba+Ca^2=(n-1)\,b,$$
$$Bb+2Cab=(m+1)\,a,$$
$$Cb^2=(m+1)\,b-(n-1)\,b,$$

giving

$$A=-(n-1)\frac{a^2-b^2}{b},\quad B=(2n-m-3)\,\frac{a}{b},\quad C=\frac{m-n+2}{b}.$$

Hence

$$\frac{\sin^{m+1}x}{(a+b\cos x)^{n-1}}=-(n-1)\,\frac{a^2-b^2}{b}\,I_n+(2n-m-3)\,\frac{a}{b}\,I_{n-1}+\frac{m-n+2}{b}\,I_{n-2},$$

and the reduction formula required is

$$I_n=-\frac{1}{n-1}\,\frac{b}{a^2-b^2}\,\frac{\sin^{m+1}x}{(a+b\cos x)^{n-1}}+\frac{2n-m-3}{n-1}\,\frac{a}{a^2-b^2}\,I_{n-1}+\frac{m-n+2}{n-1}\,\frac{1}{a^2-b^2}\,I_{n-2},$$

of which the formula of Art. 186 is a particular case.

And since I_1 and I_2 have been shown integrable in finite terms when m is given, we can use the reduction formula just established to find successively I_3, I_4, etc., in terms of I_1 and I_2, and thus integrate them.

203. **Again, Integrals of form**

$$I_1'\equiv\int\frac{\sin^p\theta\cos^q\theta}{a+b\cos\theta}\,d\theta,\quad I_2'\equiv\int\frac{\sin^p\theta\cos^q\theta\,d\theta}{(a+b\cos\theta)^2},$$

$$I_3'\equiv\int\frac{\sin^p\theta\cos^q\theta\,d\theta}{(a+b\cos\theta)^3}$$

are always integrable in finite terms, p and q being positive integers.

For (1) if p be odd, $=2k+1$,

$$I_1'=-\int\frac{(1-c^2)^k c^q\,dc}{a+bc},\ \text{where } c=\cos\theta,$$

and after expansion of the numerator in descending powers of c and division by $bc+a$, we get a series of powers of c and a remainder $\dfrac{A}{a+bc}$, and each term is integrable with respect to c.

(2) If p be even, $=2k$,

$$\sin^p\theta\cos^q\theta=(1-\cos^2\theta)^k\cos^q\theta,$$

which, when expanded in descending powers of $\cos\theta$ and divided by $b\cos\theta+a$, gives a series of powers of $\cos\theta$ with a remainder of form $\dfrac{A}{a+b\cos\theta}$, and each term is integrable with respect to θ by Arts. 117, 173.

And the same argument holds good for I_2', I_3', except that the remainders to be integrated involve such terms as

$$A\int\frac{d\cos\theta}{a+b\cos\theta}+B\int\frac{d\cos\theta}{(a+b\cos\theta)^2}+C\int\frac{d\cos\theta}{(a+b\cos\theta)^3}$$

or

$$A'\int\frac{d\theta}{a+b\cos\theta}+B'\int\frac{d\theta}{(a+b\cos\theta)^2}+C'\int\frac{d\theta}{(a+b\cos\theta)^3},$$

according as p is odd or even, and such integrations have been already considered.

204. We may then obtain a reduction formula for

$$I'_n=\int\frac{\sin^p\theta\cos^q\theta}{(a+b\cos\theta)^n}\,d\theta.$$

Let
$$P=\frac{\sin^{p+1}\theta\cos^{q+1}\theta}{(a+b\cos\theta)^{n-1}}.$$

Then

$$\frac{dP}{d\theta}=\frac{\begin{array}{r}[(p+1)\sin^p\theta\cos^{q+2}\theta-(q+1)\sin^{p+2}\theta\cos^q\theta](a+b\cos\theta)\\+(n-1)b\sin^{p+2}\theta\cos^{q+1}\theta\end{array}}{(a+b\cos\theta)^n}$$

$$=\frac{\sin^p\theta\cos^q\theta}{(a+b\cos\theta)^n}\left[\{-(q+1)+(p+q+2)\cos^2\theta\}(a+b\cos\theta)\right.$$
$$\left.+(n-1)b(1-\cos^2\theta)\cos\theta\right]$$

$$=\frac{\sin^p\theta\cos^q\theta}{(a+b\cos\theta)^n}\left[A+B(a+b\cos\theta)+C(a+b\cos\theta)^2\right.$$
$$\left.+D(a+b\cos\theta)^3\right],\text{ say.}$$

$$\left.\begin{aligned} \text{where}\quad A+Ba+Ca^2+Da^3 &= -(q+1)\,a,\\ Bb+2Cab+3Da^2b &= -(q+1)\,b+(n-1)\,b,\\ Cb^2+3Dab^2 &= \ \ (p+q+2)\,a,\\ Db^3 &= \ \ (p+q+2)\,b-(n-1)\,b, \end{aligned}\right\}$$

whence
$$A=(n-1)\,a\frac{a^2-b^2}{b^2},$$
$$B=(n-q-2)-(3n-p-q-5)\frac{a^2}{b^2},$$
$$C=(3n-2p-2q-7)\frac{a}{b^2},$$
$$D=\frac{(p+q-n+3)}{b^2},$$

and the reduction formula is

$$\frac{\sin^{p+1}\theta\cos^{q+1}\theta}{(a+b\cos\theta)^{n-1}}=AI'_n+BI'_{n-1}+CI'_{n-2}+DI'_{n-3},$$

from which I'_n can be expressed in terms of three integrals of the next lower orders and ultimately made to depend upon I'_1, I'_2, I'_3, whose integration has been discussed.

205. **General Conclusion.**

From what has been said in Art. 204, it will now appear that any integral of form

$$\int\frac{f(\sin\theta,\cos\theta)\,d\theta}{(a+b\cos\theta)^n}$$

can be integrated when n is a positive (or negative) integer, and $f(x, y)$ is a rational integral algebraic function of $\sin\theta$, $\cos\theta$; for $f(\sin\theta, \cos\theta)$ is then the sum of a number of terms of type $$A\,.\sin^p\theta\cos^q\theta.$$

206. HERMITE (*Proc. Lond. Math. Soc.* 1872) has shown how to integrate any expression of form

$$\frac{f(\sin\theta,\cos\theta)}{\sin(\theta-a_1)\sin(\theta-a_2)\sin(\theta-a_3)\dots\sin(\theta-a_n)},$$

where $f(x, y)$ is any homogeneous function of x, y of $(n-1)$ dimensions.

For by the ordinary rules of partial fractions,

$$\frac{f(t,1)}{(t-a_1)(t-a_2)\dots(t-a_n)}=\sum_{r=1}^{r=n}\frac{f(a_r,1)}{(a_r-a_1)(a_r-a_2)\dots(a_r-a_n)}\cdot\frac{1}{t-a_r}$$

(the factor a_r-a_r being omitted in the denominator of the above coefficient).

Writing $t=\tan\theta$, $a_1=\tan\alpha_1$, $a_2=\tan\alpha_2$, etc., this becomes

$$\frac{f(\sin\theta,\ \cos\theta)}{\prod_{r=1}^{r=n}\sin(\theta-\alpha_r)}=\sum\frac{A_r}{\sin(\theta-\alpha_r)},$$

where $$A_r=\frac{f(\sin\alpha_r,\ \cos\alpha_r)}{\sin(\alpha_r-\alpha_1)\sin(\alpha_r-\alpha_2)\ldots\sin(\alpha_r-\alpha_n)},$$

the factor $\sin(\alpha_r-\alpha_r)$ being omitted in the denominator.

Thus, $$\int\frac{f(\sin\theta,\ \cos\theta)}{\prod_1^n\sin(\theta-\alpha_r)}\,d\theta=\sum_1^n A_r\log\tan\frac{\theta-\alpha_r}{2}.$$

207. (i) Thus, for example, we have

$$\frac{\sin^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}=\Sigma\frac{\sin^2 a}{\sin(a-b)\sin(a-c)}\,\frac{1}{\sin(x-a)};$$

$$\therefore\int\frac{\sin^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}\,dx=\Sigma\frac{\sin^2 a}{\sin(a-b)\sin(a-c)}\log\tan\frac{x-a}{2}.$$

(ii) Similarly,

$$\int\frac{\cos^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}\,dx=\Sigma\frac{\cos^2 a}{\sin(a-b)\sin(a-c)}\log\tan\frac{x-a}{2}.$$

(iii) Hence adding,

$$\int\frac{dx}{\sin(x-a)\sin(x-b)\sin(x-c)}=\Sigma\frac{1}{\sin(a-b)\sin(a-c)}\log\tan\frac{x-a}{2},$$

(iv) or subtracting,

$$\int\frac{\cos 2x\,dx}{\sin(x-a)\sin(x-b)\sin(x-c)}=\Sigma\frac{\cos 2a}{\sin(a-b)\sin(a-c)}\log\tan\frac{x-a}{2}.$$

(v) It is easy to show that

$$\frac{\sin x}{\sin(x-a)\sin(x-b)\sin(x-c)}=\Sigma\frac{\sin a}{\sin(a-b)\sin(a-c)}\cot(x-a);$$

$$\therefore\int\frac{\sin x}{\sin(x-a)\sin(x-b)\sin(x-c)}\,dx=\Sigma\frac{\sin a}{\sin(a-b)\sin(a-c)}\log\sin(x-a).$$

EXAMPLES.

1. Integrate

(i) $\displaystyle\int \frac{a\sin\theta + b\cos\theta}{c\sin\theta + e\cos\theta}\,d\theta.$ [α, 1883.] (ii) $\displaystyle\int \frac{d\theta}{\cos\theta - \sin\theta}.$ [I. C. S., 1880.]

(iii) $\displaystyle\int \frac{\alpha + \beta\sin\theta}{a + b\cos\theta}\,d\theta.$ [TRIN. H. AND MAGD.]

(iv) $\displaystyle\int \frac{dx}{\cos\alpha + \cos x}.$ [I.C.S., 1889.] (v) $\displaystyle\int \frac{dx}{a\cos x + b\sin x}.$ [COLL., 1876.]

(vi) $\displaystyle\int \frac{1 - \tan\theta}{1 + \tan\theta}\,d\theta.$ [TRIN., 1884.] (vii) $\displaystyle\int \frac{\sec\theta}{1 + \operatorname{cosec}\theta}\,d\theta.$

[OX. I. P., 1889.]

(viii) $\displaystyle\int \frac{dx}{3(1 - \sin x) - \cos x}.$ [α, 1881.]

(ix) $\displaystyle\int \frac{\sqrt{2}\,dx}{2\sqrt{2} + \cos x + \sin x}.$ [OX. I. P., 1888.] (x) $\displaystyle\int \frac{dx}{a + b\tan x}.$

[ST. JOHN'S, 1888.]

(xi) Apply the transformation $t = \tan\frac{1}{2}x$ to the integrals

$$\int \frac{4\,dx}{5 + 3\cos x}, \quad \int \frac{4\,dx}{3 + 5\cos x}.$$

Hence or otherwise, evaluate these integrals to the nearest hundredth, when the limits are $x = 0$ and $\frac{1}{2}\pi$. Prove in any way that the second is the greater of the two integrals, when taken between 0 and $\frac{1}{2}\pi$. [MATH. TRIP. I., 1913.]

(xii) Prove that

$$\int_0^a \frac{dx}{x + (a^2 - x^2)^{\frac{1}{2}}} = \tfrac{1}{4}\pi, \quad \int_0^a \frac{a\,dx}{\{x + (a^2 - x^2)^{\frac{1}{2}}\}^2} = \frac{1}{\sqrt{2}}\log_e(1 + \sqrt{2}),$$

the positive sign being taken for the radical in each of the subjects of integration. [MATH. TRIP. II., 1913.]

2. Evaluate

(i) $\displaystyle\int_0^{\pi} \frac{dx}{4 + 5\sin x}.$ [I. C. S., 1889.] (ii) $\displaystyle\int_0^{\pi} \frac{d\theta}{a + c\cos\theta}$ $(c < a)$.

[I. C. S., 1879.]

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x}.$ [ST. JOHN'S, 1882.] (iv) $\displaystyle\int_0^{\pi} \frac{dx}{1 - 2a\cos x + a^2}.$

[I. C. S., 1888.]

3. Show that $\displaystyle\int_0^{\alpha} \frac{dx}{1 - \cos\alpha\cos x} = \frac{\pi}{2}\operatorname{cosec}\alpha,$ [OX. II. P., 1889; TRIN., 1887.]

and integrate $\displaystyle\int \frac{\cos\alpha\cos x + 1}{\cos\alpha + \cos x}\,dx.$

4. Evaluate (i) $\int \frac{dx}{\alpha^2 - \beta^2 \cos^2 x} \quad (\alpha > \beta)$. [Ox. II. P., 1887.]

(ii) $\int \frac{dx}{\sin x \cos x + \sin x + \cos x - 1}$. [$\gamma$, 1899.]

(iii) $\int \frac{dx}{\sin x + \sin 2x}$. [$\beta$, 1891.]

5. Evaluate $\int \frac{dx}{(e^x + e^{-x})^2}$. [Ox. I. P., 1889.]

6. Integrate (i) $\int \frac{dx}{(4 + 5 \cos x)^2}$. [$\alpha$, 1883.]

(ii) $\int \frac{dx}{(a + b \cos x)^2}$. [St. John's, 1884.]

(iii) $\int \frac{d\theta}{(a \cos \theta + b \sin \theta)^2}$. [$\alpha$, 1881.]

(iv) $\int \frac{dx}{(a + b \cos x + c \sin x)^2}$. [Coll., 1892.]

(v) Employ the substitution

$\tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}$ to evaluate the integral $\int_0^\theta \frac{d\theta}{(1 + e \cos \theta)^2}$.

[Math. Trip. I., 1909.]

7. Prove that

(i) $\int_0^\pi \frac{d\theta}{(1 + \cos \alpha \cos \theta)^2} = \pi \operatorname{cosec}^3 \alpha$, [Coll., 1886; St. John's, 1886.]

(ii) $\int_0^\pi \frac{\sin x \, dx}{(1 + \cos \alpha \sin x)^2} = \frac{2(\sin \alpha - \alpha \cos \alpha)}{\sin^3 \alpha}$, [$\beta$, 1887.]

and evaluate

$$\int_0^\pi \frac{x \sin x \, dx}{(1 + \cos \alpha \cos x)^3} \text{ and } \int_0^\pi \frac{x \sin x \, dx}{(1 + \cos \alpha \cos x)^4},$$

if α be less than $\frac{\pi}{2}$.

8. Evaluate the integral

$$\int \cos 2\theta \log(1 + \tan \theta) d\theta.$$

[γ, 1882.]

9. Find the values of the following integrals: $\left(\alpha < \frac{\pi}{2}\right)$

(i) $\int_0^\infty \frac{dx}{2 \cos \alpha + e^x + e^{-x}}$, (ii) $\int_0^\infty \frac{dx}{2 + \cos \alpha (e^x + e^{-x})}$.

[Trin., 1882.

10. Evaluate (i) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}$. [Ox. I. P., 1889.]

(ii) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{4+5\sin^2\theta}$. [I. C. S., 1885.]

(iii) $\int_0^{\frac{\pi}{2}} \frac{a\sin^2\theta+b\cos^2\theta}{c\sin^2\theta+d\cos^2\theta}\,d\theta$ (c and d positive). [St. John's.]

(iv) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)^2}$. [Ox. II. P., 1887.]

(v) $\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta\, d\theta}{\sin^4\theta+\cos^4\theta}$, [I. C. S., 1891.]

and (vi) Shew that if $c > a > 0$,

$$\int_0^a \frac{\sqrt{a^2-x^2}}{c^2-x^2}\,dx = \pi(c-\sqrt{c^2-a^2})/2c.$$

[Math. Trip. I., 1908.]

11. Prove that $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$,

where $a > b$, and deduce or otherwise obtain the value of

$$\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^3}.$$

[γ, 1899.]

12. Prove that if $a > b$,

$$\int_0^{\pi} \frac{dx}{(a+b\cos x)^4} = \frac{\pi}{2}\left\{\frac{5a^3}{(a^2-b^2)^{\frac{7}{2}}} - \frac{3a}{(a^2-b^2)^{\frac{5}{2}}}\right\}.$$

[γ, 1888.]

Evaluate $\int_0^{\pi} \frac{dx}{(1+e\cos x)^4}$, where $e < 1$. [St. John's, 1892.]

13. Evaluate $\int \frac{dx}{a+be^x+ce^{-x}}$, where $a^2 < 4bc$. [I. C. S., 1897.]

14. Prove that if $\alpha < \frac{\pi}{6}$,

$$\int_0^{\alpha} \frac{\sin x}{\cos 3x}\,dx = \frac{1}{6}\log\frac{\cos^3\alpha}{\cos 3\alpha}$$

[C. S., 1896.]

15. Prove that $\int_0^{2\pi} \frac{d\phi}{a+b\cos\phi+c\sin\phi} = \frac{2\pi}{\sqrt{a^2-r^2}}$,

where $r^2 = b^2+c^2$ and $r < a$. [C. S., 1900.]

16. Integrate

(i) $\int \frac{\sqrt{\tan x}}{\sin x\cos x}\,dx$. (ii) $\int \frac{\sec x\,dx}{a+b\tan x}$. (iii) $\int \frac{\sec^2 x\,dx}{a+b\tan x}$.

17. Integrate

(i) $\displaystyle\int \frac{d\theta}{15\sin^2\theta - 16\cos\theta}$. (ii) $\displaystyle\int_0^\pi \frac{x}{1+\sin x}dx$.

(iii) $\displaystyle\int \frac{\cot\theta - 3\cot 3\theta}{3\tan 3\theta - \tan\theta}d\theta$. [Ox. I. P., 1888.]

(iv) $\displaystyle\int \frac{\sin 2x\,dx}{(a+b\cos x)^2}$. [$\alpha$, 1889.]

18. Integrate

(i) $\displaystyle\int \cos 2\theta \log \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}d\theta$. (ii) $\displaystyle\int \frac{\sin\theta - \cos\theta}{\sqrt{\sin 2\theta}}d\theta$.

(iii) $\displaystyle\int \sqrt{\frac{1-\cos\theta}{\cos\theta(1+\cos\theta)(2+\cos\theta)}}\,d\theta$.

19. Integrate $\displaystyle\int \sqrt{\frac{1+\sin x}{1-\sin x}\cdot\frac{2+\sin x}{2-\sin x}}\,dx$.

20. Integrate

$$\int \frac{\sin\theta - \cos\theta}{(\sin\theta + \cos\theta)\sqrt{\sin\theta\cos\theta + \sin^2\theta\cos^2\theta}}d\theta.$$

21. Integrate $\displaystyle\int \frac{\sin^3\theta\,d\theta}{(1+\cos^2\theta)\sqrt{1+\cos^2\theta+\cos^4\theta}}$.

22. Integrate

(i) $\displaystyle\int \sqrt{1+\sin x}\,dx$. (ii) $\displaystyle\int \frac{\sin x}{\sqrt{1+\sin x}}dx$. (iii) $\displaystyle\int \frac{\tan x}{\sqrt{a+b\tan^2 x}}dx$.

23. Integrate $\displaystyle\int \frac{\sinh x \sin x - \cosh x}{1-\cos x}dx$.

24. Integrate $\displaystyle\int \frac{x^2\,dx}{(x\sin x + \cos x)^2}$.

25. Integrate $\displaystyle\int \frac{\sec x \operatorname{cosec} x}{\log\tan x}dx$.

26. Integrate

(i) $\displaystyle\int \sin^{-1}\frac{2x}{1+x^2}dx$. (ii) $\displaystyle\int \tan^{-1}\frac{3x-x^3}{1-3x^2}dx$.

(iii) $\displaystyle\int \tan^{-1}\frac{-1+\sqrt{1+x^2}}{x}dx$.

27. Integrate $\displaystyle\int \frac{\sqrt{1+x^2}}{1-x^2}dx$.

28. Integrate $\int \frac{\sin x}{\sin 2x}dx$, $\int \frac{\sin x}{\sin 3x}dx$, $\int \frac{\sin x}{\sin 4x}dx$,

and prove that

$$5\int \frac{\sin x}{\sin 5x}dx = \sin\frac{2\pi}{5}\log\left\{\frac{\sin\left(x-\frac{2\pi}{5}\right)}{\sin\left(x+\frac{2\pi}{5}\right)}\right\} - \sin\frac{\pi}{5}\log\left\{\frac{\sin\left(x-\frac{\pi}{5}\right)}{\sin\left(x+\frac{\pi}{5}\right)}\right\}.$$

[Trin. Coll., 1892.]

29. Integrate $\int \frac{d\theta}{\sin^2\theta - \sin^2\alpha}$ and $\int \frac{\cos\theta}{\sin^2\theta - \sin^2\alpha}d\theta$.

Show that $\frac{\sin\theta}{\sin n\theta}$ can be expressed in partial fractions of type

$$\frac{1}{\sin^2\theta - \sin^2\alpha} \text{ or } \frac{\cos\theta}{\sin^2\theta - \sin^2\alpha},$$

according as n is an odd or an even integer and can thereby be integrated.

30. Integrate

$$\text{(i)} \int_0^{\frac{\pi}{2}} \frac{\sin 3x\,dx}{(a+b\cos x)^2}. \qquad \text{(ii)} \int_0^{\frac{\pi}{2}} \frac{\sin^3 x\,dx}{a+b\cos x}.$$

$$\text{(iii)} \int_0^{\frac{\pi}{2}} \frac{\sin^3 x\,dx}{(a+b\cos x)^2}. \qquad \text{(iv)} \int_0^{\frac{\pi}{2}} \frac{\sin 3x\,dx}{(a+b\cos x)^n} \quad (n>3).$$

31. Show how to effect the integration of

$$\int \frac{\cos^p x}{\sin 2nx}dx, \quad \int \frac{\cos^p x}{\cos nx}dx,$$

p and n being integers. [ϵ, 1883, and Coll., 1879.]

32. Integrate $\int \cot(x-\alpha)\cot(x-\beta)\,dx$, [$\gamma$, 1891.]

and show that

$$\int \cot(x-a)\cot(x-b)\cot(x-c)\,dx = \Sigma\cot(a-b)\cot(a-c)\log\sin(x-a).$$

[Trinity, 1891.]

33. Show that

$$\int \sin x \sec(x-\alpha)\sec(x-\beta)\,dx$$
$$= \frac{1}{\sin(\beta-\alpha)}\left[\cos\alpha\cosh^{-1}\sec(x-\alpha) - \cos\beta\cosh^{-1}\sec(x-\beta)\right].$$

[Trinity, 1889.]

34. Prove that

$$\int_0^\beta x\sec x\sec(\beta - x)\,dx = \beta\,\mathrm{cosec}\,\beta\log\sec\beta.$$

[Oxf. II. P., 1901.]

35. Prove that, if α and β be positive quantities,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(\alpha\cos^2 x+\beta\sin^2 x)^{n+1}}=\frac{\pi}{2}\frac{(-1)^n}{n!}\left(\frac{d}{d\alpha}+\frac{d}{d\beta}\right)^n(\alpha\beta)^{-\frac{1}{2}}.$$

[α, 1884.]

36. Prove that, if

$$f(x)=(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$

and
$$P=\prod_{r=1}^{r=n}(1-2\alpha_r\cos\theta+\alpha_r{}^2),$$

where $\alpha_1,\ \alpha_2,\ \dots\ \alpha_n$ denote real quantities, then

$$\int_0^\pi \frac{d\theta}{P}=\pi\sum_1^n \epsilon_r/A_r,$$

where $A_r=\alpha_r f(1/\alpha_r)f'(\alpha_r)$, and $\epsilon=-1$, or $+1$, according as α_r is numerically greater or less than 1. [ST. JOHN'S, 1886.]

37. If c be less than $a\sin\theta$, show that the coefficient of c^m in the expansion of

$$\frac{2c}{\sqrt{c^2-a^2}}\tan^{-1}\left\{\sqrt{\frac{c-a}{c+a}}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

is
$$(-1)^m\frac{1}{a^m}\int\frac{d\theta}{\sin^m\theta}+A_m,$$

where A_m is independent of θ. [COLL., 1892.]

38. Show that if n be a positive integer,

$$\int\frac{\cos n\theta-\cos n\alpha}{\cos\theta-\cos\alpha}d\theta=\theta\frac{\sin n\alpha}{\sin\alpha}+\frac{2}{\sin\alpha}\sum_{r=1}^{r=n-1}\frac{\sin r\theta}{r}\sin(n-r)\alpha.$$

[HERMITE.]

39. Prove that

$$\int_0^x(1+\cos x)^n dx=\frac{1}{2^n}\cdot{}^{2n}C_n x+\frac{1}{2^{n-1}}\sum_1^n{}^{2n}C_{n-r}\frac{\sin rx}{r}.$$

40. Show that

$$\int\prod_1^n\cot(\theta-\alpha_r)d\theta=\theta\cos\frac{n\pi}{2}+\sum_1^n A_r\log\sin(\theta-\alpha_r),$$

where
$$A_r=\cot(\alpha_r-\alpha_1)\cot(\alpha_r-\alpha_2)\dots\cot(\alpha_r-\alpha_n),$$

the factor $\cot(\alpha_r-\alpha_r)$ being omitted. [HERMITE.]

41. (1) Show that

$$\int\frac{dx}{a-\sin x}=\frac{2}{\sqrt{a^2-1}}\tan^{-1}\left(\frac{a\tan\frac{x}{2}-1}{\sqrt{a^2-1}}\right)\quad(a>1).$$

(2) Differentiate with regard to x,

$$\tan^{-1}\sqrt{\frac{a+1}{a-1}\cdot\frac{1-\sin x}{1+\sin x}}.$$

Deduce from (1) and (2) that

$$\tan^{-1}\sqrt{\frac{a+1}{a-1}\cdot\frac{1-\sin x}{1+\sin x}}+\tan^{-1}\frac{a\tan\frac{x}{2}-1}{\sqrt{a^2-1}}$$

is independent of x, and verify your conclusion. [C. S., 1898.]

42. Integrate $$\int\frac{dx}{1+ab-a\cos x-b\sec x},$$

where $a<1$, $b>1$, $ab\neq 1$. [Oxf. I. P., 1917.]

43. Integrate (i) $e^x(x\cos x+\sin x)$. (ii) $(x^2+1)^{-\frac{5}{2}}$.

(iii) $(x^4+2x^3+2x+1)/(x^2+x+1)^3$. [Oxf. I. P., 1918.]

44. Integrate $$\int\frac{\cos^3 x\,dx}{\sin(x-\alpha)\sin(x-\beta)\sin(x-\gamma)}.$$

45. Deduce from the identity

$$\int_0^{\frac{\pi}{2}}\cos n\theta\,d\theta=\int_0^{\frac{\pi}{2}}d\theta\left\{1-\frac{n^2}{2!}\sin^2\theta-\frac{n^2(2^2-n^2)}{4!}\sin^4\theta-\frac{n^2(2^2-n^2)(4^2-n^2)}{6!}\sin^6\theta-\dots\right\}$$

the expression for $\sin x$ as an infinite product. [Oxf. II. P., 1887.]

46. Evaluate the integrals

(i) $\int(x+a\log x)^2\,dx$. (ii) $\int\frac{(a+x)\,dx}{x^2+ax\log x}$.

(iii) $\int\frac{(1-\log x)\,dx}{(x+a\log x)^2}$. [Math. Tripos, 1885.]

47. Show that

$$\int_0^x\frac{x^2+a(a-1)}{(x\sin x+a\cos x)^2}\,dx=\frac{a\sin x-x\cos x}{x\sin x+a\cos x}.$$

[Trin. Coll., 1891.]

48. Evaluate the indefinite integrals

(i) $$\int\frac{(\sin x+\cos x)^2}{\{(x-1)\cos x-(x+1)\sin x\}^2}\,dx.$$

(ii) $$\int\frac{x^2}{\{(x-1)\cos x-(x+1)\sin x\}^2}\,dx.$$

[Colleges, 1886.]

49. If $T\equiv x+\sqrt{1+x^2}$, show that

$$\int T^r dx=\tfrac{1}{2}\left\{\frac{T^{r-1}}{r-1}+\frac{T^{r+1}}{r+1}\right\}.$$

50. Integrate (i) $\int e^x \frac{x^3 - x + 2}{(1+x^2)^2} dx.$ [Ox. II. P., 1899.]

(ii) $\int \frac{\log(\cos\theta + \sqrt{\cos 2\theta})}{1 - \cos^2\theta} d\theta.$ [Coll. α, 1891.]

51. Prove that, if n be an integer,

$$\int_0^\pi \frac{\cos nx}{1 + \cos\alpha\cos x} dx = \pi \operatorname{cosec}\alpha(\tan\alpha - \sec\alpha)^n,$$

and deduce the value of

$$\int_0^\pi \frac{\cos nx}{(1+\cos\alpha\cos x)^2} dx.$$ [Colleges γ, 1891.]

52. From considering the integral

$$\int_0^\pi \frac{\cos n\theta}{a - \cos\theta} d\theta,$$

show that

$$1 + \frac{n+2}{1}\frac{\cos^2\alpha}{2^2} + \frac{(n+3)(n+4)}{1.2}\frac{\cos^4\alpha}{2^4} + \dots$$
$$= 2^n(\sec\alpha - \tan\alpha)^n \sec^n\alpha \operatorname{cosec}\alpha.$$

53. Prove that, if $0 < \alpha < \frac{\pi}{2}$ and n be a positive integer,

$$\int_0^\pi \sin n\phi \tan^{-1}\left(\frac{\tan\frac{\phi}{2}}{\tan\frac{\alpha}{2}}\right) d\phi = \frac{\pi}{2n}[(\sec\alpha - \tan\alpha)^n - (-1)^n].$$

54. Show that

$$\int \frac{\sin n\theta}{\sin\theta} \frac{d\theta}{\cos n\theta - \cos n\alpha} = \frac{1}{n}\sum_{r=0}^{r=n-1} \operatorname{cosec}\beta_r \log\frac{\sin\frac{1}{2}(\beta_r + \theta)}{\sin\frac{1}{2}(\beta_r - \theta)};$$

where $\beta_r = \alpha + \frac{2r\pi}{n}$.

55. Discuss the integration of

$$(a)\ \int \frac{\sin p\theta}{\sin q\theta} d\theta, \quad (b)\ \int \sin\theta \frac{\sin p\theta}{\sin q\theta} d\theta,$$

where p and q are positive integers.

56. With the help of the substitution $x^{-1} = \sqrt{t^2 - 1}$, or otherwise prove that

$$\int_0^\infty \frac{dx}{(9 + 25x^2)\sqrt{1+x^2}} = \frac{1}{12}\tan^{-1}\frac{4}{3}.$$

[Math. Trip., Pt. II., 1920.]