CHAPTER V.

RATIONAL ALGEBRAIC FRACTIONAL FORMS.

127. Integration of

$$\frac{1}{a^2-x^2} \quad (x<a) \quad \text{and} \quad \frac{1}{x^2-a^2} \quad (x>a).$$

Either of these forms should be thrown into Partial Fractions, which can be done by inspection.

$$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx$$

$$=\frac{1}{2a}[\log(a+x)-\log(a-x)]=\frac{1}{2a}\log\frac{a+x}{a-x}$$

or $$=\frac{1}{a}\tanh^{-1}\frac{x}{a} \quad (x<a).$$

$$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx$$

$$=\frac{1}{2a}[\log(x-a)-\log(x+a)]=\frac{1}{2a}\log\frac{x-a}{x+a}$$

or $$=-\frac{1}{a}\coth^{-1}\frac{x}{a} \quad \text{or} \quad \frac{-1}{a}\tanh^{-1}\frac{a}{x} \quad (x>a).$$

The Partial Fractions are so simple that the results are not usually committed to memory.

128. These inverse hyperbolic forms should be compared with

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}=\frac{1}{a}\cos^{-1}\frac{a}{\sqrt{a^2+x^2}}=\frac{1}{a}\sec^{-1}\frac{\sqrt{a^2+x^2}}{a}.$$

The three results are:

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \quad \text{or} \quad -\frac{1}{a}\cot^{-1}\frac{x}{a}.$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} \quad (x<a),$$

$$\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\coth^{-1}\frac{x}{a} \quad (x>a),$$

or

$$-\frac{1}{a}\tanh^{-1}\frac{a}{x}.$$

129. Extension of above rule.

In the same way, α and β being real,

$$\int \frac{dx}{\beta^2+(x+\alpha)^2} = \frac{1}{\beta}\tan^{-1}\frac{x+\alpha}{\beta},$$

$$\int \frac{dx}{\beta^2-(x+\alpha)^2} = \frac{1}{\beta}\tanh^{-1}\frac{x+\alpha}{\beta}, \quad \textit{i.e.}\ \frac{1}{2\beta}\log\frac{\beta+(x+\alpha)}{\beta-(x+\alpha)} \quad (x+\alpha<\beta),$$

$$\int \frac{dx}{(x+\alpha)^2-\beta^2} = -\frac{1}{\beta}\coth^{-1}\frac{x+\alpha}{\beta}$$

or

$$-\frac{1}{\beta}\tanh^{-1}\frac{\beta}{x+\alpha}, \quad \textit{i.e.}\ \frac{1}{2\beta}\log\frac{x+\alpha-\beta}{x+\alpha+\beta} \quad (x+\alpha>\beta).$$

130. Integration of

$$I \equiv \int \frac{dx}{ax^2+bx+c}.$$

Since ax^2+bx+c can always be written as

$$a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right], \quad \textit{i.e.}\ \text{of form } a\left[(x+\alpha)^2+\beta^2\right],$$

or as

$$a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right], \quad \textit{i.e.}\ \text{of form } a\left[(x+\alpha)^2-\beta^2\right];$$

taking the first or the second according as $b^2<4ac$ or $b^2>4ac$ the rules of the former article apply.

Thus

131. CASE I. $b^2 < 4ac.$

$$I \equiv \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}}$$

$$= \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}}$$

or $$= -\frac{2}{\sqrt{4ac-b^2}} \cot^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}}$$

or $$= \frac{2}{\sqrt{4ac-b^2}} \sec^{-1} \frac{\sqrt{4a}\sqrt{ax^2+bx+c}}{\sqrt{4ac-b^2}}, \text{ etc.}$$

132. CASE II. $b^2 > 4ac.$

$$I \equiv \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}}$$

$$= \frac{1}{\sqrt{b^2-4ac}} \log \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}$$

or $$= -\frac{2}{\sqrt{b^2-4ac}} \coth^{-1} \frac{2ax+b}{\sqrt{b^2-4ac}}$$

or $$= -\frac{2}{\sqrt{b^2-4ac}} \operatorname{cosech}^{-1} \frac{\sqrt{4a}\sqrt{ax^2+bx+c}}{\sqrt{b^2-4ac}},$$

which is a real form if $2ax+b > \sqrt{b^2-4ac}$,

or $$I \equiv -\frac{1}{a}\int \frac{dx}{\frac{b^2-4ac}{4a^2} - \left(x+\frac{b}{2a}\right)^2}$$

$$= -\frac{1}{\sqrt{b^2-4ac}} \log \frac{\sqrt{b^2-4ac}+(2ax+b)}{\sqrt{b^2-4ac}-(2ax+b)},$$

$$\text{i.e. } -\frac{2}{\sqrt{b^2-4ac}} \tanh^{-1} \frac{2ax+b}{\sqrt{b^2-4ac}} = \text{etc.},$$

which is a real form if $2ax+b < \sqrt{b^2-4ac}$.

133. Of these several forms the real one is to be chosen in each numerical case. The general forms are equivalent, except that they differ by a constant which may be unreal.

134. Another Method.

As the factors in the second case are real, say

$$a(x-x_1)(x-x_2),$$

the usual proceeding is to write the work as follows without the formal completing of the square in the denominator:

$$\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dx}{(x-x_1)(x-x_2)}$$

$$=\frac{1}{a(x_1-x_2)}\int\frac{dx}{x-x_1}+\frac{1}{a(x_2-x_1)}\int\frac{dx}{x-x_2}$$

$$=\frac{1}{a(x_1-x_2)}\log(x-x_1)+\frac{1}{a(x_2-x_1)}\log(x-x_2)$$

$$=\frac{1}{a(x_1-x_2)}\log\frac{x-x_1}{x-x_2}.$$

135. Other forms of the above results.

Other forms of these results may be exhibited. For instance, taking $R\equiv ax^2+bx+c$, and $4ac-b^2=4a^2\kappa^2=-4a^2\kappa'^2$; then

$$2\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}=\sin^{-1}\left(\kappa\frac{2ax+b}{ax^2+bx+c}\right)\qquad=\sin^{-1}\left(\kappa\frac{d}{dx}\log R\right)$$

and

$$2\tanh^{-1}\frac{2ax+b}{\sqrt{b^2-4ac}}=\sinh^{-1}\left(-\kappa'\frac{2ax+b}{ax^2+bx+c}\right)=-\sinh^{-1}\left(\kappa'\frac{d}{dx}\log R\right)$$

whence $I=\dfrac{1}{2a\kappa}\sin^{-1}\left(\kappa\dfrac{d}{dx}\log R\right)$ or $\dfrac{1}{2a\kappa'}\sinh^{-1}\left(\kappa'\dfrac{d}{dx}\log R\right)$

the real form to be chosen.

136. Integrals of expressions of the form

$$\frac{px+q}{ax^2+bx+c},\quad i.e.\ \frac{px+q}{R},$$

can be obtained at once by throwing $px+q$ into the form

$$px+q\equiv\lambda R'+\mu,\quad i.e. \equiv\lambda(2ax+b)+\mu,$$

where λ, μ are constants to be found;

for then

$$I=\int\frac{px+q}{ax^2+bx+c}\,dx=\int\frac{\lambda R'+\mu}{R}\,dx=\lambda\int\frac{R'}{R}\,dx+\mu\int\frac{dx}{R}$$

$$=\lambda\log R+\mu\int\frac{dx}{R},$$

and the second member of the right side has been discussed.

137. This transformation is one very frequently required. It may be performed either by inspection, or by comparing coefficients.

(i) By inspection,

$$px+q\equiv\frac{p}{2a}(2ax+b)+\left(q-\frac{pb}{2a}\right).$$

(ii) By comparing coefficients,

$$\left.\begin{aligned}2a\lambda&=p,\\ b\lambda+\mu&=q,\end{aligned}\right\}\text{ giving }\lambda=\frac{p}{2a}\text{ and }\mu=q-\frac{pb}{2a}.$$

Thus

$$\int\frac{px+q}{ax^2+bx+c}\,dx=\frac{p}{2a}\int\frac{2ax+b}{ax^2+bx+c}\,dx+\left(q-\frac{pb}{2a}\right)\int\frac{dx}{ax^2+bx+c}$$

$$=\frac{p}{2a}\log(ax^2+bx+c)+\left(q-\frac{pb}{2a}\right)\int\frac{dx}{ax^2+bx+c}.$$

It is *essential* that the numerator of the first partial fraction shall be the *differential coefficient of the denominator*, and that the *x's of the numerator of the given fraction are thereby exhausted.*

138. Ex. 1. $$\int\frac{9-7x}{x^2+12x+38}\,dx=\int\frac{51-\frac{7}{2}(2x+12)}{(x+6)^2+2}\,dx$$

$$=51\int\frac{dx}{2+(x+6)^2}-\frac{7}{2}\int\frac{2x+12}{x^2+12x+38}\,dx$$

$$=\frac{51}{\sqrt{2}}\tan^{-1}\frac{x+6}{\sqrt{2}}-\frac{7}{2}\log(x^2+12x+38).$$

Ex. 2. $$\int\frac{9-7x}{35+2x-x^2}\,dx=\int\frac{9-7x}{(7-x)(5+x)}\,dx$$

$$=\int\left(-\frac{10}{3}\,\frac{1}{7-x}+\frac{11}{3}\,\frac{1}{5+x}\right)dx$$

$$=\frac{10}{3}\log(7-x)+\frac{11}{3}\log(5+x).$$

This difference is to be noted in such examples as the two preceding; in the first the form of the result is real for all real values of x; in the second the form given is only real if x lies between -5 and $+7$. For values of $x > 7$ we should write it

$$\frac{10}{3}\log(x-7)+\frac{11}{3}\log(x+5),$$

and for values of $x < -5$,

$$\frac{10}{3}\log(7-x)+\frac{11}{3}\log(-5-x).$$

These three forms differ by unreal constants.

EXAMPLES.

1. $\int\frac{x\,dx}{x^2+2x+3}$.
2. $\int\frac{x\,dx}{x^2+2x+1}$.
3. $\int\frac{x+1}{x^2+4x+5}dx$.
4. $\int\frac{(x+1)\,dx}{3+2x-x^2}$.
5. $\int\frac{(x-1)^2\,dx}{x^2+2x+2}$.
6. $\int\frac{2x^2+3x+4}{x^2+6x+10}dx$.
7. $\int\frac{dx}{(ax+b)^2+(cx+d)^2}$.
8. $\int\frac{dx}{(ax+b)^2-(cx+d)^2}$.
9. $\int\frac{x\,dx}{(ax^2+b)^2+(cx^2+d)^2}$.
10. $\int\frac{x\,dx}{(ax^2+b)^2+(cx^2+d)^2+(ex^2+f)^2}$.
11. $\int\frac{dx}{x\left(ax+\frac{b}{x}\right)\left(cx+\frac{d}{x}\right)}$.
12. $\int\frac{e^{2x}\,dx}{e^{2x}+2e^x+3}$.

NOTE ON PARTIAL FRACTIONS.

139. In the author's *Differential Calculus* (p. 72) a Note was inserted on the methods to be pursued in the case of finding the n^{th} Differential Coefficient of an algebraical fraction when it was necessary to resolve the fraction into its simple or partial fractions. It is now necessary to repeat this Note, with some additions and alterations, as success in the integration of complicated rational algebraic fractions will depend upon the ability of the student to obtain the equivalent partial fractions with facility. Moreover, many subsequent articles will depend upon the general theory.

140. Let $\frac{f(x)}{\phi(x)}$ be the fraction in its lowest terms which is to be resolved into its simple component or partial fractions, $f(x)$ and $\phi(x)$ being supposed rational integral algebraic

functions of x, the coefficients being real and, unless the contrary be stated, rational.

Then if the degree of $f(x)$ be not already less than the degree of $\phi(x)$, we can, by ordinary division, express $\frac{f(x)}{\phi(x)}$ in the form

$$a_0x^n+a_1x^{n-1}+\dots+a_n+\frac{\chi(x)}{\phi(x)},$$

where $a_0x^n+a_1x^{n-1}+\dots+a_n$ is the quotient, and $\chi(x)$ is the remainder, of lower degree than $\phi(x)$.

Hence the integration of

$$\int\frac{f(x)}{\phi(x)}dx \text{ is } \frac{a_0x^{n+1}}{n+1}+a_1\frac{x^n}{n}+\dots+a_nx+\int\frac{\chi(x)}{\phi(x)}dx,$$

and we only have to attend to $\int\frac{\chi(x)}{\phi(x)}dx$.

Hence we may confine our attention to the case when $f(x)$ is of lower degree than $\phi(x)$.

Also we may, without loss of generality, consider the coefficient of the highest power of x in $\phi(x)$ to be unity.

141. It is proved in Theory of Equations that if $\phi(x)=0$ be a rational algebraical equation of degree n,

(1) there are n roots, real or imaginary,

(2) that imaginary roots occur in pairs, $\alpha\pm\iota\beta$, $\gamma\pm\iota\delta$, etc.

Any of these roots may be repeated.

Then the general form of ϕ is of the nature

$$\phi\equiv(x-a)(x-b)^p\{(x-\alpha)^2+\beta^2\}\{(x-\gamma)^2+\delta^2\}^q,$$

where we have taken the case of

(1) a real linear factor *occurring once only;*

(2) a real linear factor *occurring p times;*

(3) a *pair of unreal factors, each occurring once;*

(4) a *pair of unreal factors, each occurring q times.*

Any other factors which there may be in ϕ must be of one or other of these categories.

We consider these four cases separately.

And as we are going to suppose that $\frac{f(x)}{\phi(x)}$ is a fraction in its lowest terms, none of the factors described above will be factors of $f(x)$ also.

142. I. To obtain the partial fraction corresponding to the factor $x-a$ occurring once only.

Let $\phi(x) \equiv (x-a)\psi(x)$ for short. Then $\psi(x)$ does not contain $x-a$ as a factor, and $\psi(a)$ does not vanish.

Let $\dfrac{f(x)}{(x-a)\psi(x)} = \dfrac{A}{x-a} + \dfrac{\chi(x)}{\psi(x)}$, an assumption justifiable if we succeed in finding A, supposed independent of x.

Then $\dfrac{f(x)}{\psi(x)} = A + \dfrac{\chi(x)}{\psi(x)}(x-a)$ is an identity and true for all values of x.

Hence putting $x=a$, $\dfrac{f(a)}{\psi(a)} = A$.

Therefore
$$\frac{f(x)}{(x-a)\psi(x)} = \frac{f(a)}{(x-a)\psi(a)} + \frac{\chi(x)}{\psi(x)}.$$

Hence our rule to find A is,

"*Write a for x in every portion of the fraction* $\dfrac{f(x)}{(x-a)\psi(x)}$ *except in the factor* $(x-a)$ *itself.*"

And this process may be applied to every partial fraction corresponding to a factor of $\phi(x)$, *which only occurs once.*

Moreover, since

$$\phi(x) = (x-a)\psi(x), \quad \phi'(x) = (x-a)\psi'(x) + \psi(x),$$

and $\psi'(a)$ is finite, $\therefore \ \phi'(a) = \psi(a)$.

Hence we may also write A in the form $\dfrac{f(a)}{\phi'(a)}$.

143. Ex. 1. $\dfrac{x}{(x-1)(x-2)(x-3)}$

$$= \frac{1}{(x-1)(1-2)(1-3)} + \frac{2}{(2-1)(x-2)(2-3)} + \frac{3}{(3-1)(3-2)(x-3)}$$

$$= \frac{1}{2(x-1)} - \frac{2}{x-2} + \frac{3}{2(x-3)}.$$

Thus, here, three partial fractions must occur. No others can occur. For if there were a fourth fraction $\dfrac{D}{x-\delta}$, say, the denominator of their sum must be $(x-1)(x-2)(x-3)(x-\delta)$, which is not so.

Hence we have obtained the whole expression.

Ex. 2. $\dfrac{x^3}{(x-a)(x-b)}$. Here the numerator not being of lower degree than the denominator, we must divide by the denominator. The result will then be expressible as

$$\frac{x^3}{(x-a)(x-b)}=x+(a+b)+\frac{A}{x-a}+\frac{B}{x-b},$$

where A and B are to be found.

Since $\dfrac{x^3}{x-b}\equiv(x-a)[x+a+b]+A+\dfrac{B}{x-b}(x-a)$, putting $x=a$ we get $A=\dfrac{a^3}{a-b}$, and similarly $B=\dfrac{b^3}{b-a}$.

We may here stop to remark that A and B can be written down by the rule "Put $x=a$ everywhere except in $x-a$ itself" just as well *in the original expression* $\dfrac{x^3}{(x-a)(x-b)}$ as in $\dfrac{x^3}{(x-a)(x-b)}-(x+a+b)$.

This remark is general, and will usually save much trouble.

Thus $$\frac{x^3}{(x-a)(x-b)}\equiv x+(a+b)+\frac{a^3}{a-b}\,\frac{1}{x-a}+\frac{b^3}{b-a}\cdot\frac{1}{x-b}.$$

Ex. 3. Let the roots of $x^n=1$ be $\alpha, \beta, \gamma, \dots$ and $F(x)$ a rational integral algebraic expression of degree lower than n; then, by the second rule of Art. 142,

$$\frac{F(x)}{x^n-1}=\frac{F(\alpha)}{n\alpha^{n-1}}\,\frac{1}{x-\alpha}+\frac{F(\beta)}{n\beta^{n-1}}\,\frac{1}{x-\beta}+\dots$$
$$=\frac{1}{n}\left(\frac{\alpha F(\alpha)}{x-\alpha}+\frac{\beta F(\beta)}{x-\beta}+\dots\right)=\frac{1}{n}\,\Sigma\frac{\alpha F(\alpha)}{x-\alpha},$$

where the summation is for all the roots.

This may be also further expressed as

$$\frac{1}{2n}\Sigma\frac{(x+\alpha)-(x-\alpha)}{x-\alpha}F(\alpha),$$

or $$\frac{1}{2n}\Sigma F(\alpha)\frac{x+\alpha}{x-\alpha}-\frac{1}{2n}\Sigma F(\alpha).$$

If $F(x)$ be written as $Ax^m+Bx^{m-1}+\dots+K\,(m<n)$, then, since the sum of the r^{th} powers of the n^{th} roots of unity is zero when $0<r<n$, we have

$$\Sigma F(\alpha)=nK=nF(0);$$
$$\therefore\ \frac{F(x)}{x^n-1}=\frac{1}{n}\,\Sigma\frac{\alpha F(\alpha)}{x-\alpha}=\frac{1}{2n}\,\Sigma\frac{x+\alpha}{x-\alpha}F(\alpha)-\frac{1}{2}F(0).$$

By taking $F(x)=x$ and putting $x=e^{2\iota\theta}$, deduce that

$$\frac{\sin(n-2)x}{\sin nx}=-\frac{1}{n}\sum_{r=1}^{r=(n-1)}\sin\frac{2r\pi}{n}\cot\left(x-\frac{r\pi}{n}\right).$$

[MATH. TRIP., PART II., 1919.]

144. II. Next suppose **the factor $(x-a)$ in the denominator to be repeated r times** and no more, so that we may write

$$\phi(x)=(x-a)^r\psi(x) \text{ where } \psi(a) \text{ does not vanish.}$$

Put $x-a=y$.

Then $\dfrac{f(x)}{\phi(x)}=\dfrac{1}{y^r}\cdot\dfrac{f(a+y)}{\psi(a+y)}$, or expanding each function by any means in *ascending powers of y*,

$$=\frac{1}{y^r}\frac{A_0+A_1y+A_2y^2+\dots}{B_0+B_1y+B_2y^2+\dots}.$$

Divide out thus:

$$B_0+B_1y+B_2y^2+\dots)\,A_0+A_1y+A_2y^2+\dots(\,C_0+C_1y+C_2y^2+\dots$$

etc.,

and let the division be *continued until y^r is a factor of the remainder.*

Let the remainder be $y^r\chi(y)$.

Hence

$$\frac{f(x)}{\phi(x)}=\frac{C_0}{y^r}+\frac{C_1}{y^{r-1}}+\frac{C_2}{y^{r-2}}+\dots+\frac{C_{r-1}}{y}+\frac{\chi(y)}{\psi(a+y)}$$

$$=\frac{C_0}{(x-a)^r}+\frac{C_1}{(x-a)^{r-1}}+\frac{C_2}{(x-a)^{r-2}}+\dots+\frac{C_{r-1}}{x-a}+\frac{\chi(x-a)}{\psi(x)}.$$

Hence the partial fractions corresponding to $(x-a)^r$ are determined by a "long division" sum.

145. Ex. (i). Take $\dfrac{x^2}{(x-1)^3(x+1)}$. Put $x-1=y$.

Then the fraction $=\dfrac{1}{y^3}\cdot\dfrac{(1+y)^2}{2+y}$.

$$\begin{array}{l} 2+y\,)\,1+2y+y^2\,(\,\tfrac{1}{2}+\tfrac{3}{4}y+\tfrac{1}{8}y^2-\tfrac{1}{8}\dfrac{y^3}{2+y} \\ \qquad\underline{1+\tfrac{1}{2}y} \\ \qquad\quad \tfrac{3}{2}y+y^2 \\ \qquad\quad \underline{\tfrac{3}{2}y+\tfrac{3}{4}y^2} \\ \qquad\qquad \tfrac{1}{4}y^2 \\ \qquad\qquad \underline{\tfrac{1}{4}y^2+\tfrac{1}{8}y^3} \\ \qquad\qquad\quad -\tfrac{1}{8}y^3 \end{array}$$

Therefore the fraction $=\dfrac{1}{2y^3}+\dfrac{3}{4y^2}+\dfrac{1}{8y}-\dfrac{1}{8(2+y)}$

$$=\frac{1}{2(x-1)^3}+\frac{3}{4(x-1)^2}+\frac{1}{8(x-1)}-\frac{1}{8(x+1)}.$$

146. Remarks.

(1) In practice it is desirable to perform the division by the "detached coefficients" method, and the above work appears as

$$\begin{array}{l} 2+1\,)\,1+2+1\,(\,\tfrac{1}{2}+\tfrac{3}{4}+\tfrac{1}{8} \\ \qquad\underline{1+\tfrac{1}{2}} \\ \qquad\quad \tfrac{3}{2}+1 \\ \qquad\quad \underline{\tfrac{3}{2}+\tfrac{3}{4}} \\ \qquad\qquad \tfrac{1}{4} \\ \qquad\qquad \underline{\tfrac{1}{4}+\tfrac{1}{8}} \\ \qquad\qquad\quad -\tfrac{1}{8} \end{array}$$

(2) In cases where there is but one other linear or quadratic factor in the denominator $\phi(x)$ and that not a repeated one, this process will *finish the whole operation.*

Ex. (ii). $\dfrac{x^2+2x}{(x-1)^5(x^2+1)}$. Put $x=1+y$.

$$\text{The fraction}=\frac{1}{y^5}\,\frac{3+4y+y^2}{2+2y+y^2}.$$

$$\begin{array}{l}
2+2+1\,)\,3+4+1\,(\,\tfrac{3}{2}+\tfrac{1}{2}-\tfrac{3}{4}+\tfrac{1}{2}-\tfrac{1}{8}\\
\quad\quad\quad\;\; \underline{3+3+\tfrac{3}{2}}\\
\quad\quad\quad\quad\;\; 1-\tfrac{1}{2}\\
\quad\quad\quad\quad\;\; \underline{1+1+\tfrac{1}{2}}\\
\quad\quad\quad\quad\quad -\tfrac{3}{2}-\tfrac{1}{2}\\
\quad\quad\quad\quad\quad \underline{-\tfrac{3}{2}-\tfrac{3}{2}-\tfrac{3}{4}}\\
\quad\quad\quad\quad\quad\quad\quad 1+\tfrac{3}{4}\\
\quad\quad\quad\quad\quad\quad\quad \underline{1+1+\tfrac{1}{2}}\\
\quad\quad\quad\quad\quad\quad\quad\quad -\tfrac{1}{4}-\tfrac{1}{2}\\
\quad\quad\quad\quad\quad\quad\quad\quad \underline{-\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{8}}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad -\tfrac{1}{4}+\tfrac{1}{8}
\end{array}$$

$$\text{Hence}\quad\text{the fraction}=\frac{3}{2y^5}+\frac{1}{2y^4}-\frac{3}{4y^3}+\frac{1}{2y^2}-\frac{1}{8y}-\frac{\frac{1}{4}-\frac{1}{8}y}{2+2y+y^2}$$

$$=\frac{3}{2(x-1)^5}+\frac{1}{2(x-1)^4}-\frac{3}{4(x-1)^3}+\frac{1}{2(x-1)^2}-\frac{1}{8(x-1)}+\frac{1}{8}\,\frac{x-3}{1+x^2},$$

and is then ready for integration.

Ex. (iii). $\dfrac{x}{(x-1)^3(x-2)^2}$. In such a case we find the three partial fractions corresponding to $x-1$, and then, either *from the remainder* or *beginning over again*, the two corresponding to $(x-2)^2$.

147. Instead of expanding out $f(a+y)$ and $\psi(a+y)$ separately, as shown above (which is however usually best in practical cases), we may expand $\dfrac{f(a+y)}{\psi(a+y)}$ as though it were $F(a+y)$ by Taylor's theorem, or otherwise, which shows a compact theoretical form for the several coefficients, C_0, C_1, C_2, ..., of Art. 144.

Thus

$$\frac{f(a+y)}{\psi(a+y)}=\frac{f(a)}{\psi(a)}+y\frac{d}{da}\left(\frac{fa}{\psi a}\right)+\ldots+\frac{y^r}{\lfloor r}\,\frac{d^r}{da^r}\left(\frac{fa}{\psi a}\right)+\ldots,$$

So that

$$C_0=\frac{f(a)}{\psi(a)},\quad C_1=\frac{d}{da}\left(\frac{fa}{\psi a}\right),\quad C_2=\frac{1}{\lfloor 2}\,\frac{d^2}{da^2}\left(\frac{fa}{\psi a}\right)\ldots,$$

$$C_{r-1}=\frac{1}{\lfloor r-1}\,\frac{d^{r-1}}{da^{r-1}}\left(\frac{fa}{\psi a}\right).$$

148. Nothing has been assumed so far as to the reality of the several roots, a, b, etc., of $\phi(x)=0$. Hence the rules obtained equally apply for unreal or for real roots.

If then
$$\phi(x)\equiv(x-a)^p(x-b)^q(x-c)^r\ldots,$$
whether a, b, c be real or unreal, so that $p+q+r+\ldots=n$, the degree of $\phi(x)$, we obtain, by methods explained above, a result of form
$$\frac{f(x)}{\phi(x)}=\frac{A_0}{(x-a)^p}+\frac{A_1}{(x-a)^{p-1}}+\frac{A_2}{(x-a)^{p-2}}+\ldots+\frac{A_{p-1}}{x-a}$$
$$+\frac{B_0}{(x-b)^q}+\frac{B_1}{(x-b)^{q-1}}+\frac{B_2}{(x-b)^{q-2}}+\ldots+\frac{B_{q-1}}{x-b}$$
$$+\frac{C_0}{(x-c)^r}+\frac{C_1}{(x-c)^{r-1}}+\frac{C_2}{(x-c)^{r-2}}+\ldots+\frac{C_{r-1}}{x-c}$$
$$+\ldots;$$
and imagining these fractions to be reduced to a common denominator and added up to get back to the form $\dfrac{f(x)}{\phi(x)}$, the coefficient of x^{n-1} is obviously $A_{p-1}+B_{q-1}+C_{r-1}+\ldots$.

The integral will be
$$\int\frac{f(x)}{\phi(x)}dx$$
$$=-\frac{A_0}{(p-1)(x-a)^{p-1}}-\frac{A_1}{(p-2)(x-a)^{p-2}}-\ldots-\frac{A_{p-2}}{x-a}+A_{p-1}\log(x-a)$$
$$-\frac{B_0}{(q-1)(x-b)^{q-1}}-\frac{B_1}{(q-2)(x-b)^{q-2}}-\ldots-\frac{B_{q-2}}{x-b}+B_{q-1}\log(x-b)$$
$$-\frac{C_0}{(r-1)(x-c)^{r-1}}-\frac{C_1}{(r-2)(x-c)^{r-2}}-\ldots-\frac{C_{r-2}}{x-c}+C_{r-1}\log(x-c),$$
etc.,

i.e. in general partly algebraic and partly logarithmic.

149. The conditions necessary that the integral should be purely algebraic are clearly
$$A_{p-1}=B_{q-1}=C_{r-1}=\ldots=0,$$
and in number the same as the number of *different* roots of $\phi(x)=0$. But the coefficient of x^{n-1} in $f(x)/\phi(x)$ has been seen to be
$$A_{p-1}+B_{q-1}+C_{r-1}+\ldots,$$
and this must vanish when the above conditions are satisfied.

Hence the index of the highest power of x in the numerator must be at least 2 less than that of the highest power of x in the denominator.

If then the number of different roots of $\phi(x)=0$, viz. $a, b, c, \ldots$, be k, say; and if the degree of $f(x)$ be lower by 2 than the degree of $\phi(x)$, we must *necessarily* have

$$A_{p-1}+B_{q-1}+C_{r-1}+\ldots=0,$$

and one of the k conditions, $A_{p-1}=B_{q-1}=\ldots=0$, must be included in the others, and there are then only $k-1$ independent conditions to be satisfied for $\int\frac{f(x)}{\phi(x)}dx=0$ to be entirely algebraic

150. III. Consider next the **case of an irreducible quadratic factor,**

$$(x-a)^2+\beta^2,$$

not repeated, occurring in the denominator, $\phi(x)$, and let

$$\phi(x)\equiv[(x-a)^2+\beta^2]\psi(x).$$

Then the partial fractions of $\frac{f(x)}{\phi(x)}$, *i.e.* of

$$\frac{f(x)}{(x-a-\iota\beta)(x-a+\iota\beta)\psi(x)},$$

corresponding to these unreal factors, are

$$\frac{f(a+\iota\beta)}{(2\iota\beta)\psi(a+\iota\beta)}\frac{1}{x-a-\iota\beta}+\frac{f(a-\iota\beta)}{(-2\iota\beta)\psi(a-\iota\beta)}\frac{1}{x-a+\iota\beta},$$

or, separating out the real and unreal parts of $\frac{f(a+\iota\beta)}{(2\iota\beta)\psi(a+\iota\beta)}$ as $P+\iota Q$, these partial fractions are

$$\frac{P+\iota Q}{x-a-\iota\beta}+\frac{P-\iota Q}{x-a+\iota\beta}, \quad\text{or}\quad \frac{2P(x-a)-2Q\beta}{(x-a)^2+\beta^2},$$

which is of form $\qquad \frac{Lx+M}{(x-a)^2+\beta^2},$

$$\left.\begin{aligned}\text{where}\quad P&=\frac{1}{4}\left[\frac{f(a+\iota\beta)}{\iota\beta\psi(a+\iota\beta)}-\frac{f(a-\iota\beta)}{\iota\beta\psi(a-\iota\beta)}\right]\\ \text{and}\quad Q&=-\frac{1}{4}\left[\frac{f(a+\iota\beta)}{\beta\psi(a+\iota\beta)}+\frac{f(a-\iota\beta)}{\beta\psi(a-\iota\beta)}\right],\end{aligned}\right\}\text{ which are both real,}$$

and $\quad L=2P,\quad M=-2Pa-2Q\beta.$

151. IV. Case of the factor $(x-\alpha)^2+\beta^2$ repeated r times.

Let $\phi(x)\equiv[(x-\alpha)^2+\beta^2]^r\psi(x)$.

Then it will be possible to write

$$\frac{f(x)}{\phi(x)}\equiv\frac{f(x)}{[(x-\alpha)^2+\beta^2]^r\psi(x)}=\frac{P_rx+Q_r}{[(x-\alpha)^2+\beta^2]^r}+\frac{\chi_r(x)}{[(x-\alpha)^2+\beta^2]^{r-1}\psi(x)}.$$

For this is equivalent to determining P_r and Q_r, so that

$$f(x)-(P_rx+Q_r)\psi(x)\equiv\chi_r(x)[(x-\alpha)^2+\beta^2],$$

i.e. so that $\qquad f(x)-(P_rx+Q_r)\psi(x)$

contains $x-\alpha-\iota\beta$ and $x-\alpha+\iota\beta$ as factors, and this will be effected by taking P_r and Q_r such that

$$\frac{f(\alpha+\iota\beta)}{\psi(\alpha+\iota\beta)}=P_r(\alpha+\iota\beta)+Q_r \quad\text{and}\quad \frac{f(\alpha-\iota\beta)}{\psi(\alpha-\iota\beta)}=P_r(\alpha-\iota\beta)+Q_r;$$

and if $\dfrac{f(\alpha+\iota\beta)}{\psi(\alpha+\iota\beta)}$, when separated into real and unreal parts, becomes $A+\iota B$, then $P_r\alpha+Q_r=A$ and $P_r\beta=B$,

i.e. $\qquad P_r=\dfrac{B}{\beta}$ and $Q_r=A-\dfrac{B\alpha}{\beta}=\dfrac{A\beta-B\alpha}{\beta}$.

Thus P_r, Q_r, and therefore χ_r are determinate.

This being so, it is obvious that

$$\frac{\chi_r(x)}{[(x-\alpha)^2+\beta^2]^{r-1}\psi(x)}$$

can itself be expressed as

$$\frac{P_{r-1}x+Q_{r-1}}{[(x-\alpha)^2+\beta^2]^{r-1}}+\frac{\chi_{r-1}(x)}{[(x-\alpha)^2+\beta^2]^{r-2}\psi(x)},$$

and by continued repetition of the argument we get finally that

$$\frac{f(x)}{\phi(x)}\equiv\frac{P_rx+Q_r}{[(x-\alpha)^2+\beta^2]^r}+\frac{P_{r-1}x+Q_{r-1}}{[(x-\alpha)^2+\beta^2]^{r-1}}+\frac{P_{r-2}x+Q_{r-2}}{[(x-\alpha)^2+\beta^2]^{r-2}}+\dots$$
$$+\frac{P_1x+Q_1}{(x-\alpha)^2+\beta^2}+\frac{\chi_1(x)}{\psi(x)},$$

and the values of the r pairs of quantities,

$$P_r \text{ and } Q_r,\quad P_{r-1} \text{ and } Q_{r-1},\ \dots,\ P_1 \text{ and } Q_1,$$

are successively obtainable as described.

The general form of the result is thus established. But this mode of finding the numerical value of the P's and Q's is laborious, except when r is small.

152. It now appears that the general result of putting $\frac{f(x)}{\phi(x)}$ into partial fractions, where $\phi(x)$ is, say,

$$(x-a)(x-b)^{\lambda}(x^2+px+q)(x^2+rx+s)^{\mu},$$

the last two factors being irreducible to real linear factors, and $f(x)$ is any rational integral function of x of any degree, will be of the form

$$\frac{f(x)}{\phi(x)} = \text{an integral algebraic quotient}$$

$$+\frac{A}{x-a}$$

$$+\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\frac{B_3}{(x-b)^3}+\dots+\frac{B_\lambda}{(x-b)^\lambda}$$

$$+\frac{Px+Q}{x^2+px+q}$$

$$+\frac{R_1x+S_1}{x^2+rx+s}+\frac{R_2x+S_2}{(x^2+rx+s)^2}+\frac{R_3x+S_3}{(x^2+rx+s)^3}+\dots+\frac{R_\mu x+S_\mu}{(x^2+rx+s)^\mu}. \quad \text{I.}$$

This is the general typical form of the result. If other factors occur in $\phi(x)$, other partial fractions will occur in the result. But all others will be of the types exhibited.

153. The integration can therefore be effected.

For (1) The integrals of the algebraic terms are of type

$$\int A_s x^s\,dx = A_s\frac{x^{s+1}}{s+1}.$$

(2) The integral of $\int\frac{A}{x-a}dx$ is $A\log(x-a)$.

(3) The integral of $\int\frac{B_\lambda}{(x-b)^\lambda}dx$ is $-\frac{B_\lambda}{\lambda-1}\frac{1}{(x-b)^{\lambda-1}}$.

(4) The integration of $\int\frac{Px+Q}{x^2+px+q}dx$ has been effected in Art. 136.

(5) The integration of $\int\frac{R_\mu x+S_\mu}{(x^2+rx+s)^\mu}dx$ can be effected by means of a reduction formula, as will be explained in a subsequent article.

Hence we may then regard the integration $\int\frac{f(x)}{\phi(x)}dx$ as complete whenever $\frac{f(x)}{\phi(x)}$ is a rational algebraic function of x.

154. In practice, when irresoluble quadratic factors are present in the denominator we may first of all determine the

partial fractions corresponding to the real linear factors, single and repeated. Then, if there be only one quadratic factor, and that not repeated, it will appear without further trouble in the remainder of $\frac{f(x)}{\phi(x)}$. But if there be several such factors or a repeated factor, we may subtract the simple partial fractions when obtained and then after simplification discuss the remainder.

155. **Use of "Undetermined or Indeterminate Coefficients."** We may often with advantage apply the method of "indeterminate coefficients."

When the fraction has been reduced by division till the numerator is of lower degree than the denominator, *i.e.* of degree $n-1$ at most, and we get, as in I.,

$$\frac{f(x)}{\phi(x)} \equiv \frac{A}{x-a} + \sum_{s=1}^{s=\lambda} \frac{B_s}{(x-b)^s} + \frac{Px+Q}{x^2+px+q} + \sum_{k=1}^{k=\mu} \frac{R_k x + S_k}{(x^2+rx+s)^k}, \quad \text{II.}$$

we have, upon multiplying up by $\phi(x)$ an identity in which the right-hand side is of degree $n-1$ and consists of n terms when arranged in powers of x, and the left side is of degree $n-1$ at most, viz. $f(x)$.

Now $\phi(x)$ is of degree $1+\lambda+2+2\mu$, which must $=n$, and the number of quantities

$$\begin{array}{lccccc} & A, & (B_1, B_2, \ldots), & (P, Q,), & (R_1, S_1, R_2, S_2, \ldots) & \\ \text{is} & 1 & +\quad \lambda \quad + & 2 & +\quad 2\mu, & i.e. = n. \end{array}$$

Hence, upon equating coefficients of the n terms on the right-hand side to the corresponding coefficients in $f(x)$, we have just enough equations to obtain the n quantities, provided that these equations are all independent. But as we have established *otherwise* a means of finding these quantities we may *infer the consistence of the equations obtained by equating coefficients.*

156. Many of the coefficients, or all, may be found by the substitution in the identity of numerical values for x. Obviously any number of equations of this kind could be obtained, but only n would be independent. The most suitable values to take for this purpose will be such as will make one of the factors $x-a$, $x-b$, x^2+px+q or x^2+rx+s vanish, for such values would cause many of the terms of the identity to disappear.

In substituting roots of x^2+px+q, viz. $a\pm\iota\beta$ say, only one root need be substituted. Then the real and unreal parts on each side of the identity may be equated.

All the B's and A, *i.e.* $\lambda+1$ of the quantities, can be found by the easy rules given above (Arts. 140 to 147). Hence $\lambda+1$ of the equations obtained by equating coefficients will not be independent of the others when the values of $A, B_1, B_2, \ldots B_\lambda$, which have been found, are substituted. But there will still remain $2+2\mu$ *independent* relations from the equating of coefficients. The substitution of a root of x^2+px+q and of a root of $x^2+rx+s=0$ with the equating of real and unreal parts will furnish four other relations and reduce the number of independent "equated coefficient equations" to $2\mu-2$, which are linear and to be solved in the easiest way available. The student will perceive that in practice it will be best to combine several methods to determine the coefficients and to use redundant equations to check numerical results.*

157. If none but even powers of x occur in both numerator and denominator, we may put $x^2=y$, and thereby reduce the labour considerably. In such fractions, the quadratic factors becoming linear by this substitution, their occurrence may be termed pseudo-quadratic or quasi-linear.

Ex. 1. $\dfrac{x^2+1}{(x^2+4)(x^2+9)^2}$.

This is of form $\dfrac{y+1}{(y+4)(y+9)^2}$.

Putting, then, x^2 (or y) $=z-9$,

$$\frac{x^2+1}{(x^2+4)(x^2+9)^2}=\frac{-8+z}{z^2(-5+z)}$$

$$\begin{array}{r|l}
-5+z & -8+z \;\left(\dfrac{8}{5}+\dfrac{3z}{25}\right. \\
 & \underline{-8+\dfrac{8z}{5}} \\
 & -\dfrac{3z}{5} \\
 & \underline{-\dfrac{3z}{5}+\dfrac{3z^2}{25}} \\
 & -\dfrac{3z^2}{25}
\end{array}$$

$$\therefore \frac{x^2+1}{(x^2+4)(x^2+9)^2}=\frac{8}{5z^2}+\frac{3}{25z}-\frac{3}{25}\,\frac{1}{-5+z}$$

$$=\frac{8}{5}\,\frac{1}{(x^2+9)^2}+\frac{3}{25}\,\frac{1}{x^2+9}-\frac{3}{25}\,\frac{1}{x^2+4}.$$

* See also Art. 1891, Vol. II.

Ex. 2. $\dfrac{1}{(x-1)(x^2+1)(x^2+4)^2}$.

The partial fractions are of form

$$\frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+4)}+\frac{Fx+G}{(x^2+4)^2}.$$

Multiplying up we have the identity

$$\begin{aligned}1\equiv A(x^2+1)(x^2+4)^2&+(Bx+C)(x-1)(x^2+4)^2\\&+(Dx+E)(x-1)(x^2+1)(x^2+4)\\&+(Fx+G)(x-1)(x^2+1).\end{aligned}$$

Putting $x=1$, $\qquad 1=50A$.

Putting $x=\iota$, $\qquad 1=(B\iota+C)(\iota-1)9$;

$$\therefore\ \left.\begin{aligned}-B+C&=0,\\-B-C&=\tfrac{1}{9},\end{aligned}\right\}\text{ whence } B=C=-\tfrac{1}{18}.$$

Putting $x=2\iota$, $\qquad 1=(2F\iota+G)(2\iota-1)(-3)$;

$$\therefore\ \left.\begin{aligned}4F+G&=\tfrac{1}{3},\\F-G&=0,\end{aligned}\right\}\text{ whence } F=G=\tfrac{1}{15}.$$

Equating coefficients of x^6,

$$A+B+D=0;$$
$$\therefore\ D=-\tfrac{1}{50}+\tfrac{1}{18}=\tfrac{8}{225}.$$

Equating absolute terms,

$$16A-16C-4E-G=1,\text{ whence } E=\tfrac{8}{225};$$

$$\therefore\ \frac{1}{(x-1)(x^2+1)(x^2+4)^2}=\frac{1}{50}\,\frac{1}{x-1}-\frac{1}{18}\,\frac{x+1}{x^2+1}+\frac{8}{225}\,\frac{x+1}{x^2+4}+\frac{1}{15}\,\frac{x+1}{(x^2+4)^2}.$$

158. Case when the numerator is an odd function of x and the denominator is even.

$$I=\int\frac{f(x)}{\phi(x)}dx \text{ takes the form } \int\frac{xF(x^2)}{\Phi(x^2)}dx,$$

and putting $x^2=y$, $\qquad I=\dfrac{1}{2}\displaystyle\int\frac{F(y)}{\Phi(y)}dy,$

and the factors in the denominator which were quadratic factors in x are linear in y.

Ex. Thus
$$\begin{aligned}\int\frac{x^3+3x}{(x^2-1)(x^2+1)^2}dx&=\frac{1}{2}\int\frac{y+3}{(y-1)(y+1)^2}dy\\&=\frac{1}{2}\int\left[\frac{1}{y-1}-\frac{1}{y+1}-\frac{1}{(y+1)^2}\right]dy\\&=\frac{1}{2}\log\frac{y-1}{y+1}+\frac{1}{2}\,\frac{1}{y+1}\\&=\frac{1}{2}\log\frac{x^2-1}{x^2+1}+\frac{1}{2}\,\frac{1}{x^2+1}.\end{aligned}$$

159. Case when the denominator is odd and the numerator even. The same process may be adopted.

Thus
$$\int \frac{x^2+1}{x(x^2+4)}dx = \frac{1}{2}\int \frac{y+1}{y(y+4)}dy$$
$$= \frac{1}{8}\int\left(\frac{1}{y}+\frac{3}{y+4}\right)dy = \frac{1}{8}\log y(y+4)^3$$
$$= \frac{1}{8}\log x^2(x^2+4)^3.$$

160. Integration of $\int \frac{z^{2q}\,dz}{(z^2+a_1^2)(z^2+a_2^2)(z^2+a_3^2)\dots(z^2+a_n^2)}$, where $q < n$.

The partial fractions are of the form
$$\sum_1^n \frac{A_r}{z^2+a_r^2},$$
and the integral is
$$\sum_1^n \frac{A_r}{a_r}\tan^{-1}\frac{z}{a_r}.$$

The value of A_r is
$$\frac{(-a_r^2)^q}{(a_1^2-a_r^2)(a_2^2-a_r^2)\dots(a_{r-1}^2-a_r^2)(a_{r+1}^2-a_r^2)\dots(a_n^2-a_r^2)}.$$

The denominator factorized may be written as
$$(a_1-a_r)(a_2-a_r)\dots(a_{r-1}-a_r)\quad(a_{r+1}-a_r)(a_{r+2}-a_r)\dots(a_n-a_r)$$
$$\times(a_1+a_r)(a_2+a_r)\dots(a_{r-1}+a_r)\quad(a_{r+1}+a_r)(a_{r+2}+a_r)\dots(a_n+a_r).$$

Taking the case when $a_1, a_2, a_3, \dots a_n$ form an A.P., with common difference b, this denominator D, say, is
$$D=(-1)^{r-1}(r-1)b(r-2)b\dots 2b.b,\quad b.2b.3b\dots(n-r)b\times\prod_{k=1}^{k=n}(a_k+a_r)/2a_r,$$
where in forming the product of the factors in the lower line the missing term (a_r+a_r) has been supplied;
$$D=(-1)^{r-1}b^{r-1}(r-1)!\,b^{n-r}(n-r)!\prod_{k=1}^{k=n}(a_k+a_r)/2a_r$$
and
$$A_r=(-1)^{q-r+1}2a_r^{2q+1}/b^{n-1}(r-1)!(n-r)!\prod_{k=1}^{k=n}(a_k+a_r).$$

If $b=a_1$, we have
$$a_k+a_r=(r+k)a_1,$$
and
$$\prod_1^n(a_k+a_r)=a_1^n(r+1)(r+2)\dots(r+n)=a_1^n\frac{(r+n)!}{r!},$$
giving for this case the partial fractions
$$\frac{2}{a_1^{2n-2q-2}}\sum_{r=1}^{r=n}(-1)^{q-r+1}\frac{r^{2q+2}}{(n+r)!\,(n-r)!}\frac{1}{z^2+a_r^2},$$
and the integral
$$\frac{2}{a_1^{2n-2q-1}}\sum_{r=1}^{r=n}(-1)^{q-r+1}\frac{r^{2q+1}}{(n+r)!\,(n-r)!}\tan^{-1}\frac{z}{a_r},$$

161. Obviously we should also have in the same case

$$\int \frac{z^{2q+1}\,dz}{(z^2+a_1^2)(z^2+a_2^2)\dots(z^2+a_n^2)}$$

$$=\frac{2}{a_1^{2n-2q-2}}\int \sum_{r=1}^{r=n}(-1)^{q-r+1}\frac{r^{2q+2}}{(n+r)!\,(n-r)!}\frac{z}{z^2+a_r^2}$$

$$=\frac{1}{a_1^{2n-2q-2}}\sum_{r=1}^{r=n}(-1)^{q-r+1}\frac{r^{2q+2}}{(n+r)!\,(n-r)!}\log(z^2+a_r^2).$$

162. Taking the case $a_1=2$, $b=2$, and therefore $a_r=2r$,

$$\frac{z^{2q}}{(z^2+2^2)(z^2+4^2)(z^2+6^2)\dots(z^2+2^2n^2)}$$

$$=\frac{1}{(2n)!}\sum_{n}^{1}(-1)^{q-r+1}\frac{r^{2q+2}}{2^{2n-2q-3}}\,{}^{2n}C_{n-r}\frac{1}{z^2+2^2r^2}$$

$$=\frac{(-1)^{q+n-1}}{(2n)!}\frac{1}{2^{2n-1}}\Big[{}^{2n}C_0\frac{(2n)^{2q+2}}{z^2+2^2n^2}-{}^{2n}C_1\frac{(2n-2)^{2q+2}}{z^2+(2n-2)^2}+{}^{2n}C_2\frac{(2n-4)^{2q+2}}{z^2+(2n-4)^2}$$

$$+\dots+(-1)^{n-1}\,{}^{2n}C_{n-1}\frac{2^{2q+2}}{z^2+2^2}\Big];$$

and its integral

$$=\frac{(-1)^{q+n-1}}{(2n)!}\frac{1}{2^{2n-1}}$$

$$\Big[{}^{2n}C_0(2n)^{2q+1}\tan^{-1}\frac{z}{2n}-{}^{2n}C_1(2n-2)^{2q+1}\tan^{-1}\frac{z}{2n-2}$$

$$+\dots+(-1)^{n-1}\,{}^{2n}C_{n-1}\,2^{2q+1}\tan^{-1}\frac{z}{2}\Big]. \quad\dots\dots(\text{A})$$

163. And similarly, if the index of z in the numerator had been $2q+1$ instead of $2q$, the same work shows

$$\frac{z^{2q+1}}{(z^2+2^2)(z^2+4^2)\dots(z^2+2^2n^2)}$$

$$=\frac{(-1)^{q+n-1}}{(2n)!}\frac{1}{2^{2n-1}}$$

$$\Big[{}^{2n}C_0(2n)^{2q+2}\frac{z}{z^2+2^2n^2}-{}^{2n}C_1(2n-2)^{2q+2}\frac{z}{z^2+(2n-2)^2}$$

$$+\dots+(-1)^{n-1}\,{}^{2n}C_{n-1}\,2^{2q+2}\frac{z}{z^2+2^2}\Big];$$

and its integral

$$=\frac{(-1)^{q+n-1}}{(2n)!}\frac{1}{2^{2n}}$$

$$[{}^{2n}C_0(2n)^{2q+2}\log(z^2+2^2n^2)-{}^{2n}C_1(2n-2)^{2q+2}\log\{z^2+(2n-2)^2\}$$

$$+\dots+(-1)^{n-1}\,{}^{2n}C_{n-1}2^{2q+2}\log(z^2+2^2)]. \quad\dots(\text{B})$$

164. Taking the case $a_1=1$, $b=2$, and therefore $a_r=2r-1$,

$$\frac{z^{2q}}{(z^2+1^2)(z^2+3^2)(z^2+5^2)\dots[z^2+(2n-1)^2]}$$

$$=\frac{1}{(2n-1)!}\sum_n^1(-1)^{q-r+1}\frac{(2r-1)^{2q+1}}{2^{2n-2}}\,{}^{2n-1}C_{n-r}\frac{1}{z^2+(2r-1)^2}$$

$$=\frac{(-1)^{q-n+1}}{(2n-1)!}\frac{1}{2^{2n-2}}\Big[{}^{2n-1}C_0\frac{(2n-1)^{2q+1}}{z^2+(2n-1)^2}-{}^{2n-1}C_1\frac{(2n-3)^{2q+1}}{z^2+(2n-3)^2}$$

$$+{}^{2n-1}C_2\frac{(2n-5)^{2q+1}}{z^2+(2n-5)^2}-\dots+(-1)^{n-1}\,{}^{2n-1}C_{n-1}\frac{1^{2q+1}}{z^2+1^2}\Big];$$

and its integral

$$=\frac{(-1)^{q-n+1}}{(2n-1)!}\frac{1}{2^{2n-2}}$$

$$\Big[{}^{2n-1}C_0(2n-1)^{2q}\tan^{-1}\frac{z}{2n-1}-{}^{2n-1}C_1(2n-3)^{2q}\tan^{-1}\frac{z}{2n-3}$$

$$+\dots+(-1)^{n-1}\,{}^{2n-1}C_{n-1}1^{2q}\tan^{-1}\frac{z}{1}\Big]. \quad \text{(C)}$$

165. And for $\frac{z^{2q+1}}{(z^2+1^2)\dots[z^2+(2n-1)^2]}$ the integral will be

$$=\frac{(-1)^{q-n+1}}{(2n-1)!}\frac{1}{2^{2n-1}}\big[{}^{2n-1}C_0(2n-1)^{2q+1}\log\{z^2+(2n-1)^2\}$$

$$-{}^{2n-1}C_1(2n-3)^{2q+1}\log\{z^2+(2n-3)^2\}$$

$$+\dots+(-1)^{n-1}\,{}^{2n-1}C_{n-1}1^{2q+1}\log\{z^2+1^2\}\big]. \quad \text{(D)}$$

166. Consider the integral $\int\frac{x^m\,dx}{x^{2n}-2a^nx^n\cos n\alpha+a^{2n}}$. $\quad (m<2n)$

Here $f(x)=x^m$, $\phi(x)=x^{2n}-2a^nx^n\cos n\alpha+a^{2n}$ (Art. 142)

$$=\prod_{r=0}^{r=n-1}\left[x^2-2ax\cos\left(\alpha+\frac{2r\pi}{n}\right)+a^2\right],$$

$$\phi'(x)=2nx^{n-1}(x^n-a^n\cos n\alpha).$$

Let $\qquad \alpha+\frac{2r\pi}{n}=\chi.$

The factor $\quad x^2-2ax\cos\chi+a^2\equiv(x-ae^{\iota\chi})(x-ae^{-\iota\chi})$,
and gives rise to the partial fractions

$$\frac{f(ae^{\iota\chi})}{\phi'(ae^{\iota\chi})}\frac{1}{x-ae^{\iota\chi}}+\frac{f(ae^{-\iota\chi})}{\phi'(ae^{-\iota\chi})}\frac{1}{x-ae^{-\iota\chi}}.$$

Now $\quad \frac{f(ae^{\iota\chi})}{\phi'(ae^{\iota\chi})}=\frac{a^me^{\iota m\chi}}{2na^{2n-1}e^{\iota(n-1)\chi}(e^{\iota n\chi}-\cos n\alpha)}$

$$=\frac{a^me^{\iota m\chi}}{2na^{2n-1}e^{\iota(n-1)\chi}\iota\sin n\alpha}=\frac{e^{-\iota(n-m-1)\chi}}{2\iota na^{2n-m-1}\sin n\alpha}.$$

Hence the two partial fractions

$$=\frac{1}{2\iota n a^{2n-m-1}\sin n\alpha}\left[\frac{e^{-\iota(n-m-1)\chi}}{x-ae^{\iota\chi}}-\frac{e^{\iota(n-m-1)\chi}}{x-ae^{-\iota\chi}}\right]$$

$$=\frac{1}{2\iota n a^{2n-m-1}\sin n\alpha}\left[\frac{e^{-\iota(n-m-1)\chi}(x-ae^{-\iota\chi})-e^{\iota(n-m-1)\chi}(x-ae^{\iota\chi})}{x^2-2ax\cos\chi+a^2}\right]$$

$$=\frac{1}{2\iota n\sin n\alpha\, a^{2n-m-1}}\left[\frac{2a\iota\sin(n-m)\chi-2x\iota\sin(n-m-1)\chi}{x^2-2ax\cos\chi+a^2}\right];$$

$$\therefore\ \frac{x^m}{x^{2n}-2a^nx^n\cos n\alpha+a^{2n}}$$

$$\equiv\frac{1}{2n\sin n\alpha\, a^{2n-m-1}}\sum_{r=1}^{r=n}\left[\frac{2a\sin\chi\cos(n-m-1)\chi-2(x-a\cos\chi)\sin(n-m-1)\chi}{(x-a\cos\chi)^2+a^2\sin^2\chi}\right]$$

Hence $\displaystyle\int\frac{x^m\,dx}{x^{2n}-2a^nx^n\cos n\alpha+a^{2n}}$, $(m<2n)$,

$$=\frac{1}{n\sin n\alpha}\frac{1}{a^{2n-m-1}}\sum_0^{n-1}\cos(n-m-1)\left(\alpha+\frac{2r\pi}{n}\right)\tan^{-1}\frac{x-a\cos\left(\alpha+\frac{2r\pi}{n}\right)}{a\sin\left(\alpha+\frac{2r\pi}{n}\right)}$$

$$-\frac{1}{2n\sin n\alpha}\frac{1}{a^{2n-m-1}}\sum_0^{n-1}\sin(n-m-1)\left(\alpha+\frac{2r\pi}{n}\right)\log\left[x^2-2ax\cos\left(\alpha+\frac{2r\pi}{n}\right)+a^2\right].$$

In the same way $x^{p-1}/(x^n\pm a^n)$ may be integrated. The results are given in Exs. 39 and 40, pages 166 and 167.

167. Ex. Calculate $\displaystyle\int_0^\infty\frac{dx}{x^4+2a^2x^2\cos 2\beta+a^4}$.

Here (Art. 166) $\qquad \beta=\dfrac{\pi}{2}-\alpha,\quad m=0,\quad n=2.$

The indefinite integral is

$$\frac{1}{2\sin 2\beta}\frac{1}{a^3}\left[\cos\left(\frac{\pi}{2}-\beta\right)\tan^{-1}\frac{x-a\cos\left(\frac{\pi}{2}-\beta\right)}{a\sin\left(\frac{\pi}{2}-\beta\right)}\right.$$

$$+\cos\left(\frac{3\pi}{2}-\beta\right)\tan^{-1}\frac{x-a\cos\left(\frac{3\pi}{2}-\beta\right)}{a\sin\left(\frac{3\pi}{2}-\beta\right)}$$

$$-\tfrac{1}{2}\sin\left(\frac{\pi}{2}-\beta\right)\log\left\{x^2-2ax\cos\left(\frac{\pi}{2}-\beta\right)+a^2\right\}$$

$$\left.-\tfrac{1}{2}\sin\left(\frac{3\pi}{2}-\beta\right)\log\left\{x^2-2ax\cos\left(\frac{3\pi}{2}-\beta\right)+a^2\right\}\right]$$

$$=\frac{1}{2a^3\sin 2\beta}\left[\sin\beta\tan^{-1}\frac{x-a\sin\beta}{a\cos\beta}+\sin\beta\tan^{-1}\frac{x+a\sin\beta}{a\cos\beta}\right.$$

$$\left.-\tfrac{1}{2}\cos\beta\log(x^2-2ax\sin\beta+a^2)+\tfrac{1}{2}\cos\beta\log(x^2+2ax\sin\beta+a^2)\right],$$

and taken between limits 0 and ∞

$$=\frac{1}{2a^3\sin 2\beta}\left[\sin\beta\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right]=\frac{\pi}{4a^3\cos\beta}.$$

The indefinite integral may also be written as

$$\frac{1}{2a^3\sin 2\beta}\left[\sin\beta\tan^{-1}\frac{2ax\cos\beta}{a^2-x^2}+\cos\beta\tanh^{-1}\frac{2ax\sin\beta}{a^2+x^2}\right].$$

168. An integral of the form

$$\int\frac{a+bx^{\frac{p}{q}}}{c+dx^{\frac{r}{s}}}dx$$

can always be integrated as follows:

Let l be the L.C.M. of q and s, and let $\frac{p}{q}=\frac{\lambda}{l}$ and $\frac{r}{s}=\frac{\mu}{l}$.

Let $x=z^l$, $dx=lz^{l-1}dz$.

Then $$\int\frac{a+bx^{\frac{p}{q}}}{c+dx^{\frac{r}{s}}}dx=l\int\frac{a+bz^\lambda}{c+dz^\mu}z^{l-1}dz,$$

and the expression to be integrated is now rational, and when expressed in partial fractions each term can be integrated.

Ex. $\int\frac{1+x^{\frac{1}{3}}}{1+x^{\frac{1}{2}}}dx$ (Let $x=z^6$.) $=\int\frac{1+z^2}{1+z^3}6z^5\,dz$

$$=6\int\frac{z^5+z^7}{1+z^3}dz=6\int\left(z^4+z^2-z-\frac{z^2-z}{z^3+1}\right)dz$$

$$=6\int\left[z^4+z^2-z-\frac{\frac{2}{3}}{z+1}-\frac{1}{6}\,\frac{2z-1-3}{z^2-z+1}\right]dz$$

$$=\frac{6z^5}{5}+2z^3-3z^2-4\log(z+1)-\log(z^2-z+1)+2\sqrt{3}\tan^{-1}\frac{2z-1}{\sqrt{3}}$$

$$=\tfrac{6}{5}x^{\frac{5}{6}}+2x^{\frac{1}{2}}-3x^{\frac{1}{3}}-4\log(1+x^{\frac{1}{6}})-\log(1-x^{\frac{1}{6}}+x^{\frac{1}{3}})$$

$$+2\sqrt{3}\tan^{-1}\frac{2x^{\frac{1}{6}}-1}{\sqrt{3}}.$$

169. In exactly the same way the integration of

$$\int\frac{a+b(\alpha+\beta x)^{\frac{p}{q}}}{c+d(\alpha+\beta x)^{\frac{r}{s}}}dx,$$

can be effected by putting $\alpha+\beta x=z^l$ when l is the L.C.M. of q and s, and more generally that of

$$\int\frac{f[(\alpha+\beta x)^{\frac{p}{q}}]}{\phi[(\alpha+\beta x)^{\frac{r}{s}}]}dx,$$

where $f(t)$ and $\phi(t)$ are any rational algebraic functions of t; for, putting $a+\beta x=z^l$, as before, the integral becomes

$$\int \frac{f(z^\lambda)}{\phi(z^\mu)} \cdot \frac{lz^{l-1}}{\beta} dz;$$

and the integrand being now rational and algebraic, we can in any such case proceed to put it into partial fractions and then integrate.

EXAMPLES.

Integrate with regard to x the expressions in the following seven groups:

1. Linear unrepeated factors

(i) $\dfrac{1}{x(x^2-1)}$. (ii) $\dfrac{1}{(x-1)(x-3)(x-5)}$.

(iii) $\dfrac{1-3x^2}{3x-x^3}$. (iv) $\dfrac{x^2+x+1}{(x^2-1)(x^2-4)}$.

(v) $\dfrac{(x-1)(x-2)}{(x^2-9)(x^2-16)}$. (vi) $\dfrac{(x-a)(x-b)(x-c)}{(x-a_1)(x-b_1)(x-c_1)}$.

(vii) $\dfrac{(x-a)(x-b)(x-c)}{(x^2-a_1{}^2)(x^2-b_1{}^2)(x^2-c_1{}^2)}$. (viii) $\dfrac{x+1}{x^2+10x-75}$.

(ix) $\dfrac{x+1}{x^2+10x-119}$. (x) $\dfrac{x+1}{x^3-31x^2+311x-1001}$.

2. Linear repeated factors:

(i) $\dfrac{1}{(x-1)^3(x+1)}$. (ii) $\dfrac{1}{(x-1)^4(x+1)^4}$.

(iii) $\dfrac{x+1}{x^4(x-1)^4}$. (iv) $(ax^2+bx^3)^{-1}$.

(v) $(x^2-7x+12)^{-2}$. (vi) $\dfrac{x^3}{(x-a)^2(x-b)}$.

(vii) $\dfrac{x^2-3x+3}{x^3-7x^2+16x-12}$.

[I. C. S., 1900.]

3. Quasi-linear occurrence of factors. Powers of x *all even*:

(i) $\displaystyle\int\frac{dx}{(x^2+a^2)(x^2+b^2)}$. (ii) $\displaystyle\int\frac{(x^2+a^2)(x^2+b^2)}{(x^2+c^2)(x^2+d^2)}dx$.

(iii) $\displaystyle\int\frac{x^2(x^2+a^2)}{(x^2+c^2)}dx$. (iv) $\displaystyle\int\frac{x^2dx}{(x^2+1)(2x^2+1)}$.

(v) $\displaystyle\int\frac{ax^2+b}{(cx^2+d)(ex^2+f)}dx$. (vi) $\displaystyle\int\frac{ax^2+b}{x^2(cx^2+d)(ex^2+f)(gx^2+h)}dx$.

In the last two c, d, e, f, g, h may be considered positive.

4. Quasi-linear factors. Numerator an odd function, Denominator even, or Numerator even, Denominator odd:

(i) $\int \frac{dx}{x(x^2+1)}$. (ii) $\int \frac{x^2+2}{x(x^4-1)}\,dx$.

(iii) $\int \frac{dx}{x^7-6x^5+11x^3-6x}$. (iv) $\int \frac{x\,dx}{(ax^2+bx+c)^2+(ax^2-bx+c)^2}$.

5. Quadratic factors not repeated:

(i) $\int \frac{dx}{x^4+x^2+1}$. (ii) $\int \frac{(x+1)^2}{x^4+x^2+1}\,dx$.

(iii) $\int \frac{x^2+1}{x^4+1}\,dx$. (iv) $\int \frac{x^2+1}{x^4-x^2+1}\,dx$.

(v) $\int (x^2+a^2)(x^4+a^2x^2+a^4)^{-1}\,dx$. (vi) $\int (x^2-a^2)(x^4+a^2x^2+a^4)^{-1}\,dx$.

(vii) $\int \frac{x^2+3x+1}{x^4+x^2+1}\,dx$. (viii) $\int \frac{dx}{x^4+1}$.

6. Linear factors repeated. Quadratic factors not repeated.

(i) $\frac{x^2dx}{(x-1)^2(x^2-2x+4)}$. (ii) $\frac{dx}{(1+x)^2(1+2x+4x^2)}$.

(iii) $\frac{x^4dx}{(x-1)^2(x^2+4)}$. (iv) $\frac{dx}{(x+1)^2(x^2+1)}$.

(v) $\frac{dx}{(x-1)^2(x^2+1)}$. (vi) $\frac{dx}{x(x-1)^2(x^2+1)}$.

(vii) $\frac{dx}{x^3(a^2+x^2)}$. (viii) $\frac{dx}{x^3(a^2+x^2)(b^2+x^2)}$.

(ix) $\frac{dx}{(x-1)^3(x^2+x+1)}$. (x) $\frac{dx}{(2x-3)^2(4x^2+5)}$.

7. Repeated quadratic factors:

(i) $\frac{dx}{x(x^2+1)^2}$. (ii) $\frac{dx}{(x-1)^2(x^2+1)^2}$.

(iii) $\frac{(x+1)\,dx}{(x^2+1)^2}$. (iv) $\frac{(x+a)(x+b)}{(x^2+c^2)^3}\,dx$.

8. Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{\tan\theta}\,d\theta$ and $\int_0^{\frac{\pi}{4}} \sqrt{\cot\theta}\,d\theta$.

9. Evaluate (i) $\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^4 x-\cos^2 x\sin^2 x+\sin^4 x}$,

(ii) $\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^4 x+\cos^2 x\sin^2 x+\sin^4 x}$.

10. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$.

11. Show that $\int_0^{\infty} \frac{x^2\, dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(a+b)(b+c)(c+a)}$.

12. Show that $\int_{-\infty}^{+\infty} \frac{dx}{(x^2 \pm ax + a^2)(x^2 \pm bx + b^2)} = \frac{2\pi}{\sqrt{3}} \frac{a+b}{ab(a^2+ab+b^2)}$.

[γ, 1891.]

13. Show that the sum of the infinite series

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots \quad (a>0,\ b>0)$$

can be expressed as a definite integral, viz.

$$\int_0^1 \frac{t^{a-1}}{1+t^b}\, dt.$$

And hence prove that

$$\tfrac{1}{1} - \tfrac{1}{4} + \tfrac{1}{7} - \tfrac{1}{10} + \tfrac{1}{13} - \tfrac{1}{16} + \dots = \tfrac{1}{3}(\pi 3^{-\frac{1}{2}} + \log_e 2).$$

[Oxford, 1887.]

14. Integrate: (i) $\int \frac{25\, dx}{2x^4 + 3x^3 + 3x - 2}$. [Colleges, 1882.]

(ii) $\int \frac{x^5+2}{x^5-x}\, dx$. [St. John's, 1881.]

(iii) $\int \frac{(1+x^2)\, dx}{1 - 2x^2\cos\alpha + x^4}$. [Colleges, 1882.]

(iv) $\int \frac{dx}{1+x^5}$. [Colleges α, 1891.]

(v) $\int_0^1 \frac{1+x^2}{(1-x^2)^2 + a^2x^2}\, dx$.

15. Prove that $\int_0^{\infty} \frac{dx}{1+x^6} = \frac{\pi}{3}$. [St. John's, 1881.]

16. Prove that $\int \frac{dx}{(x-a)^p (x-b)^q}$

$$= \sum_{r=0}^{r=p-2} \frac{Q_r}{(a-b)^{q+r}} \frac{(x-a)^{-p+r+1}}{-p+r+1} + \sum_{r=0}^{r=q-2} \frac{P_r}{(b-a)^{p+r}} \frac{(x-b)^{-q+r+1}}{-q+r+1}$$

$$+ \frac{1}{(a-b)^{p+q-1}} Q_{p-1} \log(x-a) + \frac{1}{(b-a)^{p+q-1}} P_{q-1} \log(x-b),$$

where P_r and Q_r are the coefficients of z^r in $(1+z)^{-p}$ and $(1+z)^{-q}$ respectively.

17. Integrate (i) $\int \frac{dx}{(5x^3-3x)^2(x^2-1)}$. [MATH. TRIP., 1878.]

(ii) $\int \frac{dx}{(5x^3-3x^2)^2(x^2-1)}$.

(iii) $\int \frac{\sqrt{x}\,dx}{(1+x)(2+x)(3+x)}$. [OXFORD I., 1888.]

18. Prove (i) $\int_{-\infty}^{\infty} \frac{2x^2-x+1}{(x^2+x+1)^3}dx = \frac{10}{9}\pi\sqrt{3}$. [COLLEGES β, 1891.]

(ii) $\int_0^{\infty} \frac{\sqrt{x}}{1+x^2}dx = \frac{\pi}{\sqrt{2}}$. [TRINITY, 1882.]

(iii) $\int_x^{\frac{1}{x}} \frac{dx}{1+x^4}dx = \frac{1}{2\sqrt{2}}\left\{\pi - 2\tan^{-1}\frac{\sqrt{2}}{x^{-1}-x}\right\}$.

[TRINITY, 1895.]

19. Integrate $\int \frac{x\,dx}{x^3+1}$.

Prove that $\frac{1}{2.5}+\frac{1}{8.11}+\frac{1}{14.17}+\dots$ to $\infty = \frac{1}{9}\left[\frac{\pi}{\sqrt{3}} - \log 2\right]$.

[COLLEGES, 1896.]

20. Integrate $\int \frac{(\sqrt{\cot x}-\sqrt{\tan x})\,dx}{1+3\sin 2x}$. [COLLEGES β, 1890.]

21. Integrate (i) $\int \tan^{-1}\sqrt{\frac{a^2x-1}{b^2x-1}}\,dx$.

(ii) $\int \sqrt{a^2+\sqrt{b^2+c/x}}\,dx$. [MATH. TRIP., 1898.]

22. Integrate $\int \frac{(\sqrt{a}-\sqrt{x})^2\,dx}{(a^2+ax+x^2)\sqrt{x}}$. [COLLEGES, 1896.]

23. Integrate $\int \frac{5x^3+3x-1}{(x^3+3x+1)^3}dx$. [J. M. SCH., OX., 1904.]

24. Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos^5 x}dx$. [ST. JOHN'S, 1892.]

25. Integrate $\int \frac{dx}{x(x-a)^n}$, n being a positive integer.

[ST. JOHN'S, 1882.]

26. Integrate $\int \left(\frac{x-b}{x-a}\right)^{\frac{n}{2}} dx$. [COLLEGES α, 1885.]

27. Integrate $\int \frac{dx}{1+3e^x+2e^{2x}}$. [MATH. TRIP., 1895.]

28. Sum the series

$$\frac{x^3}{1.3}+\frac{x^5}{3.5}+\frac{x^7}{5.7}+\dots \text{ ad inf.},$$

assuming it to be convergent.

Deduce that

$$\frac{1}{1.3}\cdot\frac{1}{2^3}+\frac{1}{3.5}\cdot\frac{1}{2^5}+\frac{1}{5.7}\cdot\frac{1}{2^7}+\dots \text{ ad inf.}=\frac{1}{4}-\frac{3}{16}\log 3.$$

[I. C. S., 1899.]

29. Prove that

$$1-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\dots \text{ ad inf.}=\frac{\pi}{8}(1+\sqrt{2}).$$

[COLLEGES β, 1888.]

30. Evaluate $\int x^2\log(1-x^2)\,dx,$

and deduce that

$$\frac{1}{1.5}+\frac{1}{2.7}+\frac{1}{3.9}+\dots=\frac{8}{9}-\frac{2}{3}\log 2.$$

[COLLEGES α, 1889.]

31. Integrate $\int (a^4+x^4)^{-1}\,dx.$

Prove that

$$\frac{1}{1.5}+\frac{1}{9.13}+\frac{1}{17.21}+\frac{1}{25.29}+\dots=\frac{\sqrt{2}}{32}\left\{\pi+\log(3+2\sqrt{2})\right\}.$$

[MATH. TRIP., 1896.]

32. Show that

$$\int\frac{dx}{x(x+1)(x+2)(x+3)\dots(x+n)}=\frac{1}{\underline{|n}}\sum_{r=0}^{r=n}(-1)^r\,{}^nC_r\log(x+r).$$

33. Show that

$$\int\frac{(1+x)^n}{(1-2x)^3}dx=\frac{3^n}{2^{n+2}}\frac{1}{(1-2x)^2}-n\frac{3^{n-1}}{2^{n+1}}\frac{1}{1-2x}-\frac{n(n-1)\,3^{n-2}}{2^{n+2}}\log(1-2x)$$

+ a rational integral algebraic expression of a finite number of terms.

34. Show that if $c<1$, $\int\frac{dx}{(1-x)(1-cx)(1-c^2x)\dots\text{to }\infty}$

$$=x+\frac{1}{2}\frac{x^2}{1-c}+\frac{1}{3}\frac{x^3}{(1-c)(1-c^2)}+\frac{1}{4}\frac{x^4}{(1-c)(1-c^2)(1-c^3)}+\dots\text{ to }\infty.$$

35. Show that $\int\frac{x^{n+2}\,dx}{(x-a_1)(x-a_2)(x-a_3)\dots(x-a_n)}$

$$=\frac{x^3}{3}+H_1\frac{x^2}{2}+H_2x+\sum\frac{a_1{}^{n+2}}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)}\log(x-a_1),$$

where H_r is the sum of the homogeneous products r at a time of $a_1, a_2, \dots, a_n$.

36. Show that the part of the indefinite integral

$$\int \frac{1}{x^3} \frac{f(x)}{\phi(x)}\, dx$$

which becomes infinite when $x=0$, f and ϕ being rational integral functions of x which do not vanish when $x=0$, is

$$-\frac{1}{2x^2}\frac{f(0)}{\phi(0)} - \frac{1}{x}\frac{f'(0)\phi(0)-\phi'(0)f(0)}{[\phi(0)]^2}$$

$$+\frac{1}{2}\log x \frac{f''(0)\{\phi(0)\}^2 - 2f'(0)\phi'(0)\phi(0) - f(0)[\phi(0)\phi''(0) - 2\{\phi'(0)\}^2]}{[\phi(0)]^3}.$$

[Ox. I. P., 1901.]

37. Show that when a rational fraction is decomposed into its simple or "partial" fractions, the decomposition is unique.

38. If $F(x)$ be a function of the $(n-1)^{\text{th}}$ degree which assumes the values $u_1, u_2, u_3, \ldots, u_n$ when $x = x_0, x_1, x_2, \ldots, x_n$ respectively, show that

$$F(x) = u_1 \frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}$$

$$+ u_2 \frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ u_n \frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}.$$

39. Prove that if $p < n+1$,

$$na^{n-p}\int \frac{x^{p-1}\,dx}{x^n - a^n} = \log(x-a) + \sum_{r=1}^{r=\frac{n-1}{2}} \cos\frac{2rp\pi}{n} \log\left(x^2 - 2ax\cos\frac{2r\pi}{n} + a^2\right)$$

$$-2\sum_{r=1}^{r=\frac{n-1}{2}} \sin\frac{2rp\pi}{n} \tan^{-1}\frac{x - a\cos\frac{2r\pi}{n}}{a\sin\frac{2r\pi}{n}}$$

if n be odd,

and

$$= \log(x-a) + (-1)^p \log(x+a)$$

$$+ \sum_{r=1}^{r=\frac{n-2}{2}} \cos\frac{2rp\pi}{n} \log\left(x^2 - 2ax\cos\frac{2r\pi}{n} + a^2\right)$$

$$-2\sum_{r=1}^{r=\frac{n-2}{2}} \sin\frac{2rp\pi}{n} \tan^{-1}\frac{x - a\cos\frac{2r\pi}{n}}{a\sin\frac{2r\pi}{n}} \quad \text{if } n \text{ be even.}$$

40. Prove that if $p < n+1$,

$$na^{n-p}\int\frac{x^{p-1}}{x^n+a^n}dx = (-1)^{p-1}\log(x+a)$$

$$-\sum_{r=1}^{r=\frac{n-1}{2}}\cos(2r-1)\frac{p\pi}{n}\log\left\{x^2-2ax\cos(2r-1)\frac{\pi}{n}+a^2\right\}$$

$$+2\sum_{r=1}^{r=\frac{n-1}{2}}\sin(2r-1)\frac{p\pi}{n}\tan^{-1}\frac{x-a\cos(2r-1)\frac{\pi}{n}}{a\sin(2r-1)\frac{\pi}{n}}$$

if n be odd,

and $$= -\sum_{r=1}^{r=n}\cos(2r-1)\frac{p\pi}{n}\log\left\{x^2-2ax\cos(2r-1)\frac{\pi}{n}+a^2\right\}$$

$$+2\sum_{r=1}^{r=\frac{n}{2}}\sin(2r-1)\frac{p\pi}{n}\tan^{-1}\frac{x-a\cos(2r-1)\frac{\pi}{n}}{a\sin(2r-1)\frac{\pi}{n}}$$ if n be even.

41. Prove that

$$\int_0^x\frac{dx}{1-x^{2n}} = \frac{1}{2n}\sum_{r=0}^{r=n-1}\left(\cos\frac{r\pi}{n}\tanh^{-1}\frac{2x\cos\frac{r\pi}{n}}{1+x^2}+\sin\frac{r\pi}{n}\tan^{-1}\frac{2x\sin\frac{r\pi}{n}}{1-x^2}\right).$$

[MATH. TRIP., 1884.]

42. Show that $$\int_0^\infty\frac{t}{t^5+1}\,dt = \frac{4\pi}{5\sqrt{10+2\sqrt{5}}}.$$

43. (i) Show that the remainder left after dividing the rational integral function $f(x)$ by $(x-c)^2+b^2$ is

$$\left[f(c)-\frac{b^2}{2!}f^{(2)}(c)+\frac{b^4}{4!}f^{(4)}(c)-\ldots+(-1)^r\frac{b^{2r}}{(2r)!}f^{(2r)}(c)+\ldots\right]$$

$$+(x-c)\left[f'(c)-\frac{b^3}{3!}f^{(3)}(c)+\frac{b^5}{5!}f^{(5)}(c)-\ldots\right.$$

$$\left.+(-1)^r\frac{b^{2r+1}}{(2r+1)!}f^{(2r+1)}(c)+\ldots\right],$$

where $f^{(s)}(c)$ denotes $\frac{d^sf(c)}{dc^s}$.

(ii) If $f(x)$ and $\phi(x)$ are rational integral functions of x, and $\phi(x)$ does not contain $(x-c)^2+b^2$ as a factor, show that it is possible to determine finite values for the constants P and Q in such a manner that

$$f(x)-[P(x-c)+Q]\phi(x)$$

is divisible, without remainder, by $(x-c)^2+b^2$.

(iii) Apply the last result to show (or prove in any manner) that

$$\frac{f(x)}{[(x-c)^2+b^2]^r\phi(x)}$$

can be expressed in the form

$$\frac{\chi(x)}{\phi(x)}+\sum_{n=1}^{n=r}\frac{P_n(x-c)+Q_n}{[(x-c)^2+b^2]^n},$$

r being a positive integer, $\chi(x)$ a rational integral function of x, and P_n and Q_n constants. [I.C.S., 1892.]

44. If $$\frac{F(x)}{f(x)}\equiv\phi(x)+\frac{\psi(x)}{f(x)},$$

where F, f, ϕ, ψ are rational polynomials of degrees $m+n$, n, m, $n-1$ respectively, show that if $a_1, a_2, a_3, \ldots, a_n$ be the roots of $f(x)=0$, considered all different, $\psi(x)$ will be determinable from

$$\begin{vmatrix} \psi(x), & x^{n-1}, & x^{n-2}, & \ldots, & x, & 1 \\ F(a_1), & a_1^{n-1}, & a_1^{n-2}, & \ldots, & a_1, & 1 \\ F(a_2), & a_2^{n-1}, & a_2^{n-2}, & \ldots, & a_2, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F(a_n), & a_n^{n-1}, & a_n^{n-2}, & \ldots, & a_n, & 1 \end{vmatrix}=0.$$

Also determine $\psi(x)$ when $f(x)=0$ has equal roots.

[OXFORD I. P., 1913.]

45. Integrate $$\int\left\{\frac{ax^2+bx+c}{(x-\alpha)(x-\beta)(x-\gamma)}\right\}^2dx.$$ [OXFORD I. P., 1917.]

46. Prove that if n be a positive integer,

(i) $$\int\frac{\cos(n-2p)\theta}{\sin n\theta}\,d\theta=\frac{1}{n}\sum_{r=1}^{r=n}\cos\frac{2pr\pi}{n}\log\sin\left(\theta-\frac{r\pi}{n}\right),$$

(ii) $$\int\frac{\sin(n-2p)\theta}{\sin n\theta}\,d\theta=\frac{1}{n}\sum_{r=1}^{r=(n-1)}\sin\frac{2pr\pi}{n}\log\operatorname{cosec}\left(\theta-\frac{r\pi}{n}\right).$$

47. Integrate (i) $\displaystyle\int\frac{dx}{1+e^x}$, (ii) $\displaystyle\int\frac{dx}{\sqrt{\sin^3x\sin(x+\alpha)}}$,

and prove that (iii) $$\int_0^\infty\frac{x\,dx}{(1+x)(1+x^2)}=\frac{\pi}{4},$$

(iv) $$\int_1^\infty\frac{(x^2+3)\,dx}{x^6(x^2+1)}=\frac{1}{30}(58-15\pi).$$

[MATH. TRIP. I., 1917.]

48. Obtain the rational part of $\displaystyle\int\frac{dx}{(x^5+1)^3}$. [MATH. TRIP. II., 1915.]

49. Prove that

$$\frac{2n}{(1+x)^{2n}+(1-x)^{2n}}=\sum_{r=0}^{r=n}\frac{\sin a_r \cos^{2n-2} a_r}{\sin(2n-1)a_r}\,\frac{1}{x^2+\tan^2 a_r},$$

where $a_r=(2r+1)\pi/4n$. [Oxf. II. P., 1899.]

Write down the values of the integrals

$$\int\frac{dx}{(1+x)^{2n}+(1-x)^{2n}},\quad \int\frac{x\,dx}{(1+x)^{2n}+(1-x)^{2n}}.$$

50. Show that

$$\int_0^\infty\frac{x\,dx}{(a+x)^n-(a-x)^n}=\frac{\pi}{2na^{n-2}}\Sigma(-1)^{\lambda-1}\sin\frac{\lambda\pi}{n}\cos^{n-3}\frac{\lambda\pi}{n},$$

the summation extending from $\lambda=1$ to $\lambda=\dfrac{n-1}{2}$ or to $\lambda=\dfrac{n-2}{2}$, according as n is odd or even.

[Cf. Wolstenholme's *Problems*, No. 1912.]

Write down the value of the integral

$$\int\frac{x^2dx}{(a+x)^n-(a-x)^n}.$$

51. Show that if n be even and $x+y=1$,

$$\int\frac{dx}{x^n y^n}=\frac{1}{n-1}\left[\frac{1}{y^{n-1}}-\frac{1}{x^{n-1}}\right]+\frac{n}{1}\cdot\frac{1}{n-2}\left[\frac{1}{y^{n-2}}-\frac{1}{x^{n-2}}\right]$$
$$+\frac{n(n+1)}{1\,.\,2}\cdot\frac{1}{n-3}\left[\frac{1}{y^{n-3}}-\frac{1}{x^{n-3}}\right]+\ldots+\frac{n(n+1)\ldots(2n-2)}{1\,.\,2\ldots(n-1)}\log\frac{x}{y}.$$

[Murphy, *Camb. Tr.*, vi.]

52. Show that if $p<q$,

$$Lt_{n=\infty}\frac{\prod_{r=1}^{r=n}\left(1-\dfrac{p^2x^2}{r^2\pi^2}\right)}{\prod_{r=1}^{r=n}\left(1-\dfrac{q^2x^2}{r^2\pi^2}\right)}=\sum_{r=1}^{r=\infty}\frac{2r\pi}{pq}\,\frac{\sin\left(\dfrac{p}{q}r\pi\right)}{\cos r\pi}\,\frac{1}{x^2-\dfrac{r^2\pi^2}{q^2}}.$$

[Todhunter, *I.C.*, p. 38.]

Deduce that if $p<q$,

$$\int\frac{\sin px}{\sin qx}dx=-2\sum_{r=1}^{r=\infty}\frac{1}{q}\,\frac{\sin\left(\dfrac{p}{q}r\pi\right)}{\cos r\pi}\tanh^{-1}\frac{qx}{r\pi}\quad(-\pi<qx<\pi).$$