CHAPTER IV

INTEGRATION BY PARTS. POWERS OF SINES AND COSINES.

INTEGRATION BY PARTS.

90. Let u and w be functions of x, and let accents denote differentiations, and suffixes integrations, with respect to x.

Thus u'' stands for $\frac{d^2u}{dx^2}$ and w_2 for $\int\left[\int w\,dx\right]dx$, and so on with u''', w_3, etc.

Then $$\frac{d}{dx}(uw)=u\frac{dw}{dx}+w\frac{du}{dx}$$

which we may write as

$$(uw)'=uw'+wu'.$$

It follows that $$uw=\int uw'\,dx+\int wu'\,dx$$

or $$\int uw'\,dx=uw-\int wu'\,dx.$$

This may be put into another form.

Let $u=\phi(x)$ and $w'\left(i.e.\ \frac{dw}{dx}\right)=\psi(x)=v$, say; so that

$$w=\int\psi(x)\,dx=v_1.$$

Then the above rule may be written

$$\int\phi(x)\,\psi(x)\,dx=\phi(x)\left\{\int\psi(x)\,dx\right\}-\int\phi'(x)\left\{\int\psi(x)\,dx\right\}dx,$$

i.e. $$\int uv\,dx=uv_1-\int u'v_1\,dx,$$

or the two functions ϕ and ψ may be interchanged, and then

$$\int \phi(x)\,\psi(x)\,dx = \psi(x)\left\{\int \phi(x)\,dx\right\} - \int \psi'(x)\left\{\int \phi(x)\,dx\right\}dx;$$

i.e.
$$\int uv\,dx = vu_1 - \int v'u_1\,dx.$$

Thus, in integrating the product of two functions, if the integral be not at once obtainable, it is possible to connect the integral

$$\int \phi(x)\,\psi(x)\,dx$$

with either of two new integrals, viz. those of

$$\int \phi'(x)\left\{\int \psi(x)\,dx\right\}dx, \quad \int \psi'(x)\left\{\int \phi(x)\,dx\right\}dx,$$

and supposing that the integral of one of the two factors $\phi(x)$, $\psi(x)$ is known, one of these new integrals *may be more easily obtainable than that of the original product.*

91. The rule may be put into words thus:

Int. of Prod. $\phi\,.\,\psi = 1^{\text{st}}$ function $\times$ Integral of 2^{nd}
$-$ Integral of [Diff. Co. of $1^{\text{st}} \times$ Int. of 2^{nd}].

92. Ex.
$$\int x \sin nx\,dx.$$

Here it is important to connect if possible $\int x\sin nx\,dx$ with another *in which the factor x has been removed*. There is a choice as to whether we put $u = x$ and $v = \sin nx$
or $u = \sin nx$ and $v = x$;
but it will be observed that in the connected integral $\int u'v_1 dx$, u has been differentiated, v integrated. Hence the removal of x will be effected if we take the first alternative.

Then $\quad u = x, \quad u' = 1, \quad v = \sin nx, \quad v_1 = -\dfrac{\cos nx}{n}.$

Thus, by the rule,

$$\begin{aligned}\int x\sin nx\,dx &= x\left[-\frac{\cos nx}{n}\right] - \int 1\left[-\frac{\cos nx}{n}\right]dx \\ &= -\frac{x\cos nx}{n} + \frac{1}{n}\int \cos nx\,dx \\ &= -\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}.\end{aligned}$$

93. It is to be noted that *unity* may be regarded as one of the factors to aid an integration.

Thus
$$\int \log x\,dx = \int 1 \log x\,dx$$
$$= x\log x - \int x\frac{d}{dx}(\log x)\,dx$$
$$= x\log x - \int x\frac{1}{x}\,dx$$
$$= x\log x - \int 1\,dx$$
$$= x\log x - x = x(\log_e x - 1),$$

or as it may be written $= x\log_e\left(\dfrac{x}{e}\right)$.

94. Repetition of the Operation.

The operation of integration by parts may be repeated as often as may be considered necessary for the evaluation of the original integral.

Thus
$$\int x^4\sin nx\,dx = (x^4)\left(-\frac{\cos nx}{n}\right) - \int(4x^3)\left(-\frac{\cos nx}{n}\right)dx,$$
$$\int(4x^3)\left(-\frac{\cos nx}{n}\right)dx = (4x^3)\left(-\frac{\sin nx}{n^2}\right) - \int(4\,.\,3x^2)\left(-\frac{\sin nx}{n^2}\right)dx,$$
$$\int(4\,.\,3x^2)\left(-\frac{\sin nx}{n^2}\right)dx = (4\,.\,3x^2)\left(+\frac{\cos nx}{n^3}\right) - \int(4\,.\,3\,.\,2\,.\,x)\left(+\frac{\cos nx}{n^3}\right)dx$$
$$\int(4.3.2.x)\left(+\frac{\cos nx}{n^3}\right)dx = (4.3.2.x)\left(\frac{\sin nx}{n^4}\right) - \int(4\,.\,3\,.\,2\,.\,1)\left(\frac{\sin nx}{n^4}\right)dx,$$
$$\int(4\,.\,3\,.\,2\,.\,1)\left(\frac{\sin nx}{n^4}\right)dx = (4\,.\,3\,.\,2\,.\,1)\left(-\frac{\cos nx}{n^5}\right).$$

Then adding and subtracting alternately,
$$\int x^4\sin nx\,dx = (x^4)\left(-\frac{\cos nx}{n}\right) - (4x^3)\left(-\frac{\sin nx}{n^2}\right)$$
$$+(4\,.\,3x^2)\left(+\frac{\cos nx}{n^3}\right) - (4\,.\,3\,.\,2\,x)\left(\frac{\sin nx}{n^4}\right)$$
$$+(4\,.\,3\,.\,2\,.\,1)\left(-\frac{\cos nx}{n^5}\right).$$

The student will note that no arithmetical simplification is attempted until the whole operation is complete. The total operation is much less liable to error if simplification be postponed to the end.

We now obviously have
$$\int x^4\sin nx\,dx = P\cos nx + Q\sin nx,$$
where
$$P = -\frac{x^4}{n} + 4\,.\,3\frac{x^2}{n^3} - \frac{4\,.\,3\,.\,2\,.\,1}{n^5},$$
$$Q = 4\frac{x^3}{n^2} - 4\,.\,3\,.\,2\frac{x}{n^4}.$$

95. The General Rule.

It is obviously possible to formulate a general rule for the repeated operation. And such a method is most serviceable in practice.

The rule is

$$\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots + (-1)^{n-1}u^{(n-1)}v_n$$
$$+ (-1)^n \int u^{(n)}v_n\, dx,$$

where $u^{(n-1)}$ is written for u with $n-1$ accents, *i.e.* the $(n-1)^{\text{th}}$ differential coefficient of u.

For

$$\begin{aligned}
\int uv\, dx &= uv_1 - \int u'v_1\, dx,\\
\int u'v_1\, dx &= u'v_2 - \int u''v_2\, dx,\\
\int u''v_2\, dx &= u''v_3 - \int u'''v_3\, dx,\\
\int u'''v_3\, dx &= u'''v_4 - \int u''''v_4\, dx,\\
&\text{etc.} = \text{etc.},\\
\int u^{(n-2)}v_{n-2}\, dx &= u^{(n-2)}v_{n-1} - \int u^{(n-1)}v_{n-1}\, dx,\\
\int u^{(n-1)}v_{n-1}\, dx &= u^{(n-1)}v_n - \int u^{(n)}v_n\, dx.
\end{aligned}$$

Hence, adding and subtracting alternately,

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots + (-1)^{n-1}u^{(n-1)}v_n$$
$$+ (-1)^n \int u^{(n)}v_n\, dx.$$

Ex. 1. Thus applying this to the last example (Art. 94),

$$\int x^4 \sin nx\, dx = (x^4)\left(-\frac{\cos nx}{n}\right) - (4x^3)\left(-\frac{\sin nx}{n^2}\right)$$
$$+ (4\,.\,3x^2)\left(+\frac{\cos nx}{n^3}\right) - (4\,.\,3\,.\,2x)\left(\frac{\sin nx}{n^4}\right)$$
$$+ (4\,.\,3\,.\,2\,.\,1)\left(-\frac{\cos nx}{n^5}\right),$$

each term being derived from the preceding by the simple rule of "**diff. 1st factor and integ. 2nd**" and connecting by alternate signs. When one of the factors is a rational integral algebraic polynomial, it is ultimately destroyed by the successive differentiations.

Ex. 2. $$\int x^m e^{ax}dx = x^m\frac{e^{ax}}{a} - mx^{m-1}\frac{e^{ax}}{a^2} + m(m-1)x^{m-2}\frac{e^{ax}}{a^3}$$
$$- m(m-1)(m-2)x^{m-3}\frac{e^{ax}}{a^4} + \dots + (-1)^m m!\,\frac{e^{ax}}{a^{m+1}}.$$

96. If one of the subsidiary integrals *returns to the original form*, this fact may be utilized to infer the result of the integration.

Ex. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx \quad\text{.....................(i)}$$

and $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx. \quad\text{..................(ii)}$$

Hence, if $P \equiv \int e^{ax}\sin bx\,dx$ and $Q \equiv \int e^{ax}\cos bx\,dx$,

$$P = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left[\frac{e^{ax}}{a}\cos bx + \frac{b}{a}P\right]$$

and $$Q = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\frac{e^{ax}}{a}\sin bx - \frac{b}{a}Q\right],$$

whence $$P = e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2},$$

$$Q = e^{ax}\frac{b\sin bx + a\cos bx}{a^2+b^2}.$$

Or we might have written equations (i) and (ii) as

$$\left.\begin{aligned} aP + bQ &= e^{ax}\sin bx,\\ -bP + aQ &= e^{ax}\cos bx,\end{aligned}\right\}\text{ and then solve for } P \text{ and } Q.$$

We may write P and Q as follows:

$$\left.\begin{aligned} P &= (a^2+b^2)^{-\frac{1}{2}}e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right),\\ Q &= (a^2+b^2)^{-\frac{1}{2}}e^{ax}\cos\left(bx - \tan^{-1}\frac{b}{a}\right),\end{aligned}\right\}$$

forms which are frequently useful and which are *derivable at once from the formula for the* n^{th} *differential coefficient*, viz.

$$\frac{d^n}{dx^n}e^{ax}\begin{matrix}\sin\\ \cos\end{matrix}bx = (a^2+b^2)^{\frac{n}{2}}e^{ax}\begin{matrix}\sin\\ \cos\end{matrix}\left(bx + n\tan^{-1}\frac{b}{a}\right),$$

by putting $n = -1$. [*Diff. Calc.*, Art. 93.]

And this is what we should be led to expect. For if to differentiate $e^{ax}\begin{matrix}\sin\\ \cos\end{matrix}(bx+c)$ is the same as to multiply it by $\sqrt{a^2+b^2}$ and to increase the angle by $\tan^{-1}\frac{b}{a}$, the effect of integration, which is the inverse operation, must be to divide out by the factor $\sqrt{a^2+b^2}$ and to diminish the angle by $\tan^{-1}\frac{b}{a}$.

And it is in this form, viz.

$$\frac{e^{ax}}{\sqrt{a^2+b^2}}\frac{\sin}{\cos}\left(bx+c-\tan^{-1}\frac{b}{a}\right),$$

that the integration of $\int e^{ax}\frac{\sin}{\cos}(bx+c)\,dx$ is most easily remembered.

97. In cases of the form

$e^{ax}\sin bx\sin cx\sin dx,\quad e^{ax}\sin^p x\cos^q x,\quad e^{ax}\sin^p x\cos nx$, etc.,

p and q being positive integers, the trigonometrical factor must first be expressed as the sum of a series of sines or cosines of multiples of x by trigonometrical means, and then each term being of form $e^{ax}\frac{\sin}{\cos}mx$ can be integrated.

98. Ex. 1. $$I=\int e^x\sin 2x\cos x\,dx.$$

Now $$\sin 2x\cos x=\tfrac{1}{2}(\sin 3x+\sin x);$$

$$\therefore\ I=\tfrac{1}{2}\int e^x(\sin 3x+\sin x)\,dx$$

$$=\tfrac{1}{2}e^x\left[\frac{1}{\sqrt{10}}\sin(3x-\tan^{-1}3)+\frac{1}{\sqrt{2}}\sin\left(x-\frac{\pi}{4}\right)\right].$$

Ex. 2. $$I=\int e^{3x}\sin^2 x\cos^3 x\,dx.$$

Now $$\sin^2 x\cos^3 x=\tfrac{1}{4}\sin^2 2x\cos x=\tfrac{1}{8}(1-\cos 4x)\cos x$$

$$=\tfrac{1}{16}(2\cos x-\cos 3x-\cos 5x);$$

$$\therefore\ \int e^{3x}\sin^2 x\cos^3 x\,dx=\tfrac{1}{16}\int e^{3x}(2\cos x-\cos 3x-\cos 5x)\,dx$$

$$=\frac{e^{3x}}{16}\left[\frac{2}{\sqrt{10}}\cos\left(x-\tan^{-1}\frac{1}{3}\right)-\frac{1}{3\sqrt{2}}\cos\left(3x-\frac{\pi}{4}\right)-\frac{1}{\sqrt{34}}\cos\left(5x-\tan^{-1}\frac{5}{3}\right)\right]$$

Ex. 3. Integrate $\int\sqrt{a^2-x^2}\,dx$ by "Parts."

$$\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}-\int x\frac{d}{dx}\sqrt{a^2-x^2}\,dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx$$

[Note this step. Some such rearrangement is frequently necessary.]

$$=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}-\int\sqrt{a^2-x^2}\,dx,$$

whence, transposing and dividing by 2,

$$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a},$$

which agrees with the result of Art. 78 obtained by the method of substitution of $a\sin\theta$ for x.

99. The method of Integration by "Parts" shows immediately that whenever a direct function $\phi(x)$ can be integrated, so also can the corresponding inverse function $\phi^{-1}(x)$, *i.e.* if

$\int \phi(x)\,dx$ can be found, so also can $\int \phi^{-1}(x)\,dx$ be found.

For, putting $\phi^{-1}(x) = z$,

$$x = \phi(z) \quad \text{and} \quad dx = \phi'(z)\,dz.$$

Hence
$$\int \phi^{-1}(x)\,dx = \int z\phi'(z)\,dz$$
$$= z\phi(z) - \int \phi(z)\,dz,$$

which establishes the rule.

100. **Geometrical Consideration.**

This is no more than might have been anticipated from geometrical considerations.

Let PQ be any arc of a curve referred to rectangular axes Ox, Oy, and let the coordinates of P be (x_0, y_0) and of Q (x_1, y_1). Let the equation of the curve be $y = \phi(x)$; or if x, y be expressed in terms of a single variable t, let the equations of the curve be

$$x = f_1(t) \equiv u, \text{ say},$$
$$y = f_2(t) \equiv v, \text{ say};$$

and let t_0 and t_1 be the values of t corresponding to the values x_0, y_0 and x_1, y_1, of x and y respectively.

Let PN, QM be the ordinates and PN_1, QM_1 the abscissae of the points P, Q. Then plainly

$$\text{area } PNMQ = \text{rect. } OQ - \text{rect. } OP - \text{area } PQM_1N_1.$$

But
$$\text{area } PNMQ = \int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1} \phi(x)\,dx,$$

$$\text{area } PQM_1N_1 = \int_{y_0}^{y_1} x\,dy = \int_{y_0}^{y_1} \phi^{-1}(y)\,dy.$$

Also
$$\text{rect. } OQ = x_1y_1 \quad \text{and} \quad \text{rect. } OP = x_0y_0.$$

Thus $$\int_{x_0}^{x_1} y\,dx = (x_1 y_1 - x_0 y_0) - \int_{y_0}^{y_1} x\,dy, \quad \text{..............(1)}$$

i.e. $$\int_{x_0}^{x_1} \phi(x)\,dx = (x_1 y_1 - x_0 y_0) - \int_{y_0}^{y_1} \phi^{-1}(y)\,dy.$$

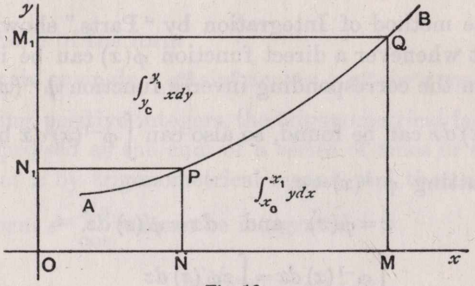

Fig. 16.

Hence the dependence of the one integral upon the other is obvious, and to establish the possibility of calculating the area $PNMQ$ is to establish incidentally the possibility of obtaining the area of PQM_1N_1.

Further, $$\int_{x_0}^{x_1} y\,dx = \int_{f_1(t_0)}^{f_1(t_1)} v\,du = \int_{t_0}^{t_1} v\frac{du}{dt}\,dt$$

and $$\int_{y_0}^{y_1} x\,dy = \int_{f_2(t_0)}^{f_2(t_1)} u\,dv = \int_{t_0}^{t_1} u\frac{dv}{dt}\,dt$$

and $$x_1 y_1 - x_0 y_0 = \Big[\,uv\,\Big]_{t_0}^{t_1}.$$

So that the equation (1) may be written

$$\int_{t_0}^{t_1} v\frac{du}{dt}\,dt = \Big[\,uv\,\Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} u\frac{dv}{dt}\,dt,$$

and thus the general rule of integration by parts is established geometrically.

The meaning of the process is therefore this: In cases where there is a difficulty in finding the area $PNMQ$, we may find instead the area PQM_1N_1 and deduce the former result from the latter.

EXAMPLES.

Integrate by parts

1. xe^{3x}, x^2e^{ax}, x^5e^{-x}, $x\cosh x$, $x^2\sinh x$.
2. $x\cos x$, $x^5\cos 2x$, $x^2\cos^2 x$, $x^2\cos 3x\sin x$, $x\sin x\sin 2x\sin 3x$.
3. $e^x\sin 2x$, $e^x\sin^2 x$, $e^{3x}\sin^3 x\cos x$, $e^{-5x}\cos x\sin^2 x\cos 3x$.
4. $x^3\log x$, $x^n\log x$, $x^n(\log x)^2$, $x^n(\log x)^3$.
5. $e^{ax}\sin px\sin qx\sin rx$, $e^{ax}\sin px\sin qx\cos rx$.
6. $e^{ax}\sin px\sin qx\cos^2 rx$, $e^{ax}\cos px\cos qx\cos^2(p+q)x$.
7. Evaluate
$$\int_0^\pi x\sin x\,dx,\quad \int_0^{\frac{\pi}{2}} x^2\cos x\,dx,\quad \int_0^{\frac{\pi}{2}} x^2\cos 2x\,dx.$$
8. Integrate
$$\int\sin^{-1}x\,dx,\quad \int x\sin^{-1}x\,dx,\quad \int x^3\sin^{-1}x\,dx,\quad \int x\tan^{-1}x\,dx.$$

101. Reduction Formulae.

It not infrequently occurs that a function which it is desired to integrate is not immediately integrable or reducible by substitution to one or other of the standard forms whose integrals have been committed to memory. But it may happen in such a case that the integral may be connected in a linear manner with the integral of another function, or with the integrals of other functions, which are simpler or easier to integrate than the original function.

Such a connecting formula is called a Reduction Formula. Thus an integration by parts makes one integral depend upon a second integral, and is a Reduction Formula.

Many Formulae of this type will be found and used in subsequent chapters.

102. We have seen how a repetition of the process of integration by parts will enable us to calculate the integrals

$$S_m=\int x^m\sin nx\,dx,\quad C_m=\int x^m\cos nx\,dx.$$

We propose to construct "Reduction Formulae" for these integrals, giving S_m, C_m in terms of S_{m-2}, C_{m-2} respectively.

Integrating by parts, we have at once

$$S_m=-x^m\frac{\cos nx}{n}+\frac{m}{n}C_{m-1}$$

and
$$C_m=x^m\frac{\sin nx}{n}-\frac{m}{n}S_{m-1}.$$

Thus,

$$S_m = -x^m\frac{\cos nx}{n} + \frac{m}{n}\left[\ x^{m-1}\frac{\sin nx}{n} - \frac{m-1}{n}S_{m-2}\right],$$

and $$C_m = \ x^m\frac{\sin nx}{n} - \frac{m}{n}\left[-x^{m-1}\frac{\cos nx}{n} + \frac{m-1}{n}C_{m-2}\right],$$

i.e. $$S_m = -x^m\frac{\cos nx}{n} + mx^{m-1}\frac{\sin nx}{n^2} - \frac{m(m-1)}{n^2}S_{m-2},$$

and $$C_m = \ x^m\frac{\sin nx}{n} + mx^{m-1}\frac{\cos nx}{n^2} - \frac{m(m-1)}{n^2}C_{m-2}.$$

Thus, when the four integrals for the cases $m=0$ and $m=1$ are found, viz.

$$S_0 = \int \sin nx\,dx = -\frac{\cos nx}{n}, \quad C_0 = \int \cos nx\,dx = \frac{\sin nx}{n},$$

$$S_1 = \int x \sin nx\,dx = -x\frac{\cos nx}{n} + \frac{\sin nx}{n^2},$$

$$C_1 = \int x \cos nx\,dx = x\frac{\sin nx}{n} + \frac{\cos nx}{n^2},$$

all others can be deduced by *successive applications* of the above formulae.

This *illustrates the use of a reduction formula.* But for expressions like $x^m \sin nx$, $x^m \cos nx$ it is ordinarily better in practice to apply the method of Art. 95 at once and avoid the successive substitutions.

Examples.

Write down the integrals of

1. $\int x^6 e^x\,dx$, $\int x^5 \sinh x\,dx$, $\int x^5 \cosh^2 x\,dx$.

2. $\int_0^{\frac{\pi}{2}} x^3 \sin x\,dx$, $\int_0^{\frac{\pi}{2}} x^3 \sin^2 x\,dx$, $\int_0^{\frac{\pi}{2}} x^4 \sin x \cos x\,dx$.

3. $\int_0^{\pi} x^5 \sin x\,dx$, $\int_0^{\pi} x^5 \cos^2 x\,dx$, $\int_0^1 x^3 \cosh x\,dx$.

4. $\int_0^{\frac{\pi}{2}} x^3(a^2\cos^2 x + b^2 \sin^2 x)\,dx$, $\int_1^3 x^3 \log x\,dx$, $\int_0^1 x \tan^{-1} x\,dx$.

5. $\int_0^{\frac{\pi}{2}} e^x \sin x \cos^2 x\,dx$, $\int_0^{\frac{\pi}{2}} x \sin x \sin 2x \sin 3x\,dx$.

103. The Determination of the Integrals

$$\int x^n e^{ax} \sin bx\, dx, \quad \int x^n e^{ax} \cos bx\, dx,$$

may be at once effected.

For remembering

$$\int e^{ax} \frac{\sin}{\cos} bx\, dx = \frac{e^{ax}}{r} \frac{\sin}{\cos} (bx - \phi),$$

where $r = \sqrt{a^2+b^2}$ and $\tan\phi = \frac{b}{a}$, we have

$$\int x^n e^{ax} \sin bx\, dx = \frac{x^n}{r} e^{ax} \sin(bx-\phi) - \frac{nx^{n-1}}{r^2} e^{ax} \sin(bx-2\phi)$$

$$+ \frac{n(n-1)}{r^3} x^{n-2} e^{ax} \sin(bx-3\phi) - \dots$$

$$+ (-1)^n \frac{n!}{r^{n+1}} e^{ax} \sin(bx - \overline{n+1}\,\phi)$$

or $$= e^{ax}(P \sin bx - Q \cos bx),$$

where

$$P \equiv \frac{x^n}{r} \cos\phi - n \frac{x^{n-1}}{r^2} \cos 2\phi + n(n-1) \frac{x^{n-2}}{r^3} \cos 3\phi - \dots,$$

$$Q \equiv \frac{x^n}{r} \sin\phi - n \frac{x^{n-1}}{r^2} \sin 2\phi + n(n-1) \frac{x^{n-2}}{r^3} \sin 3\phi - \dots.$$

Similarly,

$$\int x^n e^{ax} \cos bx\, dx = e^{ax}\{P \cos bx + Q \sin bx\}.$$

104. Integration of

$$C_n = \int e^{ax} \cos^n bx\, dx, \quad S_n = \int e^{ax} \sin^n bx\, dx.$$

We may now express $\cos^n bx$ and $\sin^n bx$ in a series of cosines or sines of multiples of bx and then integrate each term by Art. 96; or we may obtain formulae connecting C_n with C_{n-2} and S_n with S_{n-2}, thus:

$$C_n = \int e^{ax} \cos^n bx\, dx = \frac{e^{ax}}{a} \cos^n bx + \int \frac{e^{ax}}{a} . nb \cos^{n-1} bx \sin bx\, dx$$

$$= \frac{e^{ax}}{a} \cos^n bx + \frac{nb}{a} \Big[\frac{e^{ax}}{a} \cos^{n-1} bx \sin bx$$

$$- \int \frac{b}{a} e^{ax} \{\cos^n bx - (n-1) \cos^{n-2} bx \sin^2 bx\}\, dx \Big]$$

$$=\frac{e^{ax}}{x}\cos^n bx+\frac{nb}{a}\Big[\frac{e^{ax}}{a}\cos^{n-1}bx\sin bx$$

$$-\int\frac{b}{a}e^{ax}\{n\cos^n bx-(n-1)\cos^{n-2}bx\}\,dx\Big];$$

$$\therefore\ \left(1+\frac{n^2b^2}{a^2}\right)C_n=\frac{e^{ax}}{a}\cos^n bx+\frac{nb}{a^2}e^{ax}\cos^{n-1}bx\sin bx$$

$$+n(n-1)\frac{b^2}{a^2}C_{n-2}.$$

Hence

$$C_n=e^{ax}\cos^{n-1}bx\,\frac{a\cos bx+nb\sin bx}{a^2+n^2b^2}+\frac{n(n-1)b^2}{a^2+n^2b^2}C_{n-2}.$$

Similarly

$$S_n=e^{ax}\sin^{n-1}bx\,\frac{a\sin bx-nb\cos bx}{a^2+n^2b^2}+\frac{n(n-1)b^2}{a^2+n^2b^2}S_{n-2}.$$

And as $\int e^{ax}\,dx$, $\int e^{ax}\sin bx\,dx$, $\int e^{ax}\cos bx\,dx$ (that is, S_0, C_0, S_1 and C_1) can be written down (Art. 96), the integration of $\int e^{ax}\cos^n bx\,dx$ and $\int e^{ax}\sin^n bx\,dx$ can be completed, in any case where n is a positive integer, by successive reduction.

105. **Ex.** Integrate $\int e^x\sin^5 x\,dx$ (i) by the "multiple angle" method, (ii) by "reduction."

(i) Let $\cos x+\iota\sin x=y$; then $2\iota\sin x=y-\frac{1}{y}$ (see Art. 112).

$$2^5\iota^5\sin^5 x=\left(y-\frac{1}{y}\right)^5=\left(y^5-\frac{1}{y^5}\right)-5\left(y^3-\frac{1}{y^3}\right)+10\left(y-\frac{1}{y}\right)$$

$$=2\iota\sin 5x-10\iota\sin 3x+20\iota\sin x;$$

$$\therefore\ \sin^5 x=\frac{1}{2^4}(\sin 5x-5\sin 3x+10\sin x).$$

$$\therefore\int e^x\sin^5 x\,dx$$

$$=\frac{1}{2^4}\int e^x(\sin 5x-5\sin 3x+10\sin x)\,dx$$

$$=\frac{e^x}{2^4}\left[\frac{1}{\sqrt{26}}\sin(5x-\tan^{-1}5)-\frac{5}{\sqrt{10}}\sin(3x-\tan^{-1}3)+\frac{10}{\sqrt{2}}\sin\left(x-\frac{\pi}{4}\right)\right].$$

(ii) Proceeding with the reduction formula, $a=1$, $b=1$, $n=5$,

$$S_5=e^x\sin^4 x\,\frac{\sin x-5\cos x}{1^2+5^2}+\frac{5\,.\,4}{1^2+5^2}S_3.$$

Similarly $$S_3=e^x\sin^2 x\,\frac{\sin x-3\cos x}{1^2+3^2}+\frac{3\,.\,2}{1^2+3^2}S_1$$

and $$S_1=\int e^x\sin x\,dx=\frac{e^x\sin\left(x-\frac{\pi}{4}\right)}{\sqrt{1^2+1^2}};$$

$$\therefore\ S_5=e^x\left[\frac{1}{26}\sin^4 x(\sin x-5\cos x)\right.$$
$$\left.+\frac{5\,.\,4}{26}\left\{\sin^2 x\,\frac{\sin x-3\cos x}{10}+\frac{3\,.\,2}{10\sqrt{2}}\sin\left(x-\frac{\pi}{4}\right)\right\}\right].$$

106. Integrals of form $I_n=\int x^m(\log x)^n\,dx$, n being a positive integer and m not equal to -1.

Integrating by parts, we have

$$I_n=\frac{x^{m+1}}{m+1}(\log x)^n-\frac{n}{m+1}\int x^{m+1}\frac{1}{x}(\log x)^{n-1}\,dx$$

$$=\frac{x^{m+1}}{m+1}(\log x)^n-\frac{n}{m+1}\int x^m(\log x)^{n-1}\,dx,$$

i.e. $$I_n=\frac{x^{m+1}}{m+1}(\log x)^n-\frac{n}{m+1}I_{n-1}.\ \text{..............................}(1)$$

Writing l for $\log x$,

$$I_n=\frac{x^{m+1}}{m+1}l^n-\frac{n}{m+1}\left[\frac{x^{m+1}}{m+1}l^{n-1}-\frac{(n-1)}{m+1}I_{n-1}\right];$$

and proceeding in this way, we ultimately get down to I_1, which is

$$\int x^m\log x\,dx,\quad i.e.\ \frac{x^{m+1}}{m+1}l-\frac{x^{m+1}}{(m+1)^2}.$$

Hence

$$I_n=\frac{x^{m+1}}{m+1}\left[l^n-\frac{n}{m+1}l^{n-1}+\frac{n(n-1)}{(m+1)^2}l^{n-2}-\frac{n(n-1)(n-2)}{(m+1)^3}l^{n-3}\right.$$
$$\left.+\ldots+\frac{(-1)^{n-1}n!}{(m+1)^{n-1}}l+\frac{(-1)^n n!}{(m+1)^n}\right].\ \text{....}(2)$$

107. If the *definite* integral $\int_0^1 x^m(\log x)^n\,dx$ be required $(m>-1)$, note that

$$x^{m+1}(\log x)^r=0\ \text{when}\ x=1\ \text{and}\ r>0,$$

and that $$Lt_{x=0}x^{m+1}(\log x)^r=0.$$

[*Diff. Calc.*, Art. 474, Ex. 3.]

Hence

$$\Big[I_n\Big]_0^1=(-1)\frac{n}{m+1}\Big[I_{n-1}\Big]_0^1=(-1)^2\frac{n(n-1)}{(m+1)^2}\Big[I_{n-2}\Big]_0^1=\text{etc.},$$

and finally,
$$\Big[I_1\Big]_0^1=\frac{-1}{(m+1)^2}.$$

Hence
$$\Big[I_n\Big]_0^1=(-1)^n\frac{n!}{(m+1)^{n+1}},$$

$$\textit{i.e.}\ \int_0^1 x^m(\log x)^n\,dx=(-1)^n\frac{n!}{(m+1)^{n+1}},\ \ldots\ldots\ldots\ldots\ldots(3)$$

which is also directly obvious from result (2).

When $m=-1$,

$$I_n=\int\frac{1}{x}(\log x)^n\,dx=\frac{(\log x)^{n+1}}{n+1}.$$

108. The reduction formula established by integration by parts was

$$I_n=\frac{x^{m+1}}{m+1}(\log x)^n-\frac{n}{m+1}I_{n-1}.$$

We may point out that this could be obtained by the rule of "the smaller index $+1$" of Art. 217 by putting $P=x^{m+1}(\log x)^n$ and differentiating, but in this case there is no advantage in using this method, as the same formula is immediately written down by "parts" as above.

109. We may add, in passing, that $\int\frac{x^m}{\log x}dx$ cannot be integrated in finite terms except when $m=-1$. In that case, we have

$$\int\frac{1}{x\log x}dx=\log(\log x).$$

In other cases put $x=e^y$.

Then
$$\int\frac{x^m}{\log x}dx=\int\frac{e^{my}}{y}e^y\,dy=\int\frac{e^{(m+1)y}}{y}dy,$$

and expanding the exponential, we have

$$=\int\left[\frac{1}{y}+(m+1)+\frac{(m+1)^2}{\underline{|2}}y+\frac{(m+1)^3}{\underline{|3}}y^2+\ldots\right]dy$$

$$=\log y+(m+1)y+\frac{(m+1)^2}{\underline{|2}}\frac{y^2}{2}+\frac{(m+1)^3}{\underline{|3}}\frac{y^3}{3}+\ldots$$

$$=\log(\log x)+(m+1)\log x+\frac{(m-1)^2}{\underline{|2}}\frac{(\log x)^2}{2}+\frac{(m+1)^3}{\underline{|3}}\frac{(\log x)^3}{3}+\ldots,$$

and the integration is expressed as an infinite series.

110. Integrals of the form $\int x^m (\log x)^n \, dx$, where n is a negative integer, may be reduced to the above form by using the reduction formula in the reversed form, and writing n for $n-1$,

$$\int x^m (\log x)^n \, dx = \frac{x^{m+1}}{n+1} (\log x)^{n+1} - \frac{m+1}{n+1} \int x^m (\log x)^{n+1} \, dx.$$

Thus

$$\int \frac{x^2}{(\log x)^2} \, dx = -\frac{x^3}{\log x} + 3 \int \frac{x^2}{\log x} \, dx$$

$$= -\frac{x^3}{\log x} + 3\left[\log(\log x) + 3 \log x + \frac{3^2 (\log x)^2}{2 \lfloor 2} + \dots\right].$$

But as these expansions are not finite in expression, they are of but little practical importance.

111. Integrals, however, where m is negative and n is positive, can be expressed in finite terms by the reduction formulae, and present no difficulty.

Ex. $$I_3 = \int \frac{(\log x)^3}{x^{10}} \, dx.$$

$$I_3 = \frac{x^{-9}}{-9} (\log x)^3 - \frac{3}{-9} \int x^{-10} (\log x)^2 \, dx$$

$$= -\frac{1}{9} x^{-9} (\log x)^3 + \frac{3}{9}\left[\frac{x^{-9}}{-9} (\log x)^2 + \frac{2}{9} \int x^{-10} \log x \, dx\right]$$

$$= -\frac{1}{9} \frac{(\log x)^3}{x^9} - \frac{3}{9^2} \frac{(\log x)^2}{x^9} - \frac{3 \,.\, 2}{9^3} \frac{\log x}{x^9} - \frac{3 \,.\, 2 \,.\, 1}{9^4 x^9},$$

i.e. $$-\frac{1}{9x^9}\left[(\log x)^3 + \frac{3}{9} (\log x)^2 + \frac{3 \,.\, 2}{9^2} (\log x) + \frac{3 \,.\, 2 \,.\, 1}{9^3}\right].$$

Note on a Trigonometrical Process.

112. We return to the Method of Multiple Angles already introduced in Arts. 97, 105.

The process of expressing $\sin^p x \cos^q x$ in multiple angles is a matter of Trigonometry. But for the convenience of the student it is briefly indicated here, as it will be extensively required in what follows.

Remembering that

$$(\cos x + \iota \sin x)^n = \cos nx + \iota \sin nx \text{ (Demoivre)},$$

let $\quad \cos x + \iota \sin x = y; \quad$ then $\quad \cos x - \iota \sin x = \dfrac{1}{y},$

$$\cos nx + \iota \sin nx = y^n \quad \text{and} \quad \cos nx - \iota \sin nx = \frac{1}{y^n}.$$

Thus $$2\cos x = y + \frac{1}{y}, \quad 2\iota \sin x = y - \frac{1}{y},$$

$$2\cos nx = y^n + \frac{1}{y^n}, \quad 2\iota \sin nx = y^n - \frac{1}{y^n}.$$

Thus, if we require, say, $\sin^8 x$ in a series of sines or cosines of multiples of x, we proceed thus:

$$2^8 \iota^8 \sin^8 x = \left(y - \frac{1}{y}\right)^8 = y^8 + \frac{1}{y^8} - 8\left(y^6 + \frac{1}{y^6}\right) + 28\left(y^4 + \frac{1}{y^4}\right) - 56\left(y^2 + \frac{1}{y^2}\right) + 70$$

$$= 2\cos 8x - 16\cos 6x + 56\cos 4x - 112\cos 2x + 70$$

and $$\sin^8 x = \frac{1}{2^7}(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35).$$

$\sin^8 x$ thus expressed is then ready either for finding *the n^{th} differential coefficient*, or *for integration*, or for *expansion* in powers of x, as may be required.

If we required $\sin^6 x \cos^2 x$, say, in a series of sines or cosines of multiples of x, then

$$2^6 \iota^6 \sin^6 x \,.\, 2^2 \cos^2 x = \left(y - \frac{1}{y}\right)^6 \left(y + \frac{1}{y}\right)^2 \quad \text{(See the next article.)}$$

$$= y^8 + \frac{1}{y^8} - 4\left(y^6 + \frac{1}{y^6}\right) + 4\left(y^4 + \frac{1}{y^4}\right) + 4\left(y^2 + \frac{1}{y^2}\right) - 10$$

$$= 2\cos 8x - 8\cos 6x + 8\cos 4x + 8\cos 2x - 10,$$

and $$\sin^6 x \cos^2 x = \frac{1}{2^7}\left\{-\cos 8x + 4\cos 6x - 4\cos 4x - 4\cos 2x + 5\right\},$$

and is ready for integration, etc.

113. It is convenient for such examples to remember that the several sets of binomial coefficients may be quickly reproduced in the following scheme:

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

etc.,

each number being formed at once as the sum of the one immediately above it and the preceding one. Thus, in forming the seventh row,

$$0+1=1, \quad 1+5=6, \quad 5+10=15, \quad 10+10=20, \text{ etc.},$$

and in multiplying out such a product as the one in Art. 112, we only need the coefficients of $(1-t)^6(1+t)^2$, and all the work appearing will be

Coefficients of $(1-t)^6$ are $1-6+15-20+15-6+1$,

Coefficients of $(1-t)^6(1+t)$ are $1-5+\ 9-\ 5-\ 5+9-5+1$,

Coefficients of $(1-t)^6(1+t)^2$ are $1-4+\ 4+\ 4-10+4+4-4+1$,

each row of figures being formed according to the same law as before.

The student will discover the reason of this by performing the actual multiplication of

$$a+bt+ct^2+dt^3+\dots \text{ by } 1+t,$$

in which the several coefficients in the result are

$$0+a, \ a+b, \ b+c, \ c+d, \dots.$$

Similarly, if the coefficients in $(1+t)^4(1-t)^2$ were required, the work appearing would be

$$1+4+6+4+1$$

$$1+3+2-2-3-1$$

$$1+2-1-4-1+2+1,$$

and the last row gives the coefficients required.

The coefficients here are formed thus:

$$1-0=1, \quad 4-1=3, \quad 6-4=2, \quad 4-6=-2, \text{ etc.}$$

POWERS AND PRODUCTS OF SINES AND COSINES.

114. Sine or Cosine with Positive Odd Integral Index.

Any *odd positive power* of a sine or cosine can be integrated immediately thus:

To integrate

$$\int \sin^{2n+1} x\, dx, \quad \text{let } \cos x = c; \quad \therefore \ \sin x\, dx = -dc.$$

Hence

$$\int \sin^{2n+1} x\, dx = -\int (1-c^2)^n\, dc$$

$$= -\int \left[1 - nc^2 + \frac{n(n-1)}{1.2} c^4 - \dots + (-1)^n c^{2n}\right] dc$$

$$=-c+\frac{nc^3}{3}-\frac{n(n-1)}{1\,.\,2}\frac{c^5}{5}+\ldots-(-1)^n\frac{c^{2n+1}}{2n+1}$$

$$=-\cos x+{}^nC_1\frac{\cos^3 x}{3}-{}^nC_2\frac{\cos^5 x}{5}+\ldots-(-1)^n\,{}^nC_n\frac{\cos^{2n+1} x}{2n+1}.$$

Similarly, putting $\sin x=s$, and therefore $\cos x\,dx=ds$, we have

$$\int\cos^{2n+1} x\,dx=\int(1-s^2)^n\,ds$$

$$=\sin x-{}^nC_1\frac{\sin^3 x}{3}+{}^nC_2\frac{\sin^5 x}{5}-\ldots+(-1)^n\,{}^nC_n\frac{\sin^{2n+1}x}{2n+1}.$$

115. Products of form $\sin^p x\,.\cos^q x$, p or q being an odd positive integer.

In the same way as before, any product of the form $\sin^p x \cos^q x$ admits of immediate integration by the same method whenever either p or q is a positive odd integer, whatever the other may be.

Thus, to integrate $\int\sin^p x\,\cos^{2n+1} x\,dx$. Let $\sin x=s$; then

$$\cos x\,dx=ds \quad\text{and}\quad \int\sin^p x\cos^{2n+1} x\,dx=\int s^p\,(1-s^2)^n\,ds,$$

and expanding as before,

$$=\frac{\sin^{p+1} x}{p+1}-{}^nC_1\frac{\sin^{p+3} x}{p+3}+{}^nC_2\frac{\sin^{p+5} x}{p+5}-\ldots+(-1)^n\,{}^nC_n\frac{\sin^{p+2n+1} x}{p+2n+1}$$

116. **When $p+q$ is a negative even integer**, the expression $\sin^p x\,\cos^q x$ admits of immediate integration in terms of $\tan x$ or $\cot x$.

For, put $\tan x=t$, and therefore $\sec^2 x\,dx=dt$, and let

$$p+q=-2n,$$

n being positive and integral.

Thus

$$\int\sin^p x\cos^q x\,dx=\int\tan^p x\cos^{p+q+2} x\,dt=\int t^p\,(1+t^2)^{n-1}\,dt$$

$$=\int(t^p+{}^{n-1}C_1\,t^{p+2}+{}^{n-1}C_2t^{p+4}+\ldots+{}^{n-1}C_{n-1}\,t^{p+2n-2})\,dt$$

$$=\frac{\tan^{p+1} x}{p+1}+{}^{n-1}C_1\frac{\tan^{p+3} x}{p+3}+{}^{n-1}C_2\frac{\tan^{p+5} x}{p+5}+\ldots+{}^{n-1}C_{n-1}\frac{\tan^{p+2n-1} x}{p+2n-1}.$$

Similarly, if we put

$$\cot x = c, \quad \text{then} \quad -\operatorname{cosec}^2 x\,dx = dc,$$

and

$$\int \sin^p x \cos^q x\,dx = -\int \cot^q x \sin^{p+q+2} x\,dc = -\int c^q (1+c^2)^{n-1}\,dc$$

$$= -\frac{\cot^{q+1} x}{q+1} - {}^{n-1}C_1 \frac{\cot^{q+3} x}{q+3} - {}^{n-1}C_2 \frac{\cot^{q+5} x}{q+5} - \dots - {}^{n-1}C_{n-1} \frac{\cot^{q+2n-1} x}{q+2n-1}.$$

This result is the same as the former, arranged in the opposite order.

117. **Use of Multiple Angles.** $\sin^p x$, $\cos^q x$, $\sin^p x \cdot \cos^q x$, where p and q are *positive integers, either odd or even.*

To sum up then, when in $\sin^p x$, p is odd, or in $\cos^q x$, q is odd, or in $\sin^p x \cos^q x$ *one* of the two p, q is odd, the best method of procedure is that of Arts. 114, 115.

But when both p and q are positive even indices, this method cannot be adopted, for the series used are not terminating series.

We then express the function to be integrated as the sum of a series of sines or cosines of multiples of x, which can be done in all cases by the method of Art. 112, or in simple cases without having recourse to that method. We then have

$$\sin^p x, \quad \cos^q x \quad \text{or} \quad \sin^p x \cos^q x$$

expressed in the form

$$\Sigma A_n \sin nx \quad \text{or} \quad \Sigma A_n \cos nx,$$

and each term may be integrated at once, giving

$$-\Sigma A_n \frac{\cos nx}{n} \quad \text{or} \quad \Sigma A_n \frac{\sin nx}{n}$$

as the integral.

118. Ex. 1. (A small even index.) $\displaystyle\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4}.$

Ex. 2. (A small odd index.) $\displaystyle\int \cos^3 x\,dx = \int \frac{3\cos x + \cos 3x}{4}\,dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x$

or otherwise $\displaystyle = \int (1-s^2)\,ds = \sin x - \frac{\sin^3 x}{3}$

Ex. 3. (A small even index.) $$\int \cos^4 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \int \frac{1+2\cos 2x+\dfrac{1+\cos 4x}{2}}{4}\, dx$$

$$= \int \left(\tfrac{3}{8}+\tfrac{1}{2}\cos 2x+\tfrac{1}{8}\cos 4x\right) dx$$

$$= \tfrac{3}{8}x+\tfrac{1}{4}\sin 2x+\tfrac{1}{32}\sin 4x.$$

119. But for higher powers we adopt the method of Art. 112.

Ex. 4. (A large even index.) $\int \sin^8 x\, dx.$

Let $\cos x + \iota \sin x = y$, etc.

$$2^8\iota^8 \sin^8 x = \left(y-\frac{1}{y}\right)^8 = \left(y^8+\frac{1}{y^8}\right) - 8\left(y^6+\frac{1}{y^6}\right) + 28\left(y^4+\frac{1}{y^4}\right) - 56\left(y^2+\frac{1}{y^2}\right) + 70$$

$$= 2\cos 8x - 16\cos 6x + 56\cos 4x - 112\cos 2x + 70;$$

$$\therefore \int \sin^8 x\, dx = \frac{1}{2^7}\left[\frac{\sin 8x}{8} - \frac{8\sin 6x}{6} + \frac{28\sin 4x}{4} - \frac{56\sin 2x}{2} + 35x\right].$$

Ex. 5. (A large odd index.) $$\int \sin^9 x\, dx = -\int (1-c^2)^4\, dc = -\int (1-4c^2+6c^4-4c^6+c^8)\, dc$$

$$= -\cos x + \frac{4\cos^3 x}{3} - \frac{6\cos^5 x}{5} + \frac{4\cos^7 x}{7} - \frac{\cos^9 x}{9}.$$

Ex. 6. Find $\int \sin^8 x \cos^2 x\, dx$. (Both indices even.)

Then, as in Art. 112,

$$2^8\iota^8 \sin^8 x \cdot 2^2 \cos^2 x = \left(y-\frac{1}{y}\right)^8 \left(y+\frac{1}{y}\right)^2$$

[and the working of the multiplication is

Coefficients in $(1-t)^8$ $\quad 1-8+28-56+70-56+28-8+1$

Coefficients in $(1-t)^8(1+t)$ $\quad 1-7+20-28+14+14-28+20-7+1$

Coefficients in $(1-t)^8(1+t)^2$ $\quad 1-6+13-8-14+28-14-8+13-6+1$]

$$\therefore\ 2^8\iota^8 \sin^8 x \cdot 2^2\cos^2 x$$

$$= \left(y^{10}+\frac{1}{y^{10}}\right) - 6\left(y^8+\frac{1}{y^8}\right) + 13\left(y^6+\frac{1}{y^6}\right) - 8\left(y^4+\frac{1}{y^4}\right) - 14\left(y^2+\frac{1}{y^2}\right) + 28$$

$$= 2\cos 10x - 12\cos 8x + 26\cos 6x - 16\cos 4x - 28\cos 2x + 28;$$

$$\therefore \int \sin^8 x \cos^2 x\, dx$$

$$= \frac{1}{2^9}\left[\frac{\sin 10x}{10} - \frac{6\sin 8x}{8} + \frac{13\sin 6x}{6} - \frac{8\sin 4x}{4} - \frac{14\sin 2x}{2} + 14x\right]$$

$$= \frac{1}{2^9}\left[\frac{\sin 10x}{10} - \frac{3\sin 8x}{4} + \frac{13\sin 6x}{6} - 2\sin 4x - 7\sin 2x + 14x\right].$$

Ex. 7. Find $\int \sin^8 x \cos^3 x\, dx$.
(One index odd.)

$$\int \sin^8 x \cos^3 x\, dx = \int \sin^8 x\,(1-\sin^2 x)\, d\,(\sin x).$$

$$= \frac{\sin^9 x}{9} - \frac{\sin^{11} x}{11}.$$

Ex. 8. $\int e^{2x} \sin^6 x \cos^2 x\, dx$
(An exponential factor.)

$$= -\frac{1}{2^7}\int e^{2x}\,[\cos 8x - 4\cos 6x + 4\cos 4x + 4\cos 2x - 5]\, dx$$

(Art. 112)

$$= -\frac{e^{2x}}{2^7}\left[\frac{\cos(8x-\tan^{-1}4)}{\sqrt{68}} - 2\,\frac{\cos(6x-\tan^{-1}3)}{\sqrt{10}}\right.$$

$$\left. + 2\,\frac{\cos(4x-\tan^{-1}2)}{\sqrt{5}} + 2\,\frac{\cos\left(2x-\frac{\pi}{4}\right)}{\sqrt{2}} - \frac{5}{2}\right].$$

Ex. 9. Consider $I \equiv \int e^x \sin nx \cos^3 x \sin^2 x\, dx$.
(An exponential factor and a trigonometrical factor $\sin nx$, in which n is not necessarily integral.)

As before, $\quad 2^3 \cos^3 x\, 2^2 \iota^2 \sin^2 x = \left(y+\frac{1}{y}\right)^3 \left(y-\frac{1}{y}\right)^2.$

Coefficients of $(1+t)^3$ $\quad 1+3+3+1$,

Coefficients of $(1+t)^3(1-t)$ $\quad 1+2+0-2-1$,

Coefficients of $(1+t)^3(1-t)^2$ $\quad 1+1-2-2+1+1$;

$$\therefore\ \cos^3 x \sin^2 x = -\frac{1}{2^4}(\cos 5x + \cos 3x - 2\cos x);$$

$$\therefore\ \sin nx \cos^3 x \sin^2 x = -\frac{1}{2^5}[2\sin nx \cos 5x + 2\sin nx \cos 3x - 4\sin nx \cos x]$$

$$= -\frac{1}{2^5}[\sin(n+5)x + \sin(n-5)x + \sin(n+3)x + \sin(n-3)x$$

$$- 2\sin(n+1)x - 2\sin(n-1)x];$$

whence $\quad \int e^x \sin nx \cos^3 x \sin^2 x\, dx$

$$= -\frac{1}{2^5}e^x\left[\frac{\sin\{(n+5)x-\tan^{-1}(n+5)\}}{\sqrt{(n+5)^2+1}} + \frac{\sin\{(n-5)x-\tan^{-1}(n-5)\}}{\sqrt{(n-5)^2+1}}\right.$$

$$+ \frac{\sin\{(n+3)x-\tan^{-1}(n+3)\}}{\sqrt{(n+3)^2+1}} + \frac{\sin\{(n-3)x-\tan^{-1}(n-3)\}}{\sqrt{(n-3)^2+1}}$$

$$\left. - 2\,\frac{\sin\{(n+1)x-\tan^{-1}(n+1)\}}{\sqrt{(n+1)^2+1}} - 2\,\frac{\sin\{(n-1)x-\tan^{-1}(n-1)\}}{\sqrt{(n-1)^2+1}}\right].$$

120. Integral Powers of a Secant or Cosecant.

Even positive powers of a secant or cosecant are even negative powers of a cosine or a sine, and come under the head discussed in Art. 116.

Thus, $\int \sec^2 x\, dx = \tan x$,

$$\int \sec^4 x\, dx = \int (1+\tan^2 x)\, d\tan x$$
$$= \tan x + \frac{\tan^3 x}{3},$$
$$\int \sec^6 x\, dx = \int (1+2\tan^2 x+\tan^4 x)\, d\tan x$$
$$= \tan x + \frac{2\tan^3 x}{3} + \frac{\tan^5 x}{5},$$

and generally

$$\int \sec^{2n+2} x\, dx = \int (1+t^2)^n\, dt, \text{ where } t = \tan x,$$
$$= \tan x + {}^nC_1 \frac{\tan^3 x}{3} + {}^nC_2 \frac{\tan^5 x}{5} + \dots + {}^nC_n \frac{\tan^{2n+1} x}{2n+1}.$$

Similarly,

$$\int \operatorname{cosec}^2 x\, dx = -\cot x,$$
$$\int \operatorname{cosec}^4 x\, dx = -\int (1+\cot^2 x)\, d\cot x$$
$$= -\cot x - \frac{\cot^3 x}{3},$$

etc.,

and generally

$$\int \operatorname{cosec}^{2n+2} x\, dx = -\cot x - {}^nC_1 \frac{\cot^3 x}{3} - {}^nC_2 \frac{\cot^5 x}{5} - \dots - {}^nC_n \frac{\cot^{2n+1} x}{2n+1}.$$

121. Exactly in the same way

$$\int \sec^p x \operatorname{cosec}^q x\, dx$$

can be integrated when $p+q$ is a positive even integer, either in terms of $\tan x$ or of $\cot x$.

This has been done already in Art. 116, for it may be written

$$\int \cos^{-p} x \sin^{-q} x\, dx,$$

where $-p-q$ is a negative even integer.

122. **Odd Powers.**

But for odd positive powers of a secant or a cosecant, we have to adopt another method, because the Binomial Series used would be non-terminating.

We now proceed as follows:

By differentiation,

$$(n+1)\sec^{n+2}x \quad -n\sec^n x \quad = \quad \frac{d}{dx}(\tan x\sec^n x)$$

and $$(n+1)\operatorname{cosec}^{n+2}x-n\operatorname{cosec}^n x=-\frac{d}{dx}(\cot x\operatorname{cosec}^n x);$$

whence

$$\left.\begin{aligned}(n+1)\int\sec^{n+2}x\,dx &= \quad\tan x\sec^n x \quad +n\int\sec^n x\,dx\\ \text{and}\quad (n+1)\int\operatorname{cosec}^{n+2}dx &= -\cot x\operatorname{cosec}^n x+n\int\operatorname{cosec}^n x\,dx.\end{aligned}\right\}\quad(A)$$

Hence, changing n to $n-2$,

$$\int\sec^n x\,dx \quad = \quad \frac{\tan x\sec^{n-2}x}{n-1}+\frac{n-2}{n-1}\int\sec^{n-2}x\,dx,$$

or

$$\int\operatorname{cosec}^n x\,dx=-\frac{\cot x\operatorname{cosec}^{n-2}x}{n-1}+\frac{n-2}{n-1}\int\operatorname{cosec}^{n-2}x\,dx.$$

Now $$\int\sec x\,dx \quad =\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\operatorname{gd}^{-1}x,$$

$$\int\operatorname{cosec}x\,dx=\log\tan\frac{x}{2}.$$

Hence

$$\int\sec^3 x\,dx=\frac{\tan x\sec x}{2}+\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \quad \text{(see Art. 79)},$$

$$\int\sec^5 x\,dx=\frac{\tan x\sec^3 x}{4}+\frac{3}{4}\,\frac{\tan x\sec x}{2}+\frac{3}{4}\,\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right),$$

etc.,

and generally

$$\begin{aligned}\int\sec^n x\,dx=\frac{\tan x\sec^{n-2}x}{n-1}&+\frac{n-2}{n-1}\,\frac{\tan x\sec^{n-4}x}{n-3}\\ &+\frac{(n-2)(n-4)}{(n-1)(n-3)}\,\frac{\tan x\sec^{n-6}x}{n-5}+\dots\\ &+\frac{(n-2)(n-4)\dots3.1}{(n-1)(n-3)\dots4.2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\end{aligned}$$

(n odd).

The same formula would equally apply if n be even, except that it would terminate differently, viz. the last term would be

$$\frac{(n-2)(n-4)\dots 4.2}{(n-1)(n-3)\dots 5.3}\tan x \quad (n \text{ even}).$$

In the same way

$$\int \operatorname{cosec}^3 x\,dx = -\frac{\cot x \operatorname{cosec} x}{2} + \frac{1}{2}\log\tan\frac{x}{2},$$

$$\int \operatorname{cosec}^5 x\,dx = -\frac{\cot x \operatorname{cosec}^3 x}{4} - \frac{3}{4}\frac{\cot x \operatorname{cosec} x}{2} + \frac{3}{4}\frac{1}{2}\log\tan\frac{x}{2},$$

and generally,

$$\int \operatorname{cosec}^n x\,dx = -\frac{\cot x \operatorname{cosec}^{n-2} x}{n-1} - \frac{n-2}{n-1}\frac{\cot x \operatorname{cosec}^{n-4} x}{n-3}$$
$$-\frac{(n-2)(n-4)}{(n-1)(n-3)}\frac{\cot x \operatorname{cosec}^{n-6} x}{n-5} - \dots$$
$$+\frac{(n-2)(n-4)\dots 3.1}{(n-1)(n-3)\dots 4.2}\log\tan\frac{x}{2} \quad (n \text{ odd})$$

or

$$-\frac{(n-2)(n-4)\dots 4.2}{(n-1)(n-3)\dots 5.3}\cot x. \quad (n \text{ even.})$$

But as explained above, if n be even we should not in general employ this method, but that of Art. 120.

123. Since positive or negative powers of secants and cosecants are negative or positive powers respectively of cosines and sines, it will appear that so long as p is an integer, whether positive or negative,

$$\int \sin^p x\,dx, \quad \int \cos^p x\,dx, \quad \int \sec^p x\,dx, \quad \int \operatorname{cosec}^p x\,dx$$

can be integrated. Also it appears that $\int \sin^p x \cos^q x\,dx$ can always be integrated *directly* if p and q are positive integers; also that, even if one of the two p or q be negative or fractional, the integration can still be *directly* effected if the other be a positive odd integer. And further, this integration can be *directly* effected if $p+q$ be a negative even integer, even though both p and q may be fractional.

For other cases of $\int \sin^p x \cos^q x\,dx$, where p, q are negative integers, a reduction formula is in general required (see Art. 228).

124. If the student has any difficulty in reproducing the formulae of connection marked (A), they may be obtained at once by integration by parts thus:

$$\int \sec^{n+2}x\,dx = \int \sec^n x \frac{d\tan x}{dx}\,dx$$

$$= \sec^n x \tan x - \int n \sec^n x \tan^2 x\,dx$$

$$= \sec^n x \tan x - n\int (\sec^{n+2}x - \sec^n x)\,dx,$$

i.e. $$(n+1)\int \sec^{n+2}x\,dx = \sec^n x \tan x + n\int \sec^n x\,dx.$$

And similarly for $\int \operatorname{cosec}^{n+2}x\,dx$,

$$(n+1)\int \operatorname{cosec}^{n+2}x\,dx = -\operatorname{cosec}^n x \cot x + n\int \operatorname{cosec}^n x\,dx.$$

125. **Integral Powers of tangents or cotangents.**

Any integral powers of tangents or cotangents may be readily integrated.

For $$\int \tan^n x\,dx = \int \tan^{n-2}x(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2}x\,d\tan x - \int \tan^{n-2}x\,dx$$

$$= \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx.$$

And since $\int \tan x\,dx = \log\sec x$

an $$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x,$$

we may integrate successively $\tan^3 x$, $\tan^4 x$, $\tan^5 x$, etc.

Thus we have

$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \log\sec x,$$

$$\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \tan x + x,$$

$$\int \tan^5 x\,dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log\sec x,$$

$$\int \tan^6 x\,dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x,$$

etc.,

and generally

$$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \frac{\tan^{n-3}x}{n-3} + \frac{\tan^{n-5}x}{n-5} - \ldots + (-1)^{\frac{n+1}{2}}\frac{\tan^2 x}{2}$$

$$+(-1)^{\frac{n-1}{2}}\log\sec x \quad (n \text{ odd})$$

or

$$= \frac{\tan^{n-1}x}{n-1} - \frac{\tan^{n-3}x}{n-3} + \ldots + (-1)^{\frac{n+2}{2}}\tan x + (-1)^{\frac{n}{2}}x$$

$(n$ even$)$.

126. Similarly for cotangents,

$$\int \cot^n x\,dx = \int \cot^{n-2}x(\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\frac{\cot^{n-1}x}{n-1} - \int \mathrm{cct}^{n-2}x\,dx,$$

whilst $\displaystyle\int \cot x\,dx = \log\sin x,$

$$\int \cot^2 x\,dx = \int(\operatorname{cosec}^2 x - 1)\,dx = -\cot x - x.$$

Thus we have successively

$$\int \cot^3 x\,dx = -\frac{\cot^2 x}{2} - \log\sin x,$$

$$\int \cot^4 x\,dx = -\frac{\cot^3 x}{3} + \frac{\cot x}{1} + x,$$

$$\int \cot^5 x\,dx = -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log\sin x,$$

and generally

$$\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} + \frac{\cot^{n-3}x}{n-3} - \frac{\cot^{n-5}x}{n-5} + \ldots$$

$$-(-1)^{\frac{n+1}{2}}\frac{\cot^2 x}{2} - (-1)^{\frac{n+1}{2}}\log\sin x \quad (n \text{ odd})$$

or

$$= -\frac{\cot^{n-1}x}{n-1} + \frac{\cot^{n-3}x}{n-3} - \frac{\cot^{n-5}x}{n-5} + \ldots + (-1)^{\frac{n}{2}}\frac{\cot x}{1} + (-1)^{\frac{n}{2}}x$$

$(n$ even$)$.

Hence any odd or even positive or negative power of a tangent or cotangent can be integrated readily.

EXAMPLES.

1. Integrate $\sin^2 x$, $\sin^3 x$, $\sin^4 x$, $\sin^5 x$, $\sin^8 x$, $\sin^9 x$, $\sin^{2n}x$, $\sin^{2n+1}x$, doing those with odd indices in two ways.

2. Integrate $\sin^2 x\cos^2 x$, $\sin^3 x\cos^3 x$, $\sin^4 x\cos^4 x$, $\sin^3 x\cos^6 x$, $\sin^6 x\cos^3 x$, $\sin^6 x\cos^4 x$.

3. Integrate $\dfrac{\sin^2 x}{\cos^4 x}$, $\cos^2 x \operatorname{cosec}^4 x$, $\sec^2 x \operatorname{cosec}^2 x$, $\dfrac{1}{\sin^4 x \cos^4 x}$.

4. Evaluate $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$, $\int_0^{\frac{\pi}{4}} \cos^5 x\,dx$, $\int_0^{\frac{\pi}{4}} \cos^6 x\,dx$.

5. Integrate $\sin ax \cos^2 bx$, $\sin 3x \cos^3 x$, $\sin nx \cos^2 x$.

6. Show that

$$\int \sin x \sin 2x \sin 3x\,dx = -\frac{1}{8}\cos 2x - \frac{1}{16}\cos 4x + \frac{1}{24}\cos 6x.$$

7. Show that

(i) $\displaystyle\int \sin mx \cos nx\,dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}$.

(ii) $\displaystyle\int \sin mx \sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$.

(iii) $\displaystyle\int \cos mx \cos nx\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}$.

Deduce from (ii) and (iii) the values of

$$\int \sin^2 mx\,dx \quad \text{and} \quad \int \cos^2 mx\,dx,$$

and verify the results by independent integration.

8. Prove that $\int_0^{\pi} \sin mx \sin nx\,dx$ and $\int_0^{\pi} \cos mx \cos nx\,dx$ are both zero so long as m and n are integral and unequal. But if m and n are equal integers their values are each equal to $\frac{\pi}{2}$.

GENERAL EXAMPLES.

1. Prove that $\displaystyle\int u\frac{d^2v}{dx^2}dx = u\frac{dv}{dx} - v\frac{du}{dx} + \int v\frac{d^2u}{dx^2}dx$.

2. Perform the following integrations:

(i) $\int \cos^{-1} x\,dx$. (ii) $\int \cos^{-1}\frac{1}{x}\,dx$.

(iii) $\int x^3 \tan^{-1} x\,dx$. (iv) $\int x \sec^2 x\,dx$.

(v) $\int x \sec x \tan x\,dx$. (vi) $\int (ax+b)\log(cx+d)\,dx$.

(vii) $\int \tan^{-1}\sqrt{1-x^2}\,dx$. [St. John's, 1886.]

(viii) $\displaystyle\int \frac{x^4}{1+x^2}\tan^{-1} x\,dx$. [Ox. II. P., 1889.]

(ix) $\int \sin^{-1}\sqrt{\dfrac{x}{a+x}}.$ (x) $\int x\sin^{-1}\sqrt{\dfrac{2a-x}{4a}}\,dx.$

(xi) $\int \cos^{-1}\sqrt{\dfrac{a}{a+x}}\,dx.$ (xii) $\int x^n \log x\,dx.$

3. Integrate

(i) $\int e^{a\sin^{-1}x}\,dx.$ (ii) $\int \dfrac{x\sin^{-1}x}{(1-x^2)^{\frac{1}{2}}}\,dx.$ (iii) $\int \dfrac{x^3\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}\,dx.$

4. Integrate

(i) $\int \dfrac{e^{m\tan^{-1}x}}{1+x^2}\,dx.$ (ii) $\int \dfrac{e^{m\tan^{-1}x}}{(1+x^2)^{\frac{3}{2}}}\,dx.$ (iii) $\int \dfrac{e^{m\tan^{-1}x}}{(1+x^2)^2}\,dx.$

(iv) $\int \dfrac{e^{m\tan^{-1}x}}{(1+x^2)^{\frac{5}{2}}}\,dx.$ (v) $\int \dfrac{e^{m\tan^{-1}x}}{(1+x^2)^{\frac{n}{2}+1}}\,dx$ ($n \equiv$ a positive integer)

5. Integrate (i) $\int xe^{bx}\cos ax\,dx.$

(ii) $\int x^2e^{ax}\sin bx\,dx.$ [α 1888.]

(iii) $\int xe^x\sin^2 x\,dx.$

6. Integrate

(i) $\int e^{ax}(\sin bx+\cos bx)\,dx.$ (v) $\int x^2\,3^x\sin 4x\,dx.$

(ii) $\int e^{ax}(\sinh bx+\cosh bx)\,dx.$ (vi) $\int \cos\left(b\log\dfrac{x}{a}\right)dx.$

(iii) $\int e^{ax}\sinh bx\cosh ax\,dx.$ (vii) $\int \cosh\left(b\log\dfrac{x}{a}\right)dx.$

(iv) $\int e^{ax}\cosh ax\sin bx\,dx.$ (viii) $\int^{\pi}\theta\sin\theta\cosh(\cos\theta)\,d\theta.$

[α 1891.]

7. Integrate

(i) $\int \dfrac{xe^x}{(x+1)^2}\,dx.$ (ii) $\int e^x\dfrac{1+\sin x}{1+\cos x}\,dx.$ (iii) $\int e^x\dfrac{1-\sin x}{1-\cos x}\,dx.$

(iv) $\int \dfrac{\cosh x+\sinh x\sin x}{1+\cos x}\,dx.$ (v) $\int \dfrac{dx}{1+e^x}.$ [MECH. SC. TRIP.]

(vi) $\int \sqrt{1+e^{nx}}\,dx.$ (vii) $\int e^x\dfrac{1+x^2}{(1+x)^2}\,dx.$

[Ox. I. P., 1890.]

8. Integrate

(i) $\int (\log x)^2\, dx.$ [Ox. I. P., 1888.]

(ii) $\int \left(x + \frac{1}{x} + \frac{1}{x^2}\right) \log x\, dx.$ [Ox. I. P., 1889.]

(iii) $\int x^{-2} \tan^{-1} x\, dx.$ [Ox. II. P., 1887.]

(iv) $\int \log (x + \sqrt{a^2 + x^2})\, dx.$ [Math. Trip., 1882.]

(v) $\int x \log (x + \sqrt{a^2 + x^2})\, dx.$ [St. John's, 1884.]

(vi) $\int (a + x) \sqrt{a^2 + x^2} dx.$ [St. John's, 1888.]

(vii) $\int (a^2 + x^2) \sqrt{a + x}\, dx.$ [St. John's, 1888.]

(viii) $\int e^{ax} x^2 \sin (bx + c)\, dx.$ [Coll., 1892.]

(ix) $\int x^3 (1 - x^{\frac{2}{3}})^{\frac{3}{2}}\, dx.$ [Ox. I. P., 1890.]

9. Integrate

(i) $\int x\, e^{ax} \sin bx \sin cx\, dx.$

(ii) $\int x\, e^{ax} \sin bx \sin^2 cx\, dx.$

10. Show that if u be a rational integral function of x,

$$\int e^{x/a} u\, dx = a e^{x/a} \left\{ u - a\frac{du}{dx} + a^2 \frac{d^2u}{dx^2} - a^3 \frac{d^3u}{dx^3} + \dots \right\},$$

where the series within the brackets is necessarily finite.

[Trin. Coll., 1881.]

11. If $u \equiv \int e^{ax} \cos bx\, dx$, $v \equiv \int e^{ax} \sin bx\, dx$, prove that

$$\tan^{-1} \frac{v}{u} + \tan^{-1} \frac{b}{a} = bx,$$

and that $(a^2 + b^2)(u^2 + v^2) = e^{2ax}$.

12. Evaluate $\int x^2 \log (1 - x^2)\, dx$, and deduce that

$$\frac{1}{1\,.\,5} + \frac{1}{2\,.\,7} + \frac{1}{3\,.\,9} + \dots = \frac{8}{9} - \frac{2}{3} \log_e 2.$$ [α, 1889.]

13. Integrate $\int \sec^{\frac{3}{5}}\theta \operatorname{cosec}^{\frac{7}{5}}\theta\, d\theta$, $\int \sec^{\frac{3}{5}}\theta \sin\theta\, d\theta$.

14. Find the value of

$$\int \left\{ u\frac{d^n v}{dx^n} - (-1)^n v \frac{d^n u}{dx^n} \right\} dx.$$

[γ, 1890.]

15. Evaluate

$$\int \left[\frac{d^3u}{dx^3}\left(\frac{dv}{dx} - \frac{dw}{dx}\right) + \frac{d^3v}{dx^3}\left(\frac{dw}{dx} - \frac{du}{dx}\right) + \frac{d^3w}{dx^3}\left(\frac{du}{dx} - \frac{dv}{dx}\right)\right] dx.$$

[γ, 1890.]

16. Establish the following formulae for integration by parts, u and v being functions of x, and accents denoting differentiations and suffixes integrations with respect to x, and $u^{(n)}$ denoting u with n accents:

(i) $$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots + (-1)^{n-1}u^{(n-1)}v_n + (-1)^n \int u^{(n)}\,dv_{n+1}.$$

(ii) $$\iint (uv)(dx)^2 = uv_2 - 2u'v_3 + 3u''v_4 - 4u'''v_5 + \dots + (-1)^{n-1} n u^{(n-1)} v_{n+1} + (-1)^n n \int u^{(n)} v_{n+1}\,dx + (-1)^n \int dx \int u^{(n)} v_n\,dx.$$

[α, 1888.]

17. If u be a function of x, and differentiations and integrations are respectively denoted by accents and suffixes, and (n) means n accents, show that

$$\log u = 1 - \left(\frac{1}{u}\right)' u_1 + \left(\frac{1}{u}\right)'' u_2 - \left(\frac{1}{u}\right)''' u_3 + \dots + (-1)^n \int \left(\frac{1}{u}\right)^{(n)} du_n$$

[St. John's, 1889.]

18. If u, v, w be functions of x, and accents and suffixes denote differentiations and integrations respectively, show that

$$\begin{aligned} 2uvw = {} & (vw)'u_1 - (vw)''u_2 + (vw)'''u_3 - \dots + (-1)^{m-1}\int (vw)^{(m)} du_m \\ & + (wu)'v_1 - (wu)''v_2 + (wu)'''v_3 - \dots + (-1)^{n-1}\int (wu)^{(n)} dv_n \\ & + (uv)'w_1 - (uv)''w_2 + (uv)'''w_3 - \dots + (-1)^{p-1}\int (uv)^{(p)} dw_p. \end{aligned}$$

[St. John's, 1889.]

19. Prove that

$$\int_0^1 v^{vx}\,dv = 1 - \frac{x}{2^2} + \frac{x^2}{3^3} - \frac{x^3}{4^4} + \frac{x^4}{5^5} - \dots \text{ etc.}$$

[MATH. TRIP., 1878.]

20. Find the value of $\int_0^1 x^x\,dx$ correct to five decimal places.

[J. M. SCH. OX., 1904.]

21. Prove that

$$e^{a^2x^2}\int_0^x e^{-a^2x^2}\,dx = x + \tfrac{2}{3}a^2x^3 + \frac{2^2a^4}{3\,.\,5}x^5 + \dots$$

$$+ \frac{(2a^2)^n}{3\,.\,5\,.\dots(2n-1)}e^{a^2x^2}\int_0^x x^{2n}e^{-a^2x^2}\,dx.$$

[OX. I. PUB., 1899.]

22. Find the sum of the series, supposed convergent,

$$\frac{x^5}{1\,.\,3\,.\,5} - \frac{x^7}{3\,.\,5\,.\,7} + \frac{x^9}{5\,.\,7\,.\,9} - \text{ etc. to } \infty.$$

[COLL., 1881.]

23. If y and z be functions of x, and $u = yz' - zy'$, prove the following:

(i) $\int zu^{-2}(y'z'' - z'y'')\,dx = -y^{-1}(1 + y'zu^{-1})$,

(ii) the integration of $zy^{-1}u^{-2}(yz'' - zy'')$ can be reduced to that of y^{-2}.

[ST. JOHN'S, 1886.]

24. Show how the method of integration by parts may be applied to find

$$\int f(x)\frac{d^{n+1}V}{dx^{n+1}}\,dx,$$

where $f(x)$ is a rational algebraical expression of the n^{th} degree.

Prove that $\displaystyle\int_{-1}^1 f(x)\frac{d^{n+1}(x^2-1)^{n+1}}{dx^{n+1}}\,dx = 0.$

[COLL., 1876.]

25. Prove that $\int (\cos x)^n\,dx$ may be expressed by the series,

$$\sin x - N_1\frac{\sin^3 x}{3} + N_2\frac{\sin^5 x}{5} - N_3\frac{\sin^7 x}{7} + \dots, \text{ etc.},$$

N_1, N_2, N_3, ... being the coefficients of the expansion $(1+\alpha)^{\frac{n-2}{2}}$, and n having any real value positive or negative.

[SMITH'S PRIZE, 1876.]

26. Prove that

$$\int x^n e^x \sin x\,dx = e^x\sum_{r=0}^{r=n}(-1)^r\frac{n\,!}{(n-r)\,!}x^{n-r}\,2^{-\frac{r+1}{2}}\sin\left\{x - \frac{(r+1)\pi}{4}\right\}.$$

27. Express the infinite series

$$\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3}+\ldots$$

as a definite integral, and find its value. [Ox. II. P., 1902.]

28. Show that

$$2^m\int\cos mx\cos^m x\,dx=A+x+m\frac{\sin 2x}{2}+\frac{m(m-1)}{1.2}\frac{\sin 4x}{4}+\ldots$$
$$+\frac{\sin 2mx}{2m},$$

where m is an integer and A is independent of x. [Coll. α, 1885.]

29. Evaluate the integral

$$\int_0^T a_1\sin\frac{2\pi t}{T}\,.\,a_2\sin\frac{2\pi}{T}(t+\lambda)\,dt,$$

and draw curves showing how its value depends on that of λ.

[Mech. Sc. Trip., 1899.]

30. Prove that if $y=f(x)$ and $x=\phi(y)$ are equivalent relations, then, between any corresponding limits,

$$\int\sqrt{f'(x)}\,dx=\int\sqrt{\phi'(y)}\,dy.$$

Hence, or otherwise, prove that if $\tan\beta=\sqrt{1-c}\tan\alpha$,

$$\int_0^\alpha\frac{dx}{\sqrt{1-c\sin^2 x}}=\int_0^\beta\frac{dx}{\sqrt{1-c\cos^2 x}}.$$ [Ox. II. P., 1886.]

31. Prove that the remainder R in the series

$$\theta=\tan\theta-\tfrac{1}{3}\tan^3\theta+\ldots+\frac{(-1)^n}{2n+1}\tan^{2n+1}\theta+R$$

may be written as a definite integral,

$$(-1)^{n+1}\int_0^\theta\tan^{2n+2}\theta\,d\theta.$$ [Coll., 1881.]

32. Show that the integrals $\int_0^x f(z)\,dz$, $\int_0^x z^n f^{(n)}(z)\,dz$ are connected thus:

$$\int_0^x f(z)\,dz=xf(x)-\frac{x^2}{2!}f'(x)+\frac{x^3}{3!}f''(x)-\ldots$$
$$+(-1)^{n-1}\frac{x^n}{n!}f^{(n-1)}(x)+(-1)^n\int_0^x z^n f^{(n)}(z)\,dz,$$

and that if one can be integrated the other can also be integrated.

[Bernoulli.]

33. Integrate

$$\int \{(2n+1)\cos(2n+\tfrac{1}{2})\theta + (2n-2)\cos(2n-\tfrac{3}{2})\theta\}(\cos\theta)^{\frac{1}{2}}\,d\theta,$$

and prove that when n is a positive integer,

$$\int_0^{\frac{\pi}{2}} \cos(2n+\tfrac{1}{2})\theta\,(\cos\theta)^{\frac{1}{2}}\,d\theta = 0.$$

[OXFORD II. PUB., 1913.]

34. Find the sum of the areas included between the axis of x and the arc of the curve $y = x\sin(x/a)$ from the ordinate $x=0$ to the ordinate $x = n\pi a$, n being any positive integer, odd or even.

[OXF. I. P., 1911.]

35. Evaluate $\int_0^{2a} \frac{x^n}{\sqrt{2ax-x^2}}\,dx$ when n is any positive integer.

[OXF. I. P., 1916.]

36. Show that $\int_0^1 x\log(1+\tfrac{1}{2}x)\,dx = \tfrac{3}{4}(1-2\log\tfrac{3}{2})$, and prove that this is less than $\int_0^1 \tfrac{1}{2}x^2\,dx$.

[MATH. TRIP., PART I., 1913.]

37. If $T_n = \int_0^x \tan^n x\,dx$, show that $(n-1)(T_n + T_{n-2}) = \tan^{n-1}x$.

Given that $\pi = 3{\cdot}141592\ldots$, $\log_e 2 = 0{\cdot}693147\ldots$, show that

$$\int_0^{\frac{\pi}{4}} \tan^5 x\,dx = 0{\cdot}09657\ldots,\quad \int_0^{\frac{\pi}{4}} \tan^4 x\,dx = 0{\cdot}11873\ldots.$$

[MATH. TRIP. I., 1915.]

38. Prove that $$\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}\,dx = \frac{\pi}{4} - \frac{1}{2}\log_e 2.$$

[MATH TRIP. I., 1917.]

39. Find the area A between the curve

$$y = a(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x)$$

and the axis of x between the limits 0 and π; and the volume V obtained by rotating this area about the axis of x.

Prove that $4V = \pi^2 aA$. [MATH. TRIP. I., 1913.]

40. Show that

$$\int_0^1 x^{2p-1}\log(1+x)\,dx = \frac{1}{2p}\left\{\frac{1}{1\,.\,2} + \frac{1}{3\,.\,4} + \ldots + \frac{1}{(2p-1)\,2p}\right\}.$$

[MATH. TRIP., PT. I., 1916.]