CHAPTER III.

CHANGE OF THE INDEPENDENT VARIABLE.

47. It will frequently facilitate integration if we change the independent variable x to a new variable z by a suitable choice of relation connecting the two.

Let $x=F(z)$ be the relation chosen, and let

$$\int V\,dx \quad \text{or} \quad \int f(x)\,dx$$

be the integral to be transformed.

Let $\qquad u=\int V\,dx.$

Then $\dfrac{du}{dx}=V.$

But $\dfrac{du}{dz}=\dfrac{du}{dx}\dfrac{dx}{dz}=V\dfrac{dx}{dz}\equiv V\,F'(z),\quad$ *i.e.* $f[F(z)]F'(z).$

$$\therefore\ u=\int V\frac{dx}{dz}\,dz \quad \text{or} \quad \int f[F(z)]\,F'(z)\,dz.$$

48. Thus, to integrate $\int\dfrac{e^{\tan^{-1}x}}{1+x^2}\,dx$, let $\tan^{-1}x=z$ or $x=\tan z$.

$$\frac{dx}{dz}=\sec^2 z\,;$$

$$\therefore\ \int\frac{e^{\tan^{-1}x}}{1+x^2}\,dx=\int\frac{e^z}{1+\tan^2 z}\,\frac{dx}{dz}\,dz=\int\frac{e^z}{1+\tan^2 z}\sec^2 z\,dz$$

$$=\int e^z\,dz=e^z=e^{\tan^{-1}x}.$$

Instead of writing $x=\tan z$, it would be a little shorter to take $\tan^{-1}x=z$, and then $\dfrac{1}{1+x^2}\,\dfrac{dx}{dz}=1$. And

$$\text{the integral}=\int e^z\,dz, \text{ at once}$$

$$=e^z=e^{\tan^{-1}x}.$$

49. In the practical use of the formula

$$\int f(x)\,dx = \int f[F(z)]\,F'(z)\,dz,$$

after having made choice of the transformation $x=F(z)$, it is usual to make use of *differentials*, and instead of writing

$$\frac{dx}{dz}=F'(z),$$

we shall write the same equation as

$$dx=F'(z)\,dz,$$

and the formula will thus be reproduced by replacing dx in the integral $\int f(x)\,dx$ by $F'(z)\,dz$, and the x by $F(z)$. (See *Diff. Cal.*, Art. 156.)

Thus, in the example above, viz. $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$, after putting $\tan^{-1}x=z$, we may write

$$\frac{dx}{1+x^2}=dz,$$

and the integral becomes $\int e^z dz = e^z = e^{\tan^{-1}x}$.

50. When the integration is a definite one between specified limits, the limits for z will not in general be the same as those for x. But supposing a and b to be the inferior and superior limits for x, those for z must be such that whilst x ranges once over its values from a to b, z passes once and once only through the corresponding range of values for z, viz. from $F^{-1}(a)$ to $F^{-1}(b)$, where $x=F(z)$ is the connecting formula.

51. The transformation of the indefinite integral is

$$\int f(x)\,dx = \int f[F(z)]\,F'(z)\,dz.$$

Let $$f(x)=\psi'(x).$$

Then, if the limits for x be a and b,

$$\int_a^b f(x) = \int_a^b \psi'(x)\,dx = \psi(b)-\psi(a).$$

Now, when $x=a$, $\quad z=F^{-1}(a)$;

and, when $x=b$, $\quad z=F^{-1}(b)$.

Also $$f[F(z)]=\frac{d}{dx}\psi[F(z)],$$

and $$f[F(z)]F'(z)=\frac{d}{dx}\psi[F(z)]\frac{dx}{dz}=\frac{d}{dz}\psi[F(z)];$$

whence $$\int_{F^{-1}(a)}^{F^{-1}(b)}f\{F(z)\}F'(z)\,dz=\int_{F^{-1}(a)}^{F^{-1}(b)}\frac{d}{dz}\{\psi[F(z)]\}\,dz$$
$$=\psi[F\{F^{-1}b\}]-\psi[F\{F^{-1}(a)\}]$$
$$=\psi(b)-\psi(a).$$

So that the result of integrating $f[F(z)]F'(z)$ with regard to z between limits $F^{-1}(a)$ and $F^{-1}(b)$ is identical with that of integrating $f(x)$ with regard to x between the limits a and b.

52. Case of a Multiple-Valued Function.

It must be noted that $F^{-1}(a)$ and $F^{-1}(b)$ may be multiple-valued functions of a and b. Thus, for instance,

$$\sin^{-1}\tfrac{1}{2}\text{ being the same thing as }n\pi+(-1)^n\frac{\pi}{6},$$

where n is any integer whatever, is a multiple-valued function. The question will thus frequently arise as to which of a variety of values of $F^{-1}(a)$ and $F^{-1}(b)$ it is proper to take as the limits in the transformed integral.

If, however, we remember the connecting formula $x=F(z)$ and imagine x continuing its march in a continuous manner, always increasing from the value of a to the value of b, then, starting with any of the values of $F^{-1}(a)$, say α, $F^{-1}(x)$ is to change in a continuous manner from α to the *first occasion* on which it takes up the value $F^{-1}(b)$, or β say, increasing along the whole march from α to β, if x and z increase together, *i.e.* if $F'(z)$ be positive from $x=a$ to $x=b$, or decreasing along the whole march from α to β if x and z are such that z decreases as x increases, *i.e.* if $F'(z)$ be negative from $x=a$ to $x=b$. Then α and β are the limits for z which correspond to a and b respectively for x.

53. For instance, let it be required to find the value of $\int_0^1\frac{dx}{\sqrt{1-x^2}}$ where we assign the positive sign to the radical $\sqrt{1-x^2}$. By the transformation $x=\sin\theta$, we have $\frac{dx}{d\theta}=\cos\theta$. And the indefinite integral is $\int(\pm 1)d\theta$ or $\pm\theta$, according as $+\sqrt{1-x^2}=+\cos\theta$ or $-\cos\theta$.

$$\left.\begin{array}{l}\text{When } x=0,\ \theta=\sin^{-1}0=n\pi.\\ \text{When } x=1,\ \theta=\sin^{-1}1=m\pi+(-1)^m\frac{\pi}{2},\end{array}\right\}\ n \text{ and } m \text{ being any integers.}$$

In the march of x from 0 to 1, $\sin\theta$ passes from 0 to 1 and is always positive. If the radius terminating θ lie in the first quadrant,

$$\theta \text{ increases from } 0 \text{ to } \frac{\pi}{2}, \text{ and } \frac{dx}{d\theta} \text{ is positive.}$$

If the terminating radius of θ lie in the second quadrant,

$$\theta \text{ decreases from } \pi \text{ to } \frac{\pi}{2}, \text{ and } \frac{dx}{d\theta} \text{ is negative.}$$

Generally, if θ starts from $2m\pi$, the next occasion on which $\sin\theta$ is 1 is at $\theta=2m\pi+\frac{\pi}{2}$ and $\sin\theta$ is *increasing* from 0 to 1.

If θ start at $(2m+1)\pi$, θ must decrease, as x increases, and therefore must pass from $(2m+1)\pi$, where $\sin\theta$ is zero, to $(2m+1)\pi-\frac{\pi}{2}$, where $\sin\theta$ is 1. Therefore it is proper to take our limits, either

$$0 \text{ to } \frac{\pi}{2},\quad \sin\theta \text{ increasing},\ \theta \text{ increasing};$$

or $$\pi \text{ to } \frac{\pi}{2},\quad \sin\theta \text{ increasing},\ \theta \text{ diminishing};$$

or $$2\pi \text{ to } \frac{5\pi}{2},\quad \sin\theta \text{ increasing},\ \theta \text{ increasing};$$

or $$3\pi \text{ to } \frac{5\pi}{2},\quad \sin\theta \text{ increasing},\ \theta \text{ diminishing};$$

etc.

But we have noted that $+\sqrt{1-x^2}=\pm\cos\theta$, the $+$ sign to be taken if $\cos\theta$ be positive, the $-$ sign if $\cos\theta$ be negative. Accordingly,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}=\Big[\theta\Big]_0^{\frac{\pi}{2}}=\frac{\pi}{2}-0=\frac{\pi}{2},$$

or $$=\Big[-\theta\Big]_\pi^{\frac{\pi}{2}}=-\frac{\pi}{2}+\pi=\frac{\pi}{2},$$

or $$=\Big[\theta\Big]_{2\pi}^{\frac{5\pi}{2}}=\frac{5\pi}{2}-2\pi=\frac{\pi}{2},$$

or $$=\Big[-\theta\Big]_{3\pi}^{\frac{5\pi}{2}}=\frac{-5\pi}{2}+3\pi=\frac{\pi}{2},$$

etc.

54. It will perhaps make the matter clearer if a graph of the transformation formula be drawn in such cases.

In the present case, $x=\sin\theta$ referred to θ, x axes is a curve of sines whose axis is the θ-axis cutting it at $O, L, M, N\dots$; x increases from 0 to 1 along any of the arcs, viz.,

$$O \text{ to } A, \quad L \text{ to } A,$$
$$M \text{ to } B, \quad N \text{ to } B,$$
etc.,

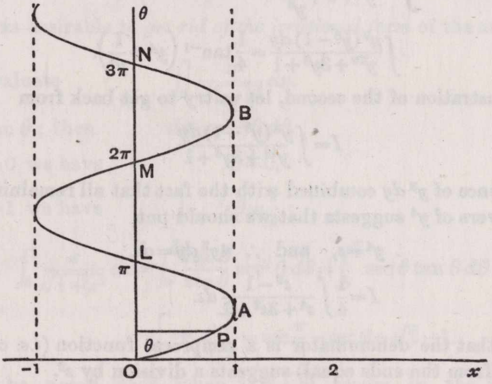

Fig. 13.

and the limits are as stated :

$$\text{along } OA \left[\theta\right]_0^{\frac{\pi}{2}}, \quad \text{along } LA \left[-\theta\right]_\pi^{\frac{\pi}{2}}, \quad \text{along } MB \left[\theta\right]_{2\pi}^{\frac{5\pi}{2}}, \text{ etc.}$$
$$\frac{dx}{d\theta} +^{\text{ve}} \qquad \frac{dx}{d\theta} -^{\text{ve}} \qquad \frac{dx}{d\theta} +^{\text{ve}}$$

55. Purpose of a Substitution.

The purpose of a substitution is two-fold.

(1) Given an elementary known integral to construct a more complex one, and thus extend one's knowledge of integrable forms.

(2) Given an integral which does not fall under the list of fundamental forms, to reduce it to such form if possible.

And it must be noted that it often happens that though one substitution may reduce to a simpler form, that a further substitution, or further substitutions, may be necessary before the integration can be effected.

56. As an illustration of the first.

Beginning with the known result $\int \frac{dx}{1+x^2} = \tan^{-1} x$, let us put

$$x = y^4 + \frac{1}{y^4}, \quad \text{say.}$$

Then $$dx=\left(4y^3-\frac{4}{y^5}\right)dy$$

whence $$4\int\frac{\left(y^3-\frac{1}{y^5}\right)dy}{y^8+3+\frac{1}{y^8}}=\tan^{-1}\left(y^4+\frac{1}{y^4}\right),$$

or $$\int\frac{y^3(y^8-1)\,dy}{y^{16}+3y^8+1}=\frac{1}{4}\tan^{-1}\left(y^4+\frac{1}{y^4}\right).$$

As an illustration of the second, let us try to get back from

$$I=\int\frac{y^3(y^8-1)\,dy}{y^{16}+3y^8+1}.$$

The presence of $y^3\,dy$ combined with the fact that all remaining powers of y are powers of y^4 suggests that we should put

$$y^4=z,\quad\text{and}\ \therefore\ 4y^3\,dy=dz.$$

Then $$I=\frac{1}{4}\int\frac{z^2-1}{z^4+3z^2+1}dz.$$

The fact that the denominator is a reciprocal function (*i.e.* coefficients equidistant from the ends equal) suggests a division by z^2.

I is then written as $$\frac{1}{4}\int\frac{\left(1-\frac{1}{z^2}\right)dz}{z^2+3+\frac{1}{z^2}},$$

which is seen to be $$\frac{1}{4}\int\frac{\left(1-\frac{1}{z^2}\right)dz}{1+\left(z+\frac{1}{z}\right)^2}.$$

The form of this suggests further that we should now put

$$z+\frac{1}{z}=u,$$

for then $$\left(1-\frac{1}{z^2}\right)dz=du.$$

I now becomes $$\frac{1}{4}\int\frac{du}{1+u^2},$$

i.e. $$=\frac{1}{4}\tan^{-1}u=\frac{1}{4}\tan^{-1}\left(z+\frac{1}{z}\right)=\frac{1}{4}\tan^{-1}\left(y^4+\frac{1}{y^4}\right).$$

57. Choice of Substitution.

It will be obvious that a proper choice of substitution can only be the result of experience. No general rules can be given, but the student may learn something as to the proper course to be taken from observation of the worked-out cases which follow and from the accompanying remarks.

Ex. 1. Evaluate $\int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx.$

Let $x=z^2$; then $dx=2z\,dz.$

$$\therefore \int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx = \int \frac{1}{z} \cos z \,.\, 2z\, dz = 2\int \cos z\, dz = 2\sin z = 2\sin\sqrt{x}.$$

Here it was desirable to *get rid of the irrational form* of the angle.

Ex. 2. Evaluate $\int_0^1 \frac{x}{\sqrt{1+x^2}}\, dx.$

Put $x=\tan\theta$; then $dx=\sec^2\theta\, d\theta.$

When $x=0$ we have $\theta=0,$

" $x=1$ we have $\theta=\frac{\pi}{4}$;

$$\therefore \int_0^1 \frac{x}{\sqrt{1+x^2}}\, dx = \int_0^{\frac{\pi}{4}} \frac{\tan\theta}{\sec\theta} \sec^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \sec\theta\tan\theta\, d\theta$$

$$= \Big[\sec\theta\Big]_0^{\frac{\pi}{4}} = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2}-1.$$

It is to be noted that when $\sqrt{a^2+x^2}$ occurs in the integrand, $x=a\tan\theta$ or $x=a\cot\theta$ or $x=a\sinh z$ *are likely substitutions, for they rationalize the radical.*

When $\sqrt{x^2-a^2}$ occurs, $x=a\sec\theta$, $x=a\,\mathrm{cosec}\,\theta$ or $x=a\coth z$, are good substitutions.

Ex. 3. Evaluate $I=\int \frac{dx}{x\sqrt{x^{2n}-a^{2n}}}.$

Let $x^n=a^n z^{-1}$; then $n\frac{dx}{x}=-\frac{dz}{z},$

and
$$I=-\frac{1}{n}\int \frac{dz}{z\sqrt{\frac{a^{2n}}{z^2}-a^{2n}}}$$

$$=-\frac{1}{na^n}\int \frac{dz}{\sqrt{1-z^2}}$$

$$=-\frac{1}{na^n}\sin^{-1} z = -\frac{1}{na^n}\sin^{-1}\left(\frac{a^n}{x^n}\right).$$

Note that $x^n=a^n z^{-1}$ *is generally a proper substitution in cases when* $\sqrt{a^{2n}\pm x^{2n}}$ *occurs.*

Also, $x^n=a^n\tan\theta$ or $a^n\cot\theta$ or $a^n\sinh z$ for $\sqrt{a^{2n}+x^{2n}}$,
or $x^n=a^n\sin\theta$ or $a^n\cos\theta$ or $a^n\,\mathrm{sech}\, z$ for $\sqrt{a^{2n}-x^{2n}}$,
might be used.

When $\sqrt{x^{2n}-a^{2n}}$ occurs, $x^n=a^n\sec\theta$ or $a^n\,\mathrm{cosec}\,\theta$, or $x^n=a^n\coth z$, would be useful.

Ex. 4. *When $\sqrt{2ax-x^2}$ occurs, a useful trial is*

$$x=a(1-\cos\theta), \quad i.e.\ x=a\,\mathrm{vers}\,\theta.$$

Thus, to evaluate $\qquad I=\int\frac{x}{\sqrt{2ax-x^2}}\,dx,$

$$dx=a\sin\theta\,d\theta, \qquad \sqrt{2ax-x^2}=a\sin\theta;$$

$$\therefore\ I=\int\frac{a(1-\cos\theta)\,a\sin\theta\,d\theta}{a\sin\theta}$$

$$=a\int(1-\cos\theta)\,d\theta$$

$$=a(\theta-\sin\theta)$$

$$=a\,\mathrm{vers}^{-1}\frac{x}{a}-\sqrt{2ax-x^2}.$$

Ex. 5. *When $\sqrt{a-x}$ or $\sqrt{\frac{a-x}{a+x}}$ occurs in the integrand the substitution $x=a\cos\theta$ will often be found useful, or perhaps better, $x=a\cos 2\theta$.*

Evaluate $\qquad I=\int x\sqrt{\frac{a-x}{a+x}}\,dx.$

Let $x=a\cos 2\theta$; then

$$dx=-2a\sin 2\theta\,d\theta.$$

$$I=-2a^2\int\cos 2\theta\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\sin 2\theta\,d\theta$$

$$=-2a^2\int(\cos 2\theta\tan\theta\,2\sin\theta\cos\theta)\,d\theta$$

$$=-2a^2\int\cos 2\theta\,(1-\cos 2\theta)\,d\theta$$

$$=-2a^2\int\left(\cos 2\theta-\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$=-2a^2\left(\frac{\sin 2\theta}{2}-\frac{\theta}{2}-\frac{\sin 4\theta}{8}\right)$$

$$=\frac{a^2}{4}(4\theta-4\sin 2\theta+\sin 4\theta)$$

$$=\frac{a^2}{4}\left[2\cos^{-1}\frac{x}{a}-\frac{4}{a}\sqrt{a^2-x^2}+\frac{2x}{a^2}\sqrt{a^2-x^2}\right]$$

$$=\frac{a^2}{2}\cos^{-1}\frac{x}{a}+\frac{(x-2a)}{2}\sqrt{a^2-x^2}.$$

58. When an inverse function occurs in the integrand such as $\sin^{-1}\frac{x}{a}$, $\cos^{-1}\frac{x}{a}$, $\tan^{-1}\frac{x}{a}$, $\mathrm{vers}^{-1}\frac{x}{a}$, it is usually helpful to put $x=a\sin\theta$, $a\cos\theta$, $a\tan\theta$, or $a\,\mathrm{vers}\,\theta$, as the case may be, and work with the direct functions.

Many other forms of substitution will occur in due course, but what has been said will suffice for present purposes.

EXAMPLES.

1. Evaluate (i) $\int \frac{3x^2}{1+x^3}\,dx$. Put $x^3=z$.

(ii) $\int_0^1 \frac{dx}{x^2+4x+5}$. Put $x+2=z$.

(iii) $\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}}$. Put $x-1=z$; $\therefore \frac{dx}{x-1}=\frac{dz}{z}$, etc.

(iv) $\int_0 \frac{dx}{e^x+e^{-x}}$. Put $e^x=z$.

(v) $\int \frac{e^x\,dx}{2e^{2x}+2e^x+1}$. Put $e^x=\frac{z}{1-z}$.

(vi) $\int \tan^2 x\sec^2 x\,dx$. (vii) $\int \frac{dx}{\cosh^2 mx}$.

2. Evaluate (i) $\int_0^a \sqrt{a^2-x^2}\,dx$. Put $x=a\sin\theta$.

(ii) $\int_0^{2a} \sqrt{2ax-x^2}\,dx$. Put $x=a(1-\cos\theta)$.

Draw graphs to illustrate these two integrations.

3. Find the values of

(i) $\int_0^a x\sqrt{a^2-x^2}\,dx$. (ii) $\int_0^a x^2\sqrt{a^2-x^2}\,dx$.

Interpret the meaning of these integrations.

4. Integrate $\int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$. Put $ax+\frac{b}{x}=z$.

5. Integrate $\int \frac{x^{2n}\,dx}{(a^2+x^2)^{n+\frac{3}{2}}}$. Put $x=a\tan\theta$. [ST. JOHN'S, 1883.]

6. Integrate $\int \sec^{\frac{6}{5}}\theta\,\mathrm{cosec}^{\frac{4}{5}}\theta\,d\theta$. Put $\tan\theta=z$. [ST. JOHN'S, 1883.]

7. Integrate $\int \sqrt{\frac{\sin x}{\cos^5 x}}\,dx$. Put $\tan x=z$. [TRINITY, 1883.]

8. Integrate

(i) $\int \frac{dx}{x\sqrt{x^4-1}}$. (ii) $\int \frac{dx}{x\sqrt{1-x^4}}$. (iii) $\int \frac{dx}{x\sqrt{1+x^4}}$.

9. Integrate

(i) $\int \left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,dx$. (ii) $\int \frac{ax^2-b}{x^2+(ax^2+b)^2}\,dx$.

(iii) $\int \frac{ax^2-b}{x^{n+2}}(ax^2+b)^n\,dx$. (iv) $\int \frac{1}{(c+ex)^2}\cos\frac{a+bx}{c+ex}\,dx$.

(v) $\int \frac{e^{a\tan^{-1}x}}{1+x^2}\,dx$. (vi) $\int \frac{e^{a\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$. (vii) $\int \frac{\sin x\cos x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$.

10. Integrate

(i) $\int \{\phi(x)\psi'(x)+\phi'(x)\psi(x)\}\,dx.$ (ii) $\int \frac{\phi(x)\psi'(x)-\phi'(x)\psi(x)}{[\phi(x)]^2}\,dx.$

(iii) $\int \frac{\phi'(x)\,dx}{1+[\phi(x)]^2}.$ (iv) $\int e^{\phi(x)}\phi'(x)\,dx.$

(v) $\int e^{-\psi(x)}\frac{\phi'(x)-\phi(x)\psi'(x)\log\phi(x)}{\phi(x)}\,dx.$

11. Show that

$$\int (x-a)\sqrt{\frac{x-4a}{x}}\,dx = a^2(\sinh 2u - 2\sinh u)$$

where $$x = 4a\cosh^2\frac{u}{2}$$ [Ox. I. Pub., 1899.]

59. The Hyperbolic Functions.

To avoid complexity of form in many integrations and to secure symmetry in the results of integrations of expressions of similar algebraic form, it is customary to make full use of the hyperbolic functions and their inverses. (*Diff. Calc.*, Art. 23.)

By analogy with the exponential values of the sine, cosine, tangent, etc., the exponential functions

$$\frac{e^x-e^{-x}}{2},\quad \frac{e^x+e^{-x}}{2},\quad \frac{e^x-e^{-x}}{e^x+e^{-x}},\ \text{etc.,}$$

are respectively written

$$\sinh x,\quad \cosh x,\quad \tanh x,\ \text{etc.,}$$

or sometimes more shortly as $\text{sh}\,x$, $\text{ch}\,x$, $\text{th}\,x$, etc.

By further analogy with the inverse circular functions,

if $u = \sinh x$ or $\cosh x$ or $\tanh x$, etc.,

we write the inverse hyperbolic functions

$x = \sinh^{-1}u$ or $\cosh^{-1}u$ or $\tanh^{-1}u$, etc., respectively,

or sometimes as $\text{sh}^{-1}x$, $\text{ch}^{-1}x$, $\text{th}^{-1}x$.

This notation is now commonly adopted by modern writers. Professor Sir George Greenhill (*Chapter on the Integral Calculus*, 1888) indicates it as being common amongst American writers, and as being frequently employed by writers on Applied Mathematics. The earlier notation used by Bertrand, viz.

sect sin hyp x, sect cos hyp x, sect tan hyp x, etc.,

is far too cumbrous for free use.

The properties of these functions are now usually discussed at some length in books on Trigonometry [see Dr. Hobson's *Trigonometry*, pages 303-316]. It is therefore unnecessary to repeat them here fully. But for the convenience of students who have not already sufficient familiarity with their use, we reproduce those of the elementary properties which we shall require for the immediate purpose in hand.

60. Elementary Properties.

We clearly have

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1,$$

analogous to $\cos^2\theta + \sin^2\theta = 1$;

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x,$$

analogous to $\cos^2\theta - \sin^2\theta = \cos 2\theta$;

whence $2\cosh^2 x = 1 + \cosh 2x,$

analogous to $2\cos^2\theta = 1 + \cos 2\theta$;

$$2\sinh^2 x = \cosh 2x - 1,$$

analogous to $2\sin^2\theta = 1 - \cos 2\theta$;

$$\operatorname{sech}^2 x + \tanh^2 x = \left(\frac{2}{e^x + e^{-x}}\right)^2 + \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = 1,$$

analogous to $\sec^2\theta - \tan^2\theta = 1$;

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}, \quad \text{analogous to } \tan\theta = \frac{\sin\theta}{\cos\theta};$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\cosh x}{\sinh x}, \quad \text{analogous to } \cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$= \frac{1}{\tanh x},$$

etc.

It is unnecessary to point out methods of proof or analogies further, and the following results may be proved by the student as exercises, and will form a convenient list for reference:

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y,$$

$$\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y,$$

$$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y,$$

$$\cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y,$$

$$\sinh 2x=2\sinh x\cosh x,$$

$$\cosh 2x=\cosh^2 x+\sinh^2 x.$$

$$=2\cosh^2 x-1$$

$$=1+2\sinh^2 x,$$

$$\tanh 2x=\frac{2\tanh x}{1+\tanh^2 x},$$

$$\left.\begin{array}{c}\sinh x+\sinh y=2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2},\\ \text{etc.,}\end{array}\right\}$$

$$\cosh x=1+\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}+\ldots,$$

$$\sinh x=x+\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}+\ldots,$$

$$\tanh^{-1}x=x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots.$$

61. It should be remarked that such expressions as $\sin\theta$, $\cos\theta$, etc., where θ is complex, *i.e.* of the form $u+\iota v$, do not come under the heading of the sines and cosines defined geometrically in the early parts of trigonometry. They are *re-defined now by the exponential values*

$\sin\theta$, standing for $\dfrac{e^{\iota\theta}-e^{-\iota\theta}}{2\iota}$; $\cos\theta$, standing for $\dfrac{e^{\iota\theta}+e^{-\iota\theta}}{2}$;

etc., for any value of θ real or complex.

Then writing $\theta=\iota x$, where $\iota=\sqrt{-1}$,

$$\sin\iota x=\iota\sinh x,$$

$$\cos\iota x=\cosh x,$$

$$\tan\iota x=\iota\tanh x,$$

$$\cot\iota x=-\iota\coth x,$$

Also the ordinary formulae of trigonometry can be proved *from these definitions*, viz., we have

$$\cos^2\theta+\sin^2\theta=1,$$

$$\cos^2\theta-\sin^2\theta=\cos 2\theta,$$

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi,$$

etc.,

and the restriction of the reality of θ and ϕ is removed.

Then, having proved the addition formulae for the sines and cosines from these definitions, we have

$$\sin(u+\iota v)=\sin u\cos\iota v+\cos u\sin\iota v$$

$$=\sin u\cosh v+\iota\cos u\sinh v,$$

etc.

62. Inverse Hyperbolic Functions.

We are, in the Integral Calculus, more particularly interested in the inverse forms.

Let us search for the meaning of the inverse function $\sinh^{-1}\dfrac{x}{a}$

Put $$\sinh^{-1}\frac{x}{a}=y.$$

Then $$\frac{x}{a}=\sinh y=\frac{e^y-e^{-y}}{2};$$

$$\therefore\ e^{2y}-2\frac{x}{a}e^y-1=0,$$

and $$e^y=\frac{x\pm\sqrt{a^2+x^2}}{a},$$

and remembering that $e^{\pm\iota 2\lambda\pi}$, where λ is an integer,

$$=\cos 2\lambda\pi\pm\iota\sin 2\lambda\pi=1,$$

we may, to retain generality, write this as

$$e^{y-2\iota\lambda\pi}=\frac{x\pm\sqrt{a^2+x^2}}{a},$$

or $$y=2\iota\lambda\pi+\log\frac{x\pm\sqrt{a^2+x^2}}{a}.$$

$$\text{Now}\quad 2\iota\lambda\pi+\log\frac{x-\sqrt{a^2+x^2}}{a}=2\iota\lambda\pi-\log\frac{a}{x-\sqrt{a^2+x^2}}$$

$$=2\iota\lambda\pi-\log\left(-\frac{x+\sqrt{a^2+x^2}}{a}\right)$$

$$=2\iota\lambda\pi-\log(-1)-\log\frac{x+\sqrt{a^2+x^2}}{a}$$

$$=2\lambda'-1)\iota\pi-\log\frac{x+\sqrt{a^2+x^2}}{a},$$

for $\log(-1)=\log e^{(2n+1)\iota\pi}=(2n+1)\iota\pi$,
n and λ' being integers.

Thus, $$y=\mu\iota\pi+(-1)^{\mu}\log\frac{x+\sqrt{a^2+x^2}}{a},$$
where μ is an integer.

The "principal value" of y is then $\log\dfrac{x+\sqrt{a^2+x^2}}{a}$, and it is usual to take this as synonymous with $\sinh^{-1}\dfrac{x}{a}$, omitting the generality obtained by the addition of unreal constants.

63. Similarly putting

$$\cosh^{-1}\frac{x}{a}=y,$$

$$\frac{x}{a}=\cosh y=\frac{e^y+e^{-y}}{2},$$

$$e^{2y}-2\frac{x}{a}e^y+1=0$$

and $$e^y=\frac{x\pm\sqrt{x^2-a^2}}{a},$$

and omitting as before the generality derived from the unreal constants, we shall take the solution

$$y=\log\frac{x+\sqrt{x^2-a^2}}{a},$$

viz. the "principal value" of y with the positive sign as $\cosh^{-1}\dfrac{x}{a}$, and therefore $\cosh^{-1}\dfrac{x}{a}$ is to be understood as synonymous with

$$\log\frac{x+\sqrt{x^2-a^2}}{a}.$$

64. Again, putting $\tanh^{-1}\frac{x}{a}=y$,

$$\frac{x}{a}=\tanh y=\frac{e^y-e^{-y}}{e^y+e^{-y}},$$

and therefore $$e^{2y}=\frac{a+x}{a-x},$$

and omitting generalities as before,

$$\tanh^{-1}\frac{x}{a}=\frac{1}{2}\log\frac{a+x}{a-x}.$$

65. Similarly, $$\coth^{-1}\frac{x}{a}=\frac{1}{2}\log\frac{x+a}{x-a},$$

$$\operatorname{sech}^{-1}\frac{x}{a}=\log\frac{a+\sqrt{a^2-x^2}}{x},$$

$$\operatorname{cosech}^{-1}\frac{x}{a}=\log\frac{a+\sqrt{a^2+x^2}}{x}.$$

66. We shall therefore consider

$\sinh^{-1}\frac{x}{a}$ as meaning $\log\frac{x+\sqrt{x^2+a^2}}{a}$,

$\cosh^{-1}\frac{x}{a}$ as meaning $\log\frac{x+\sqrt{x^2-a^2}}{a}$,

$\tanh^{-1}\frac{x}{a}$ as meaning $\frac{1}{2}\log\frac{a+x}{a-x}$,

$\coth^{-1}\frac{x}{a}$ as meaning $\frac{1}{2}\log\frac{x+a}{x-a}$,

$\operatorname{sech}^{-1}\frac{x}{a}$ as meaning $\log\frac{a+\sqrt{a^2-x^2}}{x}$,

$\operatorname{cosech}^{-1}\frac{x}{a}$ as meaning $\log\frac{a+\sqrt{a^2+x^2}}{x}$.

67. **Periodicity of the Hyperbolic Functions.**

These hyperbolic functions are periodic. But the periodicity is imaginary.

For, since $e^{\pm\iota\lambda\pi}=\cos\lambda\pi\pm\iota\sin\lambda\pi=(-1)^\lambda$, ($\lambda\equiv$ an integer),

we have $$\cosh(x+\lambda\iota\pi)=\frac{e^{x+\lambda\iota\pi}+e^{-(x+\lambda\iota\pi)}}{2}=(-1)^\lambda\cosh x.$$

Similarly, $\sinh(x+\lambda\iota\pi)=(-1)^\lambda\sinh x$,

whence $\tanh(x+\lambda\iota\pi)=\tanh x$.

Thus, the periodicity of $\sinh x$ and $\cosh x$ is $2\pi\iota$,

that of $\tanh x$ and $\coth x$ is $\pi\iota$.

Also

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = 0, \quad \cosh 0 = \frac{e^0 + e^{-0}}{2} = 1, \quad \tanh 0 = 0, \text{ etc.,}$$

and therefore

$$\sinh \iota\pi = \frac{e^{\iota\pi} - e^{-\iota\pi}}{2} = \iota \sin \pi = 0,$$

$$\cosh \iota\pi = \cos \pi = -1,$$

etc.

Again,

$$\begin{aligned}\cosh^{-1}(-z) &= \log(-z + \sqrt{z^2-1}) \\ &= \log(z - \sqrt{z^2-1}) + \log(-1) \\ &= \log\frac{1}{z+\sqrt{z^2-1}} + \iota\pi \\ &= -\log(z+\sqrt{z^2-1}) + \iota\pi \\ &= -\cosh^{-1}z + \iota\pi,\end{aligned}$$

$$\begin{aligned}\sinh^{-1}(-z) &= \log(-z + \sqrt{z^2+1}) \\ &= \log\frac{1}{z+\sqrt{z^2+1}} \\ &= -\log(z+\sqrt{z^2+1}) \\ &= -\sinh^{-1}z,\end{aligned}$$

$$\begin{aligned}\tanh^{-1}(-z) &= \frac{1}{2}\log\frac{1-z}{1+z} = -\frac{1}{2}\log\frac{1+z}{1-z} \\ &= -\tanh^{-1}z,\end{aligned}$$

etc.,

analogous to the properties of the circular functions,

$$\cos^{-1}(-z) = -\cos^{-1}z + \pi, \quad \sin^{-1}(-z) = -\sin^{-1}z,$$
$$\tan^{-1}(-z) = -\tan^{-1}z,$$

etc.

68. Geometrical Interpretation.

Let a rectangular hyperbola $x^2 - y^2 = a^2$ and its auxiliary circle be drawn; then any point on the hyperbola may be represented by either of the parameters θ or u by putting

$$\left.\begin{aligned} x &= a \sec\theta, \\ y &= a\tan\theta, \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} x &= a\cosh u, \\ y &= a\sinh u. \end{aligned}\right\}$$

Hence θ and u are connected by the equations

$$\sec\theta = \cosh u$$

or

$$\tan\theta = \sinh u.$$

Let P be the point θ (or u) on the hyperbolic arc AP; PN the ordinate, NT the tangent from N to the auxiliary circle.

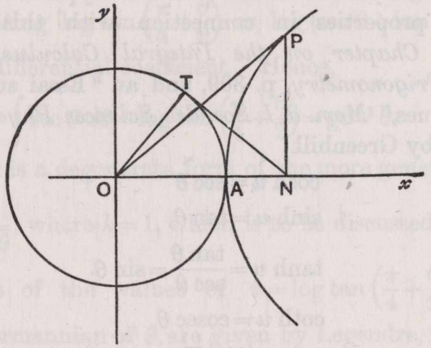

Fig. 14.

Then obviously the abscissa

$$ON = x = OT \sec NOT = a \sec NOT.$$

Hence, the angle NOT is the parameter θ.

Also, since $ON^2 - NT^2 = a^2$, it follows that $NT = y$, as is obvious, since $y = a\tan\theta$, as also $NT = a\tan\theta$.

The area of the portion NAP of the hyperbola

$$= \int_a^x y\,dx = \int_0^u a\sinh u \,.\, a\sinh u\,du,$$

$$= a^2\int_0^u \sinh^2 u\,du$$

$$= \frac{a^2}{2}\int_0^u (\cosh 2u - 1)\,du$$

$$= \frac{a^2}{2}\left(\frac{\sinh 2u}{2} - u\right) = \frac{a^2\sinh 2u}{4} - \frac{a^2u}{2}.$$

Also, area of triangle $ONP = \frac{1}{2}xy$

$$= \frac{1}{2}a\cosh u \,.\, a\sinh u = \frac{a^2\sinh 2u}{4}.$$

Hence the area of the hyperbolic sector OAP

$$= \triangle ONP - \text{area } ANP$$

$= \dfrac{a^2u}{2}$, analogous to $\dfrac{a^2\theta}{2}$ for the circular sector

This indicates the meaning of u, viz.

$$u = \frac{2 \text{ area of hyperbolic sector } CAP}{a^2}.$$

It is this connection with the hyperbola from which these transcendental functions are termed hyperbolic functions. For other properties in connection with this figure, see Greenhill's *Chapter on the Integral Calculus*, p. 27, or Hobson's *Trigonometry*, p. 309, and an "Essai sur les Fonct. Hyperboliques," *Mem. d. l. Soc. des Sciences Phys.*, Bordeaux, 1875, cited by Greenhill.

Since $\cosh u = \sec\theta$

and $\therefore \sinh u = \tan\theta,$

we have $\tanh u = \dfrac{\tan\theta}{\sec\theta} = \sin\theta.$

$$\coth u = \operatorname{cosec}\theta,$$

etc.,

which express functions of u in terms of θ. Again, expressing θ in terms of u, we obviously have

$$\sin\theta = \tanh u,$$

$$\cos\theta = \operatorname{sech} u,$$

$$\tan\theta = \sinh u,$$

$$\cot\theta = \operatorname{cosech} u, \text{ etc.}$$

69. **The Gudermannian.**

The angle θ, which may therefore be variously expressed as $\sin^{-1}(\tanh u)$, $\cos^{-1}(\operatorname{sech} u)$, $\tan^{-1}(\sinh u)$, $\cot^{-1}(\operatorname{cosech} u)$, $\sec^{-1}(\cosh u)$, or $\operatorname{cosec}^{-1}(\coth u)$, is called by Cayley the "Gudermannian" of u* (*Elliptic Functions*, p. 56), and denoted by him by the convenient notation

$$\theta = \operatorname{gd} u,$$

or inversely $u = \operatorname{gd}^{-1}\theta.$

Then $\sin\theta$, $\cos\theta$, $\tan\theta$ he denotes by $\operatorname{sg} u$, $\operatorname{cg} u$, $\operatorname{tg} u$.

Again, $\log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right) = \log(\sec\theta + \tan\theta)$

$$= \log(\cosh u + \sinh u) = \log e^u = u.$$

*So named from Gudermann, who specially discussed this function (Cayley, p. 44).

Hence, gd u is such that

$$\log\tan\left(\frac{\pi}{4}+\frac{\text{gd}\,u}{2}\right)=u,$$

or $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\text{gd}^{-1}\theta,$ which is the same thing.

Differentiating $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ or $\log(\sec\theta+\tan\theta)$, we get $\sec\theta$ as the differential coefficient. Hence,

$$\int\sec\theta\,d\theta=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\text{gd}^{-1}\theta,$$

and $\int\sec\theta\,d\theta$ is a degenerate form of the more general integral $\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, where $k=1$, which is to be discussed later.

70. Tables of the values of $u=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, *i.e.* the inverse Gudermannian of θ, are given by Legendre, *Théorie des Fonctions Elliptiques*, vol. ii., to 12 places of decimals for angles in the first quadrant. They will be found to seven places at degree intervals in Hobson's *Trigonometry*, p. 316, and to five places at degree intervals in Greenhill's *Elliptic Functions*, p. 16, whence it is easy to extract the values of u corresponding to any angle θ, or the value of θ corresponding to any given value of u, and hence from the relations $\cosh u=\sec\theta$, $\sinh u=\tan\theta$, etc., we can find the values of the hyperbolic functions $\cosh u$, $\sinh u$, etc., for any values of u by the use of the intermediary angle θ by means of the ordinary tables of secants, tangents, etc. In the absence of direct tables of the hyperbolic functions this will be the proper mode of computation to follow in numerical calculations. See Lodge's *Report to Brit. Assoc.* 1888, and remarks by Greenhill on p. 15, *Elliptic Functions*.*

71. Unless extremely close approximations are required it will be sufficient to take the values of $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ from

* The Smithsonian Institute of the City of Washington publishes a set of *Mathematical Tables of the Hyperbolic Functions*, by G. F. Becker and C. E. van Orstand.

The Harvard University Press publishes *Tables of Complex Hyperbolic and Circular Functions*, by A. E. Kennelly.

the following graph, which indicates the march of the function from $\theta=0$ to $\theta=90^\circ$. There is not much deviation from a straight line from $\theta=0$ to $\theta=45^\circ$, but beyond that the function begins to increase more rapidly, passing from 4·7413 at 89° to ∞ at 90°. For the first part of the graph, obviously the ordinary rule of proportional parts will give a fair approxi-

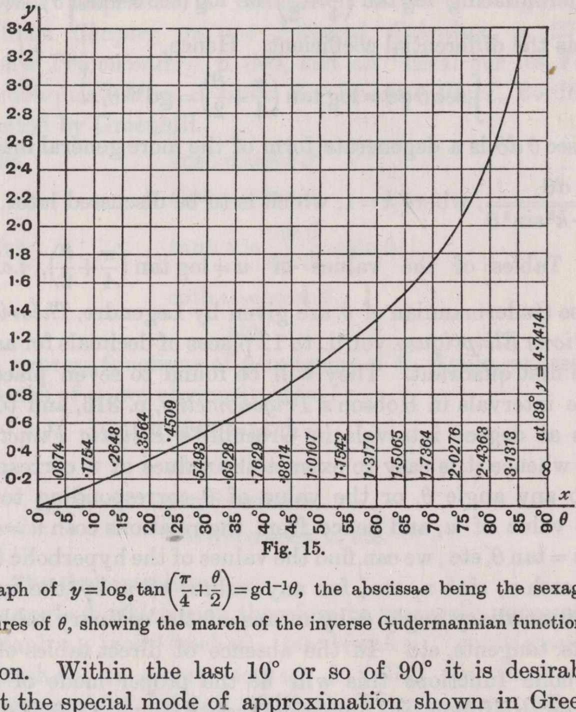

Fig. 15.

[Graph of $y=\log_e \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\text{gd}^{-1}\theta$, the abscissae being the sexagesimal measures of θ, showing the march of the inverse Gudermannian function.]

mation. Within the last 10° or so of 90° it is desirable to adopt the special mode of approximation shown in Greenhill, *loc. cit.* The ordinates are given to four places of decimals at 5° intervals, which will be sufficient for the purposes of approximation in this book. The numerical data for the graph were taken from Hobson's *Trigonometry*.

This graph is that of

$$y=\text{gd}^{-1}x,$$

the inverse Gudermannian, and equally serves to illustrate the graph of

$$x=\text{gd}\, y,$$

i.e. the march of the direct Gudermannian.

72. Let us illustrate the use of the graph—or the values tabulated in Fig. 15.

If, for instance, we require the value of sinh 1 from the tables (which should of course be $\dfrac{e^1 - e^{-1}}{2}$, and therefore we can check it).

$$u \text{ lies between } \cdot 8814 \text{ and } 1\cdot 0107,$$

i.e. $\tan^{-1}\sinh \cdot 8814 = 45°$, $\tan^{-1}\sinh 1\cdot 0107 = 50°$, $\tan^{-1}\sinh 1\cdot 0000 = \theta$.

By proportional parts,

$$\frac{\theta - 45°}{5°} = \frac{\cdot 1186}{\cdot 1293},$$

$$\theta = 4° \, 35' + 45° = 49° \, 35'.$$

$\therefore \sinh 1 = \tan 49° \, 35' = 1\cdot 1744$, from the tables of natural tangents.

To check this,

$$\sinh 1 = \frac{e - e^{-1}}{2} = \frac{2\cdot 7183 - \cdot 3679}{2} = \frac{2\cdot 3504}{2} = 1\cdot 1752,$$

which shows an error of about ·0008.

73. There is also a useful table giving the values of various powers of e, viz. $e^{\pm 1}$, $e^{\pm 2}$, $e^{\pm 3}, \ldots e^{\pm 10}$; $e^{\pm \frac{1}{2}}$, $e^{\pm \frac{3}{2}}$, $e^{\pm \frac{5}{2}}$, $e^{\pm \frac{7}{2}}$; $e^{\pm \frac{1}{4}}$, $e^{\pm \frac{1}{8}}$, $e^{\pm \frac{1}{16}}$, $e^{\pm \frac{1}{32}}$;

$e^{\pm \frac{1}{2}}$, $e^{\pm \frac{1}{3}}$, $e^{\pm \frac{1}{4}}$, $e^{\pm \frac{1}{5}}$, $e^{\pm \frac{1}{6}}$, $e^{\pm \frac{1}{7}}$, $e^{\pm \frac{1}{8}}$, $e^{\pm \frac{1}{9}}$, $e^{\pm \frac{1}{10}}$,

in Bottomley's tables, p. 56, which will be convenient in some cases.

E.g. (extracting the values from these tables)

$$\cosh \cdot 1 = \frac{e^{\cdot 1} + e^{-\cdot 1}}{2} = \frac{1\cdot 1052 + 0\cdot 9048}{2} = \frac{2\cdot 0100}{2} = 1\cdot 0050.$$

If great accuracy be required it will be necessary to use the 7, or perhaps, in cases, the 12-figure tables, but such extreme accuracy would but seldom be required in practice.

Examples.

Establish the following results:

1. $\int \cosh x \, dx = \sinh x$. 2. $\int \sinh x \, dx = \cosh x$.

3. $\int \operatorname{sech}^2 x \, dx = \tanh x$. 4. $\int \operatorname{cosech}^2 x \, dx = -\coth x$.

5. $\displaystyle\int \frac{\sinh x}{\cosh^2 x} \, dx = \int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x$.

6. $\displaystyle\int \frac{\cosh x}{\sinh^2 x} \, dx = \int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x$.

7. $\int \operatorname{cg} x \, dx = \operatorname{gd} x$. 8. $\int \operatorname{cg}^2 x \, dx = \operatorname{sg} x$. 9. $\displaystyle\int \frac{dx}{\operatorname{cg} x} = \operatorname{tg} x$.

10. (*a*) $\operatorname{sg}(u+v) = \dfrac{\operatorname{sg} u + \operatorname{sg} v}{1 + \operatorname{sg} u \operatorname{sg} v}$, (*b*) $\operatorname{cg}(u+v) = \dfrac{\operatorname{cg} u \operatorname{cg} v}{1 + \operatorname{sg} u \operatorname{sg} v}$,

(*c*) $\operatorname{sg} 2u = \dfrac{2 \operatorname{sg} u}{1 + \operatorname{sg}^2 u}$. (*d*) $\operatorname{cg} 2u = \dfrac{\operatorname{cg}^2 u}{1 + \operatorname{sg}^2 u}$,

(*e*) $\operatorname{tg} 2u = 2 \dfrac{\operatorname{sg} u}{\operatorname{cg}^2 u}$.

74. Integrals of cosec x and sec x.

Let $\tan\frac{x}{2}=z$; then, taking the logarithmic differential,

$$\frac{1}{2\tan\frac{x}{2}}\sec^2\frac{x}{2}\,dx=\frac{dz}{z},\quad i.e.\quad \frac{dx}{\sin x}=\frac{dz}{z}.$$

Thus $$\int \operatorname{cosec} x\,dx=\int\frac{dx}{\sin x}=\int\frac{dz}{z}=\log z=\log\tan\frac{x}{2}.$$

In this result put $x=\frac{\pi}{2}+y$; then $dx=dy$.

And $$\int \sec y\,dy=\log\tan\left(\frac{\pi}{4}+\frac{y}{2}\right).$$

That is, $$\int\sec x\,dx \text{ or } \int\frac{dx}{\cos x}=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\mathrm{gd}^{-1}x,$$
as we have seen before.

75. From this result we may infer the integral of
$$\int\frac{dx}{a\cos x+b\sin x}.$$

For $$a\cos x+b\sin x=R\sin(x+a),$$
where $R=\sqrt{a^2+b^2}$ and $\tan a=\frac{a}{b}$;

$$\int\frac{dx}{a\cos x+b\sin x}=\frac{1}{R}\int\frac{dx}{\sin(x+a)}$$
$$=\frac{1}{R}\int\frac{d(x+a)}{\sin(x+a)}$$
$$=\frac{1}{R}\log\tan\frac{x+a}{2}.$$

76. The **integrals of cosech x and sech x** give no trouble.

$$\int\operatorname{cosech} x\,dx=\int\frac{dx}{\sinh x}=2\int\frac{dx}{e^x-e^{-x}}=2\int\frac{e^x}{e^{2x}-1}dx$$
$$=\int\left(\frac{1}{e^x-1}-\frac{1}{e^x+1}\right)de^x$$
$$=\log\frac{e^x-1}{e^x+1}=\log\frac{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}$$
$$=\log\tanh\frac{x}{2},$$

$$\int \operatorname{sech} x\,dx = \int \frac{dx}{\cosh x} = 2\int \frac{e^x}{1+e^{2x}}\,dx = 2\int \frac{de^x}{1+e^{2x}}$$

$$= 2\tan^{-1} e^x \quad \text{or} \quad = \cos^{-1}\frac{1-e^{2x}}{1+e^{2x}}$$

$$= \cos^{-1}(-\tanh x)$$

$$= -\cos^{-1}(\tanh x) + \text{const.}$$

$$= -\sin^{-1}(\operatorname{sech} x) + \text{const.}$$

77. Integrals of $\dfrac{1}{\sqrt{x^2+a^2}}$ **and** $\dfrac{1}{\sqrt{x^2-a^2}}$.

The differential coefficient of log. $\dfrac{x+\sqrt{x^2+a^2}}{a}$ is $\dfrac{1}{\sqrt{x^2+a^2}}$.

Thus, $$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\frac{x+\sqrt{x^2+a^2}}{a} = \sinh^{-1}\frac{x}{a}.$$

Similarly, $$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\frac{x+\sqrt{x^2-a^2}}{a} = \cosh^{-1}\frac{x}{a}.$$

In the inverse hyperbolic forms which it is now possible to use, these results resemble that for the integral

$$\int \frac{dx}{\sqrt{a^2-x^2}}, \quad \text{viz.} = \sin^{-1}\frac{x}{a},$$

and the analogy is an aid to the memory.

The student will note the avoidance of complexity and the gain of symmetry referred to in Art. 59 as the result of using these forms.

We might have established these results thus:

To find $\int \dfrac{dx}{\sqrt{x^2+a^2}}$, put $x = a\sinh u$; then

$$dx = a\cosh u\,du \quad \text{and} \quad \sqrt{x^2+a^2} = a\cosh u.$$

Hence, $$\int \frac{dx}{\sqrt{x^2+a^2}} = \int du = u = \sinh^{-1}\frac{x}{a}.$$

Similarly, putting $x = a\cosh u$ we have

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh u\,du}{a\sinh u} = \int du = u = \cosh^{-1}\frac{x}{a}.$$

78. Integrals of $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$.

I. To find $\int\sqrt{a^2-x^2}\,dx$, put $x=a\sin\theta$; then

$$dx=a\cos\theta\,d\theta;$$

$$\therefore\ \int\sqrt{a^2-x^2}\,dx=a^2\int\cos^2\theta\,d\theta=\frac{a^2}{2}\int(1+\cos 2\theta)\,d\theta$$

$$=\frac{a^2}{4}\sin 2\theta+\frac{a^2}{2}\theta$$

$$=\frac{1}{2}a\sin\theta\,.\,a\cos\theta+\frac{a^2}{2}\theta$$

or
$$\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

II. To find $\int\sqrt{a^2+x^2}\,dx$, put $x=a\sinh z$; then

$$dx=a\cosh z\,dz.$$

Then, since $1+\sinh^2 z=\cosh^2 z$, we have

$$\int\sqrt{a^2+x^2}\,dx=a^2\int\cosh^2 z\,dz=\frac{a^2}{2}\int(\cosh 2z+1)\,dz$$

$$=\frac{a^2}{4}\sinh 2z+\frac{a^2z}{2}=\frac{1}{2}a\sinh z\,.\,a\cosh z+\frac{a^2}{2}z,$$

i.e.
$$\int\sqrt{a^2+x^2}\,dx=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}$$

or
$$=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\log\frac{x+\sqrt{a^2+x^2}}{a};$$

and in the latter form, if the integral be indefinite, we may drop out the a in the denominator of the logarithm, as this will only add a constant to the whole.

III. To find $\int\sqrt{x^2-a^2}\,dx$, put $x=a\cosh z$; then

$$dx=a\sinh z\,dz.$$

Then, since $\cosh^2 z-1=\sinh^2 z$,

$$\int\sqrt{x^2-a^2}\,dx=a^2\int\sinh^2 z\,dz=\frac{a^2}{2}\int(\cosh 2z-1)\,dz$$

$$=\frac{a^2}{4}\sinh 2z-\frac{a^2z}{2}=\frac{1}{2}a\sinh z\,.\,a\cosh z-\frac{a^2z}{2},$$

i.e. $$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

or $$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\frac{x+\sqrt{x^2-a^2}}{a},$$

and the a in the denominator may be omitted, as before, if the integral be indefinite.

[This last integral has already appeared in Art. 68 in finding the area of a portion of space bounded by a rectangular hyperbola, an ordinate and the x-axis].

79. From Art. 78 we may deduce the integration of $\sec^3 x$.

For, putting $\tan x = t$, $\sec^2 x\,dx = dt$, we have

$$\int \sec^3 x\,dx = \int \sqrt{1+t^2}\,dt = \frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\sinh^{-1} t$$

or $$= \frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\log\left(t+\sqrt{1+t^2}\right),$$

i.e. $$\int \sec^3 x\,dx = \frac{1}{2}\tan x\sec x + \frac{1}{2}\log(\sec x + \tan x)$$

or $$= \frac{1}{2}\tan x \sec x + \frac{1}{4}\log\frac{1+\sin x}{1-\sin x}$$

or $$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

Just in the same way, putting $\cot x = c$, $\operatorname{cosec}^2 x\,dx = -dc$, we have

$$\int \operatorname{cosec}^3 x\,dx = -\int \sqrt{1+c^2}\,dc = -\frac{c\sqrt{1+c^2}}{2} - \frac{1}{2}\log\left(c+\sqrt{1+c^2}\right)$$

$$= -\frac{1}{2}\cot x \operatorname{cosec} x - \frac{1}{2}\log(\operatorname{cosec} x + \cot x)$$

$$= -\frac{1}{2}\frac{\cos x}{\sin^2 x} - \frac{1}{4}\log\frac{1+\cos x}{1-\cos x}$$

$$= -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \frac{1}{2}\log\tan\frac{x}{2}.$$

80. We may now deduce from Art. 77 the integration of

$$\int \frac{dx}{\sqrt{R}},$$

where R is a quadratic function of x, viz.

$$R = ax^2 + 2bx + c.$$

CASE I. a Positive.

When a is positive we may write this integral as

$$\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{x^2+2\frac{b}{a}x+\frac{c}{a}}},$$

which we may arrange as

$$\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{a}\right)^2-\frac{b^2-ac}{a^2}}} \text{ or } \frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{a}\right)^2+\frac{ac-b^2}{a^2}}},$$

according as b^2 is greater or less than ac, and the real form of the integral is therefore (Art. 77)

$$\frac{1}{\sqrt{a}}\cosh^{-1}\frac{ax+b}{\sqrt{b^2-ac}} \quad\text{or}\quad \frac{1}{\sqrt{a}}\sinh^{-1}\frac{ax+b}{\sqrt{ac-b^2}},$$

according as b^2 is $>$ or $< ac$.

In either case the integral may be written in the logarithmic form

$$\frac{1}{\sqrt{a}}\log\left(ax+b+\sqrt{a}\sqrt{ax^2+2bx+c}\right), \quad i.e.\ \frac{1}{\sqrt{a}}\log\left(ax+b+\sqrt{aR}\right),$$

the constant $\frac{1}{\sqrt{a}}\log\sqrt{b^2\sim ac}$ being omitted.

Also, since $\cosh^{-1}z=\sinh^{-1}\sqrt{z^2-1}$

and $\sinh^{-1}z=\cosh^{-1}\sqrt{z^2+1}$,

$$\frac{1}{\sqrt{a}}\cosh^{-1}\frac{ax+b}{\sqrt{b^2-ac}}=\frac{1}{\sqrt{a}}\sinh^{-1}\frac{\sqrt{aR}}{\sqrt{b^2-ac}} \quad (b^2>ac)$$

and
$$\frac{1}{\sqrt{a}}\sinh^{-1}\frac{ax+b}{\sqrt{ac-b^2}}=\frac{1}{\sqrt{a}}\cosh^{-1}\frac{\sqrt{aR}}{\sqrt{ac-b^2}} \quad (b^2<ac),$$

which forms may therefore be taken when a is positive and $b^2 >$ or $< ac$ respectively.

81. CASE II. a Negative.

If in the integral $\int\frac{dx}{\sqrt{ax^2+2bx+c}}$, a be negative, write $-a=A$.

Then our integral may be written

$$\frac{1}{\sqrt{A}}\int\frac{dx}{\sqrt{-x^2+\frac{2b}{A}x+\frac{c}{A}}}$$

or $$\frac{1}{\sqrt{A}}\int\frac{dx}{\sqrt{\frac{Ac+b^2}{A^2}-\left(x-\frac{b}{A}\right)^2}}$$

or $$\frac{1}{\sqrt{A}}\sin^{-1}\frac{Ax-b}{\sqrt{Ac+b^2}},\ i.e.\ \frac{1}{\sqrt{-a}}\sin^{-1}\frac{-ax-b}{\sqrt{b^2-ac}}$$

or, omitting a constant,

$$\frac{1}{\sqrt{-a}}\cos^{-1}\frac{ax+b}{\sqrt{b^2-ac}}\left[\text{for} -\sin^{-1}z=\cos^{-1}z-\frac{\pi}{2}\right].$$

Also, since $\cos^{-1}z=\sin^{-1}\sqrt{1-z^2}$, we have

$$\cos^{-1}\frac{ax+b}{\sqrt{b^2-ac}}=\sin^{-1}\frac{\sqrt{-aR}}{\sqrt{b^2-ac}}.$$

82. To sum up then; it appears that when $R=ax^2+2bx+c$ we have the results:

$$\int\frac{dx}{\sqrt{R}}=\begin{cases}\dfrac{1}{\sqrt{-a}}\sin^{-1}\dfrac{\sqrt{-aR}}{\sqrt{b^2-ac}}, & a \text{ negative},\\[2ex] \left.\begin{array}{l}\dfrac{1}{\sqrt{a}}\sinh^{-1}\dfrac{\sqrt{aR}}{\sqrt{b^2-ac}}\\[2ex] \text{or }\ \dfrac{1}{\sqrt{a}}\cosh^{-1}\dfrac{\sqrt{aR}}{\sqrt{ac-b^2}}\end{array}\right\} & a \text{ positive},\quad \begin{array}{l} b^2>ac,\\[2ex] b^2<ac,\end{array}\end{cases}$$

and the real form is to be chosen in each case.

83. Ex. 1. Integrate $\int\frac{dx}{\sqrt{2x^2+3x+4}}$.

We may write this $\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}}=\frac{1}{\sqrt{2}}\sinh^{-1}\frac{4x+3}{\sqrt{23}}$,

or it may be written $\frac{1}{\sqrt{2}}\cosh^{-1}\frac{2\sqrt{2}}{\sqrt{23}}\sqrt{2x^2+3x+4}$

or $$\frac{1}{\sqrt{2}}\log\left(4x+3+2\sqrt{2}\sqrt{2x^2+3x+4}\right),$$

rejecting the constant $\frac{1}{\sqrt{2}}\log\frac{1}{\sqrt{23}}$.

Ex. 2. Integrate $$\int\frac{dx}{\sqrt{4+3x-2x^2}}.$$

This may be written $\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{41}{16}-(x-\frac{3}{4})^2}}$,

and therefore is $$\frac{1}{\sqrt{2}}\sin^{-1}\frac{4x-3}{\sqrt{41}},$$

which may be written as

$$\frac{1}{\sqrt{2}}\cos^{-1}\frac{2\sqrt{2}}{\sqrt{41}}\sqrt{4+3x-2x^2}.$$

84. In exactly the same way, $\int \sqrt{ax^2+2bx+c}\,dx$, when a is positive or when a is negative, can be deduced from the results of Art. 78.

It appears then that the general rule in all cases of

$$\int \frac{dx}{\sqrt{R}} \quad \text{or} \quad \int \sqrt{R}\,dx,$$

where R is quadratic, will be, "Divide out the coefficient of x^2 and then complete the square, and then make use of the suitable standard form."

85. Functions of the form $\dfrac{Ax+B}{\sqrt{ax^2+2bx+c}}$ may be integrated by first putting $Ax+B$ into the form $\lambda(ax+b)+\mu$, which may be done either by inspection or by equating the coefficients, and we obtain

$$Ax+B=\frac{A}{a}(ax+b)+\left(B-\frac{Ab}{a}\right).$$

Thus, $$\frac{Ax+B}{\sqrt{R}}=\frac{A}{a}\cdot\frac{ax+b}{\sqrt{R}}+\frac{aB-Ab}{a}\cdot\frac{1}{\sqrt{R}}.$$

The integral of the first fraction is $\dfrac{A}{a}\sqrt{R}$, and that of the second has been discussed in Arts. 80, 81.

More general forms, such as

$$\frac{f(x)}{\sqrt{ax^2+2bx+c}} \quad \text{or} \quad \frac{f(x)}{\phi(x)}\frac{1}{\sqrt{ax^2+2bx+c}},$$

where f and ϕ are rational integral algebraic polynomials in x, are to be discussed later.

86. Before leaving the integration of $\displaystyle\int \frac{dx}{\sqrt{R}}$, the student should observe other forms into which the results may be thrown.

For some purposes a 'double angle' result is preferable to that given, *e.g.*

(1) $$\int \frac{dx}{\sqrt{x(a-x)}}=\int \frac{dx}{\sqrt{\frac{a^2}{4}-\left(\frac{a}{2}-x\right)^2}}=\cos^{-1}\frac{\frac{a}{2}-x}{\frac{a}{2}}=\cos^{-1}\left(1-\frac{2x}{a}\right).$$

But we may throw this into the form $2\sin^{-1}z$ by making $z^2=\dfrac{x}{a}$ and using $2\sin^{-1}z=\cos^{-1}(1-2z^2)$.

Then $\int \frac{dx}{\sqrt{x(a-x)}} = 2\sin^{-1}\sqrt{\frac{x}{a}} = 2\cos^{-1}\sqrt{\frac{a-x}{a}} = 2\tan^{-1}\sqrt{\frac{x}{a-x}}$.

(2) $$\int \frac{dx}{\sqrt{(x+b)(a-x)}} = \int \frac{d(x+b)}{\sqrt{\{(x+b)(a+b-x+b)\}}}$$
$$= 2\sin^{-1}\sqrt{\frac{b+x}{a+b}} = 2\cos^{-1}\sqrt{\frac{a-x}{a+b}} = 2\tan^{-1}\sqrt{\frac{b+x}{a-x}}$$
$(a>x)$.

(3) $$\int \frac{dx}{\sqrt{(x-b)(a-x)}} = 2\sin^{-1}\sqrt{\frac{x-b}{a-b}} = 2\cos^{-1}\sqrt{\frac{a-x}{a-b}} = 2\tan^{-1}\sqrt{\frac{x-b}{a-x}}$$
$(a>x>b)$.

(4) $$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = \int \frac{dx}{\sqrt{\left(x+\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}} = \cosh^{-1}\frac{2x+a+b}{a-b},$$

the ordinary form; but writing this $=2\sinh^{-1}z$, *i.e.* $\cosh^{-1}(2z^2+1)$,

$$z^2 = \tfrac{1}{2}\left(\frac{2x+a+b}{a-b} - 1\right) = \frac{x+b}{a-b};$$

$$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = 2\sinh^{-1}\sqrt{\frac{x+b}{a-b}} = 2\cosh^{-1}\sqrt{\frac{x+a}{a-b}} = 2\tanh^{-1}\sqrt{\frac{x+b}{x+a}},$$
if $a>b$,

or $$2\sinh^{-1}\sqrt{\frac{x+a}{b-a}} = 2\cosh^{-1}\sqrt{\frac{x+b}{b-a}} = 2\tanh^{-1}\sqrt{\frac{x+a}{x+b}}, \text{ if } a<b,$$

and so for other cases.

87. Of such the following forms are particularly useful:

$$\left.\begin{aligned}\int \frac{dx}{\sqrt{x(a-x)}} &= 2\sin^{-1}\sqrt{\frac{x}{a}},\\ \int \frac{dx}{\sqrt{x(a+x)}} &= 2\sinh^{-1}\sqrt{\frac{x}{a}},\\ \int \frac{dx}{\sqrt{x(x-a)}} &= 2\cosh^{-1}\sqrt{\frac{x}{a}},\end{aligned}\right\}$$
and the others can be derived from these forms as shown above.

88. It will be noticed also in many cases, as, for instance, in the integral of Art. 81, viz.

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{-a}} \sin^{-1} \frac{\sqrt{-aR}}{\sqrt{b^2-ac}} \quad (R = ax^2+2bx+c),$$

that the $\sqrt{R}$ of the integrand reappears in the integral. It did not do so when the result was arrived at as

$$\frac{1}{\sqrt{-a}} \cos^{-1} \frac{ax+b}{\sqrt{b^2-ac}},$$

but was made to do so by the subsequent transformation $\cos^{-1} z = \sin^{-1}\sqrt{1-z^2}$. Examine the earlier integral (Art. 44)

$$\int \frac{dx}{\sqrt{a^2-x^2}} \text{ given as } \sin^{-1}\frac{x}{a}.$$

This could be written $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \cos^{-1}\frac{\sqrt{a^2-x^2}}{a}$,

i.e. $$\int \frac{dx}{\sqrt{R}} = \cos^{-1}\frac{\sqrt{R}}{a} \quad (R = a^2-x^2).$$

So also $$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a}$$

could be written as $$\int \frac{dx}{\sqrt{x^2+a^2}} = \cosh^{-1}\frac{\sqrt{x^2+a^2}}{a},$$

i.e. $$\int \frac{dx}{\sqrt{R}} = \cosh^{-1}\frac{\sqrt{R}}{a} \quad (R = x^2+a^2).$$

Similarly $$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} = \sinh^{-1}\frac{\sqrt{x^2-a^2}}{a}$$

could be written as $$\int \frac{dx}{\sqrt{R}} = \sinh^{-1}\frac{\sqrt{R}}{a} \quad (R = x^2-a^2).$$

And though these forms are obviously not the *simplest forms* of the various integrals, it is frequently desirable to adopt them, as they exhibit a visible relation between the integrand and the result of integration. The *simplest* forms are those tabulated to be remembered in the two lists of standard forms, Arts 44 and 89.

89. We are now in a position to make our list of

Additional Standard Forms.

1. $\displaystyle\int \cosh x\, dx = \sinh x$ and $\displaystyle\int \sinh x\, dx = \cosh x$.

2. $\displaystyle\int \operatorname{sech}^2 x\, dx = \tanh x$ and $\displaystyle\int \operatorname{cosech}^2 x\, dx = -\coth x$.

3. $\displaystyle\int \frac{\sinh x}{\cosh^2 x}\, dx = \int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x$.

4. $\displaystyle\int \frac{\cosh x}{\sinh^2 x}\, dx = \int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x$.

5. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log \frac{x+\sqrt{x^2+a^2}}{a} = \sinh^{-1}\frac{x}{a}.$ *

6. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log \frac{x+\sqrt{x^2-a^2}}{a} = \cosh^{-1}\frac{x}{a}.$

7. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

8. $\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}.$

9. $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}.$

10. $\int \frac{dx}{\sqrt{x(a-x)}} = 2\sin^{-1}\sqrt{\frac{x}{a}}.$

11. $\int \frac{dx}{\sqrt{x(a+x)}} = 2\sinh^{-1}\sqrt{\frac{x}{a}}.$

12. $\int \frac{dx}{\sqrt{x(x-a)}} = 2\cosh^{-1}\sqrt{\frac{x}{a}}.$

13. $\int \operatorname{cosec} x\,dx = \int \frac{dx}{\sin x} = \log\tan\frac{x}{2}.$

14. $\int \sec x\,dx = \int \frac{dx}{\cos x} = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \log(\sec x + \tan x)$

$$= \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} = \operatorname{gd}^{-1} x.$$

15. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a}.$

16. $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\operatorname{cosech}^{-1}\frac{x}{a}.$

17. $\int \frac{dx}{x^2-a^2}\,(x>a) = \frac{1}{2a}\log\frac{x-a}{x+a} = -\frac{1}{a}\coth^{-1}\frac{x}{a}.$

18. $\int \frac{dx}{a^2-x^2}\,(x<a) = \frac{1}{2a}\log\frac{a+x}{a-x} = \frac{1}{a}\tanh^{-1}\frac{x}{a}.$

It is customary to obtain 17 and 18 when wanted, rather than to commit them to memory. They will be discussed later (Art. 127).

* See also Art. 1890, Vol. II.

EXAMPLES.

Write down the integrals of

1. $\frac{1}{9-x^2}$, $\frac{1}{9-4x^2}$, $\frac{1}{x^2-4}$, $\frac{1}{9x^2-4}$, $\sqrt{16-9x^2}$, $\sqrt{3x^2-5}$, $\sqrt{2+3x^2}$.

2. $\frac{1}{\sqrt{x(x-4)}}$, $\frac{1}{\sqrt{x(4-x)}}$, $\frac{1}{\sqrt{x(4+x)}}$, $\frac{1}{\sqrt{2+2x-x^2}}$, $\frac{1}{\sqrt{x^2-2x+2}}$, $\sqrt{x^2+2ax}$.

3. $\frac{x}{\sqrt{9-x^2}}$, $\frac{x}{\sqrt{x^2-9}}$, $\frac{x}{\sqrt{9-4x^2}}$, $\frac{x^2}{\sqrt{1-x^2}}$, $\frac{x^2}{\sqrt{x^2+1}}$.

4. $x\sqrt{x^2+a^2}$, $(x+b)\sqrt{x^2+a^2}$, $\frac{ax+b}{\sqrt{x^2+c^2}}$.

5. $x(x^2+a^2)^{\frac{n}{2}}$, $(x+a)(x^2+2ax+b)^{\frac{n}{2}}$, $(ax-b)(ax^2-2bx+c)^{\frac{n}{2}}$.

6. $\frac{x^2+2x+3}{\sqrt{1-x^2}}$, $\frac{x^2+2x+3}{\sqrt{x^2+1}}$, $\frac{x^2+2x+3}{\sqrt{x^2+x+1}}$, $\frac{x^2+ax+b}{\sqrt{x^2+cx+d}}$.

7. $\sqrt{x^2+4x+5}$, $\sqrt{-x^2+4x+5}$, $\sqrt{4x^2+4x+5}$, $\sqrt{-4x^2+4x+5}$.

8. $\sqrt{\frac{x+a}{x-a}}$, $\sqrt{\frac{a+x}{a-x}}$, $x\sqrt{\frac{a+x}{a-x}}$, $(x+a)\sqrt{\frac{x+b}{x-b}}$, $\frac{(x+a)^{\frac{3}{2}}}{(x-a)^{\frac{1}{2}}}$.

9. $\operatorname{cosec} nx$, $\operatorname{cosec}(2x+b)$, $\frac{1}{4\cos^3 x-3\cos x}$, $\frac{1+\tan^2 x}{1-\tan^2 x}$, $\frac{1}{2}(\cot x+\tan x)$.

10. $\frac{1}{a\sin x+b\cos x}$, $\frac{1}{\sin 2x+\cos 2x}$, $\frac{a\sin x+b\cos x}{c\sin x+d\cos x}$.

11. Deduce $\int \operatorname{cosec} x\,dx=\log\tan\frac{x}{2}$ by expressing $\operatorname{cosec} x$ as

$$\frac{1}{2}\left(\cot\frac{x}{2}+\tan\frac{x}{2}\right)$$

12. Deduce $\int \sec x\,dx=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ by

(i) putting $\sin x=z$,

(ii) putting $\sec x+\tan x=z$.

Show that $\int \sec x\,dx=\cosh^{-1}(\sec x)$.

13. Integrate $\int \frac{\cos\theta\,d\theta}{\sin\theta\sqrt{1-\sin^{2n}\theta}}$.

GENERAL EXAMPLES.

1. If APB be a semicircle, centre O, and PN an ordinate to the diameter AB, and $P'N'$ a contiguous ordinate, show that

$$Lt \sum \frac{NN'}{NP} = \text{circular measure of the angle } OPN,$$

the summation being from the centre to any point N between O and B, and NN' being indefinitely diminished.

2. Find the area in the first quadrant bounded by the axes of coordinates, the ordinate $x = x_1$ and the curve $x^2 = a^2 - \dfrac{b^4}{y^2}$.

If the range $x=0$ to $x=a$ on the x-axis be divided into n equal portions of length h and rectangles be inscribed in the Newtonian manner, examine the limit of the area of the last of these rectangles when h is indefinitely diminished. Find the whole area from $x=0$ to $x=a$.

3. Find the value of $\int \sqrt{e^{2x} + ae^x}\, dx$ [R. P.]

4. Evaluate (i) $\displaystyle\int \frac{dx}{\sqrt{1-3x-x^2}}$. [I. C. S., 1884.]

(ii) $\displaystyle\int \frac{dx}{x\sqrt{x^2+x-6}}$. $\left(\text{Put } x = \dfrac{1}{y}\right)$

[Oxford Second Public Ex., 1880.]

(iii) $\displaystyle\int \frac{(1+x)\,dx}{(2x^2+3x+4)^{\frac{1}{2}}}$ [Colleges β, 1891.]

(iv) $\displaystyle\int \frac{x-1}{(x^2+2x-1)^{\frac{1}{2}}}\,dx$. [Trinity, 1892.]

(v) $\displaystyle\int \frac{3x+4}{(x^2+2x+5)^{\frac{1}{2}}}\,dx$. [Math. Trip., 1887.]

5. Show that the result of integrating $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}$ may be exhibited as

(i) $2\cos^{-1}\dfrac{\sqrt{a+x}-\sqrt{a-x}}{2\sqrt{a}}$ or as (ii) $2\sin^{-1}\dfrac{\sqrt{a+x}+\sqrt{a-x}}{2\sqrt{a}}$

or as (iii) $2\tan^{-1}\sqrt{\dfrac{a+R}{a-R}}$, where $R = \sqrt{a^2-x^2}$.

6. If $R \equiv ax^2 + 2bx + c$ and $K = \dfrac{b^2 - ac}{a}$, show that

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{a}} \tanh^{-1} \sqrt{\frac{R}{R+K}}$$

or

$$= \frac{1}{\sqrt{-a}} \tan^{-1} \sqrt{-\frac{R}{R+K}},$$

according as a is positive or negative.

7. Evaluate (i) $\displaystyle\int \frac{dx}{x\sqrt{1+x^6}}$. (Put $x^3 = \tan\theta$.) [Oxf. I., 1888.]

(ii) $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2-1}}$. (Put $x = \sec\theta$.) [Oxf. I., 1888.]

8. Show that (i) $\displaystyle\int_0^1 \frac{dx}{(1+x^2)(1-x^2)^{\frac{1}{2}}} = \frac{\pi}{2\sqrt{2}}$. [Trinity, 1888.]

(ii) $\displaystyle\int \frac{dx}{x^4\sqrt{1+x^2}} = \frac{2x^2-1}{3x^3}\sqrt{1+x^2}$. [Trinity, 1882.]

9. Integrate $\displaystyle\int \sqrt{1+e^x+e^{2x}}\,dx$.

10. A sphere of given radius a consists of an infinite number of concentric shells of very small thickness, the density at the surface of any shell varying as the n^{th} power of its radius. Find the mass of the sphere. [Ox. I. Pub., 1903.]

If any diameter AB cut one of the shells at P and the density of the shell varies inversely as (i) $OP\sqrt{AP.\,PB}$, (ii) $OP^2\sqrt{AP.\,PB}$, find the mass of the sphere in each case, O being the centre.

11. A triangle ABC is divided into strips by lines parallel to BC; a point is taken in each strip, and the square of the perpendicular from this point to BC is multiplied by the area of the strip; the same is done with all the strips, and the sum of the products is formed. Express by a definite integral the limiting value of this sum when the breadths of all the strips are diminished indefinitely, and evaluate the integral in terms of the base BC and the distance of A therefrom. [Ox. I. Pub., 1901.]

12. Prove that if $u^2 = x^2 + 2px + q$, an integral of the form

$$\int f(x, u)\,dx$$

can always be rationalized (provided f is a rational algebraic function) by one of the substitutions

$$\frac{u}{\sqrt{p^2-q}}=\frac{2y}{1-y^2} \quad \text{or} \quad \frac{u}{\sqrt{q-p^2}}=\frac{1+y^2}{1-y^2}.$$ [COLL. a, 1890.]

13. Find the relation connecting x and y, being given

$$x^2y^4\left(\frac{dy}{dx}\right)^2=a^2+b^2y^3.$$ [I. C. S., 1889.]

14. Show that $\displaystyle\int_{\sqrt{3}}^{2\sqrt{3}} z^3(z^2-3)^{\frac{3}{2}}\,dz=\frac{16038}{35}.$ [OX. I., 1888.]

15. Integrate

(i) $\displaystyle\int\frac{\sin\theta\,d\theta}{\sqrt{a\cos^2\theta+2b\cos\theta+c}}$, (ii) $\displaystyle\int\frac{\cos\theta\,d\theta}{\sqrt{a\sin^2\theta+2b\sin\theta+c}}$,

(iii) $\displaystyle\int\frac{d\theta}{\cos\theta\sqrt{a\cos^2\theta+2b\sin\theta\cos\theta+c\sin^2\theta}}$,

(iv) $\displaystyle\int\frac{d\theta}{\sin\theta\sqrt{a\cos^2\theta+2b\sin\theta\cos\theta+c\sin^2\theta}}$,

(v) $\displaystyle\int\frac{d\theta}{\sin\theta\sqrt{a\cos^2\theta+b\sin^2\theta+c}}$. [TRIN., 1888.]

16. Integrate

(i) $\displaystyle\int\frac{x^4}{(a^2-x^2)^{\frac{1}{2}}}\,dx$, (ii) $\displaystyle\int\frac{x^2\,dx}{(a+bx^2)\sqrt{c^2-x^2}}$. [TRIN., 1888.]

17. (a) Evaluate $\displaystyle\int_1^7(x^2-6x+13)\,dx$, first directly, second by putting $x^2-6x+13=y$. (Draw a graph and explain fully.)

(b) Evaluate $\displaystyle\int_0^{\frac{2b}{a}}(ax^2-2bx+c)\,dx$,

and explain by a graph the result when $2b^2=3ac$.

Obtain the same result by substituting

$$ax^2-2bx+c=y, \quad \text{taking } b^2<ac.$$

Also obtain $\displaystyle\int_0^{\frac{3b}{a}}(ax^2-2bx+c)\,dx$ by this substitution, explaining your limits for y by means of the graph.

18. Point out the fallacy in the following argument:

$$\int_{-1}^{1}\frac{dx}{1+x^2}=\Big[\tan^{-1}x\Big]_{-1}^{1}=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}.$$

But putting $x=\frac{1}{y}$, $\quad dx=-\frac{dy}{y^2}$. $\quad\left.\begin{matrix}\text{When } x=-1,\ y=-1.\\ \text{When } x=\ \ 1,\ y=\ \ 1.\end{matrix}\right\}$

$$\therefore \int_{-1}^{1}\frac{dx}{1+x^2}=-\int_{-1}^{1}\frac{dy}{1+y^2}=-\int_{-1}^{1}\frac{dx}{1+x^2},$$

for, as the result is numerical, the letter used in integration cannot affect the result.

Hence $\quad 2\int_{-1}^{1}\frac{dx}{1+x^2}=0$; $\quad$ but $\int_{-1}^{1}\frac{dx}{1+x^2}=\frac{\pi}{2}$; $\;\therefore\; \pi=0$!

19. Point out the fallacy in the following reasoning:

We have, if we put $x=e^t$,

$$\left(x\frac{d}{dx}\right)^a x^k=\left(\frac{d}{dt}\right)^a e^{kt}=k^a e^{kt}=k^a x^k.$$

But when $a=-1$, we have

$$\left(x\frac{d}{dx}\right)^{-1} x^k=\frac{1}{x}\int x^k dx=\frac{x^k}{k+1},$$

and these two results do not agree. [R.P.]

20. Prove that $\qquad \iota\,\mathrm{gd}\left(\frac{1}{\iota}\mathrm{gd}\,u\right)=u,$ [CAYLEY, *E.F.*]

and show that if $\qquad \mathrm{gd}\,u=a_1u+a_3u^3+a_5u^5+\dots,$

then will $\qquad \mathrm{gd}^{-1}u=a_1u-a_3u^3+a_5u^5-\dots.$

21. If $\qquad \sec x+\tan x=1+S_1\frac{x}{1!}+S_2\frac{x^2}{2!}+S_3\frac{x^3}{3!}+\dots,$

show that $\qquad \mathrm{gd}^{-1}x=x+S_3\frac{x^3}{3!}+S_5\frac{x^5}{5!}+\dots,$

that $\qquad S_{p+1}=S_p+\binom{p}{2}S_{p-2}S_2+\binom{p}{4}S_{p-4}S_4+\dots$

and $\qquad S_n-\binom{n}{2}S_{n-2}+\binom{n}{4}S_{n-4}-\dots+\cos\frac{n\pi}{2}=\sin\frac{n\pi}{2},$

and that $\quad S_3=2, \quad S_5=16, \quad S_7=272, \quad S_9=7936$, etc.

[*Diff. Calc.*, Art. 573, etc.]

22. Integrate $\qquad \int\frac{dx}{(a+x)(c+x)^{\frac{1}{2}}}$

by putting $\qquad c+x=(a-c)z^2$ or $(c-a)z^2$,

according as $\qquad a>$ or $<c$.

Taking the case $a>c$, consider the same integral with $a+da$ replacing a, subtract the original integral, divide by da, and take the limit when da is indefinitely diminished.

Hence obtain $\displaystyle\int \frac{dx}{(a+x)^2 (c+x)^{\frac{1}{2}}}$.

Deduce also $\displaystyle\int \frac{dx}{(a+x)(c+x)^{\frac{3}{2}}}$.

23. Evaluate $\displaystyle\int \frac{dx}{(x^2-a^2)(x^2-c^2)^{\frac{1}{2}}}$.

(i) if $a>c$,
(ii) if $a<c$. (Put $x = c\sec\phi$). [MATH. TRIP., 1878.]

24. Show that $\displaystyle\int \frac{(x-p)^{2n+1}}{(ax^2+2bx+c)^{n+\frac{3}{2}}}\,dx = \frac{1}{(b^2-ac)^{n+1}}\int (y^2-q^2)^n\,dy$,

where $q^2 \equiv ap^2+2bp+c$

and $y\sqrt{ax^2+2bx+c} = (ap+b)x+bp+c$. [COLLEGES, 1901.]

25. If $F(x) = a_1 f(x) + a_2 f(2x) + a_3 f(3x) + \dots$,

prove that $\displaystyle\frac{a_1}{1^x}+\frac{a_2}{2^x}+\frac{a_3}{3^x}+\dots = \frac{\int_0^\infty t^{x-1}F(t)\,dt}{\int_0^\infty t^{x-1}f(t)\,dt}$.

[OX. J. M. SCH., 1904.]

26. Integrate (i) $\displaystyle\int \frac{1+x^2}{1-x^2}\frac{dx}{\sqrt{1+x^4}}$.

(ii) $\displaystyle\int \frac{1-x^2}{1+x^2}\frac{dx}{\sqrt{1+x^4}}$. [EULER.]

27. Show that if $F(x, y)$ be a rational function of x and y,

$$\int F\left(x, \sqrt[n]{\frac{\alpha x+\beta}{\gamma x+\delta}}\right) dx$$

can be thrown into rational form by the substitution

$$\frac{\alpha x+\beta}{\gamma x+\delta} = z^n.$$

Hence show that

$$\int \left(\frac{1-2x}{1+2x}\right)^{\frac{3}{2}} dx = 3\tan^{-1}\sqrt{\frac{1-2x}{1+2x}} - 2\sqrt{\frac{1-2x}{1+2x}} - \frac{\sqrt{1-4x^2}}{2}.$$

28. Show that if $F(x, y)$ be any rational integral function of x and y,

$$\int F(x, \sqrt{ax^2+2bx+c})\,dx$$

can be thrown into rational form by any of the substitutions

$$(1)\quad \sqrt{ax^2+2bx+c}=\sqrt{a}\,(x+z),$$
$$(2)\quad \sqrt{ax^2+2bx+c}=xz+\sqrt{c},$$
$$(3)\quad x-x_2=z^2(x-x_1),$$

where x_1, x_2 are the roots of $ax^2+2bx+c=0$. [BERTRAND, *C.I.*, p. 39.]

Apply each of these methods to the integration of

$$\int\frac{x\,dx}{\sqrt{x^2-6x+8}},$$

showing that the result in each case reduces to

$$\sqrt{x^2-6x+8}+3\cosh^{-1}(x-3),$$

as derived by the method of Art. 85.

29. If $$x^3-3a^2x=a^2y.$$
show that $$\frac{dx}{\sqrt{x^2-4a^2}}=\frac{1}{3}\frac{dy}{\sqrt{y^2-4a^2}}.$$
and hence obtain Cardan's formula for the solution of a cubic.

[J. M. SCH. OX.]

30. Evaluate $$\int_0^{\frac{\pi}{2}}\frac{\cos x\,dx}{1-\sin^2\alpha\cos^2 x},$$
and deduce the expansion

$$\frac{2\alpha}{\sin 2\alpha}=1+\frac{2}{3}\sin^2\alpha+\frac{2\,.\,4}{3\,.\,5}\sin^4\alpha+\dots,\quad\text{where } \frac{\pi}{2}>\alpha>0.$$

[OXF. I. P., 1915.]

31. Show that

$$\int\frac{dx}{(1+x^4)\{(1+x^4)^{\frac{1}{2}}-x^2\}^{\frac{1}{2}}}=\tan^{-1}\frac{x}{\{(1+x^4)^{\frac{1}{2}}-x^2\}^{\frac{1}{2}}}.$$

[EULER, *C.I.*, iv.]

Integrate $$\int\frac{dx}{(1+x^{2n})\{(1+x^{2n})^{\frac{1}{n}}-x^2\}^{\frac{1}{2}}}.$$

32. Show that the integrals

$$\int\frac{dx}{(1-x^m)(2x^m-1)^{\frac{1}{2m}}}\quad\text{and}\quad\int\frac{x^{m-1}\,dx}{(1-x^m)(2x^m-1)^{\frac{1}{2m}}}$$

are reduced to the integration of rational fractions by the respective substitutions $2x^m-1=u^{2m}x^{2m}$ and $2x^m-1=u^{2m}$.

[LEXELL, *Actes de Pétersbourg*, 1781, ii.; LACROIX, *C.D.*, ii., p. 65.]