

794.

NUMBERS, PARTITION OF.

[From the *Encyclopædia Britannica, Ninth Edition*, vol. xvii. (1884), p. 614.]

THIS subject, created by Euler, though relating essentially to positive integer numbers, is scarcely regarded as a part of the Theory of Numbers. We consider in it a number as made up by the *addition* of other numbers: thus the partitions of the successive numbers 1, 2, 3, 4, 5, 6, &c., are as follows:—

- 1;
- 2, 11;
- 3, 21, 111;
- 4, 31, 22, 211, 1111;
- 5, 41, 32, 311, 221, 2111, 11111;
- 6, 51, 42, 411, 33, 321, 3111, 222, 2211, 21111, 111111.

These are formed each from the preceding ones; thus, to form the partitions of 6 we take first 6; secondly, 5 prefixed to each of the partitions of 1 (that is, 51); thirdly, 4 prefixed to each of the partitions of 2 (that is, 42, 411); fourthly, 3 prefixed to each of the partitions of 3 (that is, 33, 321, 3111); fifthly, 2 prefixed, not to each of the partitions of 4, but only to those partitions which begin with a number not exceeding 2 (that is, 222, 2211, 21111); and lastly, 1 prefixed to all the partitions of 5 which begin with a number not exceeding 1 (that is, 111111); and so in other cases.

The method gives *all* the partitions of a number, but we may consider different classes of partitions: the partitions into a given number of parts, or into not more than a given number of parts; or the partitions into given parts, either with repetitions or without repetitions, &c. It is possible, for any particular class of partitions, to obtain methods more or less easy for the formation of the partitions either of a given number or of the successive numbers 1, 2, 3, &c. And of course in any case, having obtained the partitions, we can count them and so obtain the number of partitions.

Another method is by Arbogast's rule of the last and the last but one; in fact, taking the value of a to be unity, and, understanding this letter in each term, the rule gives b ; c , b^2 ; d , bc , b^3 ; e , bd , c^2 , b^2c , b^4 , &c., which, if b , c , d , e , &c., denote 1, 2, 3, 4, &c., respectively, are the partitions of 1, 2, 3, 4, &c., respectively.

An important notion is that of conjugate partitions. Thus a partition of 6 is 42; writing this in the form $\begin{pmatrix} 1111 \\ 11 \end{pmatrix}$, and summing the columns instead of the lines, we obtain the conjugate partition 2211; evidently, starting from 2211, the conjugate partition is 42. If we form all the partitions of 6 into not more than three parts, these are

$$6, \quad 51, \quad 42, \quad 33, \quad 411, \quad 321, \quad 222,$$

and the conjugates are

$$111111, \quad 21111, \quad 2211, \quad 222, \quad 3111, \quad 321, \quad 33,$$

where no part is greater than 3; and so, in general, we have the theorem, the number of partitions of n into not more than k parts is equal to the number of partitions of n with no part greater than k .

We have for the number of partitions an analytical theory depending on generating functions; thus for the partitions of a number n with the parts 1, 2, 3, 4, 5, &c., without repetitions, writing down the product

$$1 + x.1 + x^2.1 + x^3.1 + x^4 \dots = 1 + x + x^2 + 2x^3 + \dots + Nx^n + \dots,$$

it is clear that, if x^α , x^β , x^γ , ... are terms of the series x , x^2 , x^3 , ... for which $\alpha + \beta + \gamma + \dots = n$, then we have in the development of the product a term x^n , and hence that, in the term Nx^n of the product, the coefficient N is equal to the number of partitions of n with the parts 1, 2, 3, ..., without repetitions; or say that the product is the generating function (G. F.) for the number of such partitions. And so in other cases we obtain a generating function.

Thus for the function

$$\frac{1}{1 - x.1 - x^2.1 - x^3 \dots}, = 1 + x + 2x^2 + \dots + Nx^n + \dots,$$

observing that any factor $1/1 - x^l$ is $= 1 + x^l + x^{2l} + \dots$, we see that, in the term Nx^n , the coefficient is equal to the number of partitions of n , with the parts 1, 2, 3, ..., with repetitions.

Introducing another letter z , and considering the function

$$1 + xz.1 + x^2z.1 + x^3z \dots = 1 + z(x + x^2 + \dots) + \dots + Nx^n z^k + \dots,$$

we see that, in the term $Nx^n z^k$ of the development, the coefficient N is equal to the number of partitions of n into k parts, with the parts 1, 2, 3, 4, ..., without repetitions.

And similarly, considering the function

$$\frac{1}{1 - xz.1 - x^2z.1 - x^3z \dots}, = 1 + z(x + x^2 + \dots) + \dots + Nx^n z^k + \dots,$$

we see that, in the term $Nx^n z^k$ of the development, the coefficient N is equal to the number of partitions of n into k parts, with the parts 1, 2, 3, 4, ..., with repetitions.

We have such analytical formulæ as

$$\frac{1}{1 - xz \cdot 1 - x^2z \cdot 1 - x^3z \dots} = 1 + \frac{zx}{1 - x} + \frac{z^2x^2}{1 - x \cdot 1 - x^2} + \dots,$$

which lead to theorems in the Partition of Numbers. A remarkable theorem is

$$1 - x \cdot 1 - x^2 \cdot 1 - x^3 \cdot 1 - x^4 \dots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots,$$

where the only terms are those with an exponent $\frac{1}{2}(3n^2 \pm n)$, and for each such pair of terms the coefficient is $(-)^n 1$. The formula shows that, except for numbers of the form $\frac{1}{2}(3n^2 \pm n)$, the number of partitions without repetitions into an odd number of parts is equal to the number of partitions without repetitions into an even number of parts, whereas for the excepted numbers these numbers differ by unity. Thus for the number 11, which is not an excepted number, the two sets of partitions are

$$11, \quad 821, 731, 641, 632, 542,$$

$$10 \cdot 1, 92, 83, 74, 65, 5321,$$

in each set 6.

We have

$$1 - x \cdot 1 + x \cdot 1 + x^2 \cdot 1 + x^4 \cdot 1 + x^8 \dots = 1;$$

or, as this may be written,

$$1 + x \cdot 1 + x^2 \cdot 1 + x^4 \cdot 1 + x^8 \dots = \frac{1}{1 - x}, = 1 + x + x^2 + x^3 + \dots,$$

showing that a number n can always be made up, and in one way only, with the parts 1, 2, 4, 8, The product on the left-hand side may be taken to k terms only: thus if $k=4$, we have

$$1 + x \cdot 1 + x^2 \cdot 1 + x^4 \cdot 1 + x^8, = \frac{1 - x^{16}}{1 - x}, = 1 + x + x^2 + \dots + x^{15},$$

that is, any number from 1 to 15 can be made up, and in one way only, with the parts 1, 2, 4, 8; and similarly any number from 1 to $2^k - 1$ can be made up, and in one way only, with the parts 1, 2, 4, ..., 2^{k-1} . A like formula is

$$\frac{1 - x^3}{x \cdot 1 - x} \cdot \frac{1 - x^9}{x^3 \cdot 1 - x^3} \cdot \frac{1 - x^{27}}{x^9 \cdot 1 - x^9} \cdot \frac{1 - x^{81}}{x^{27} \cdot 1 - x^{27}} = \frac{1 - x^{81}}{x^{40} \cdot 1 - x};$$

that is,

$$x^{-1} + 1 + x \cdot x^{-3} + 1 + x^3 \cdot x^{-9} + 1 + x^9 \cdot x^{-27} + 1 + x^{27} = x^{-40} + x^{-39} + \dots + 1 + x + \dots + x^{39} + x^{40},$$

showing that any number from -40 to $+40$ can be made up, and that in one way only, with the parts 1, 3, 9, 27 taken positively or negatively; and so in general any number from $-\frac{1}{2}(3^k - 1)$ to $+\frac{1}{2}(3^k - 1)$ can be made up, and that in one way only, with the parts 1, 3, 9, ..., 3^{k-1} taken positively or negatively.