785.

CURVE.

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THIS subject is treated here from an historical point of view, for the purpose of showing how the different leading ideas in the theory were successively arrived at and developed.

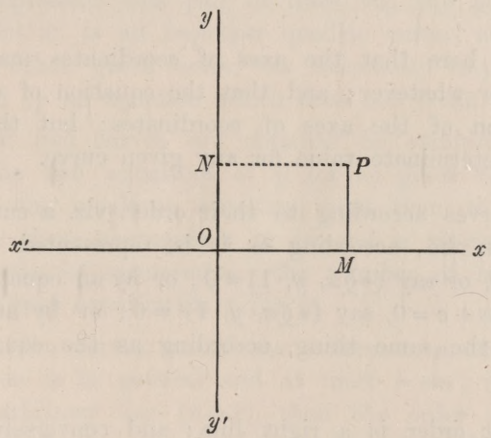
A curve is a line, or continuous singly infinite system of points. We consider in the first instance, and chiefly, a plane curve described according to a law. Such a curve may be regarded geometrically as actually described, or kinematically as in course of description by the motion of a point; in the former point of view, it is the locus of all the points which satisfy a given condition; in the latter, it is the locus of a point moving subject to a given condition. Thus the most simple and earliest known curve, the circle, is the locus of all the points at a given distance from a fixed centre, or else the locus of a point moving so as to be always at a given distance from a fixed centre. (The straight line and the point are not for the moment regarded as curves.)

Next to the circle we have the conic sections, the invention of them attributed to Plato (who lived 430 to 347 B.C.); the original definition of them as the sections of a cone was by the Greek geometers who studied them soon replaced by a proper definition *in plano* like that for the circle, viz. a conic section (or as we now say a "conic") is the locus of a point such that its distance from a given point, the focus, is in a given ratio to its (perpendicular) distance from a given line, the directrix; or it is the locus of a point which moves so as always to satisfy the foregoing condition. Similarly any other property might be used as a definition; an ellipse is the locus of a point such that the sum of its distances from two fixed points (the foci) is constant, &c., &c.

The Greek geometers invented other curves; in particular, the "conchoid," which is the locus of a point such that its distance from a given line, measured along the line drawn through it to a fixed point, is constant; and the "cissoid" which is the locus of a point such that its distance from a fixed point is always equal to the intercept (on the line through the fixed point) between a circle passing through the fixed point and the tangent to the circle at the point opposite to the fixed point. Obviously the number of such geometrical or kinematical definitions is infinite. In a machine of any kind, each point describes a curve; a simple but important instance is the "three-bar curve," or locus of a point in or rigidly connected with a bar pivotted on to two other bars which rotate about fixed centres respectively. Every curve thus arbitrarily defined has its own properties: and there was not any principle of classification.

The principle of classification first presented itself in the *Géométrie* of Descartes (1637). The idea was to represent any curve whatever by means of a relation between the coordinates (x, y) of a point of the curve, or say to represent the curve by means of its equation.

Descartes takes two lines xx', yy', called axes of coordinates, intersecting at a point O called the origin (the axes are usually at right angles to each other, and for the



present they are considered as being so); and he determines the position of a point P by means of its distances OM (or NP) $= x$, and MP (or ON) $= y$, from these two axes respectively; where x is regarded as positive or negative according as it is in the sense Ox or Ox' from O; and similarly y as positive or negative according as it is in the sense Oy or Oy' from O; or, what is the same thing,

				x	y
In the quadrant	xy,	or N.E.,	we have	$+$	$+$
„	$x'y$	„ N.W.	„	$-$	$+$
„	xy'	„ S.E.	„	$+$	$-$
„	$x'y'$	„ S.W.	„	$-$	$-$

Any relation whatever between (x, y) determines a curve, and conversely every curve whatever is determined by a relation between (x, y).

Observe that the distinctive feature is in the *exclusive* use of such determination of a curve by means of its equation. The Greek geometers were perfectly familiar with the property of an ellipse which in the Cartesian notation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of the curve; but it was as one of a number of properties, and in no wise selected out of the others for the characteristic property of the curve *.

We obtain from the equation the notion of an algebraical or geometrical as opposed to a transcendental curve, viz. an algebraical or geometrical curve is a curve having an equation $F(x, y)=0$, where $F(x, y)$ is a rational and integral algebraical function of the coordinates (x, y); and in what follows we attend throughout (unless the contrary is stated) only to such curves. The equation is sometimes given, and may conveniently be used, in an irrational form, but we always imagine it reduced to the foregoing rational and integral form, and regard this as the equation of the curve. And we have hence the notion of a curve of a *given order*, viz. the order of the curve is equal to that of the term or terms of highest order in the coordinates (x, y) conjointly in the equation of the curve; for instance, $xy-1=0$ is a curve of the second order.

It is to be noticed here that the axes of coordinates may be any two lines at right angles to each other whatever; and that the equation of a curve will be different according to the selection of the axes of coordinates; but the order is independent of the axes, and has a determinate value for any given curve.

We hence divide curves according to their order, viz. a curve is of the first order, second order, third order, &c., according as it is represented by an equation of the first order, $ax+by+c=0$, or say $(*\between x, y, 1)=0$; or by an equation of the second order, $ax^2+2hxy+by^2+2fy+2gx+c=0$, say $(*\between x, y, 1)^2=0$; or by an equation of the third order, &c.; or, what is the same thing, according as the equation is linear, quadric, cubic, &c.

A curve of the first order is a right line; and conversely every right line is a curve of the first order.

* There is no exercise more profitable for a student than that of tracing a curve from its equation, or say rather that of so tracing a considerable number of curves. And he should make the equations for himself. The equation should be in the first instance a purely numerical one, where y is given or can be found as an explicit function of x; here, by giving different numerical values to x, the corresponding values of y may be found; and a *sufficient* number of points being thus determined, the curve is traced by drawing a continuous line through these points. The next step should be to consider an equation involving literal coefficients; thus, after such curves as $y=x^3$, $y=x(x-1)(x-2)$, $y=(x-1)\sqrt{x-2}$, &c., he should proceed to trace such curves as $y=(x-a)(x-b)(x-c)$, $y=(x-a)\sqrt{x-b}$, &c., and endeavour to ascertain for what different relations of equality or inequality between the coefficients the curve will assume essentially or notably distinct forms. The purely numerical equations will present instances of nodes, cusps, inflexions, double tangents, asymptotes, &c.,—specialities which he should be familiar with before he has to consider their general theory. And he may then consider an equation such that neither coordinate can be expressed as an explicit function of the other of them (practically, an equation such as $x^3+y^3-3xy=0$, which requires the solution of a cubic equation, belongs to this class); the problem of tracing the curve here frequently requires special methods, and it may easily be such as to require and serve as an exercise for the powers of an advanced algebraist.

A curve of the second order is a conic, or as it is also called a quadric; and conversely every conic, or quadric, is a curve of the second order.

A curve of the third order is called a cubic; one of the fourth order a quartic; and so on.

A curve of the order m has for its equation $(*\between x, y, 1)^m = 0$; and when the coefficients of the function are arbitrary, the curve is said to be the general curve of the order m. The number of coefficients is $\frac{1}{2}(m+1)(m+2)$; but there is no loss of generality if the equation be divided by one coefficient so as to reduce the coefficient of the corresponding term to unity, hence the number of coefficients may be reckoned as $\frac{1}{2}(m+1)(m+2)-1$, that is, $\frac{1}{2}m(m+3)$; and a curve of the order m may be made to satisfy this number of conditions; for example, to pass through $\frac{1}{2}m(m+3)$ points.

It is to be remarked that an equation may *break up;* thus a quadric equation may be $(ax+by+c)(a'x+b'y+c')=0$, breaking up into the two equations $ax+by+c=0$, $a'x+b'y+c'=0$, viz. the original equation is satisfied if either of these is satisfied. Each of these last equations represents a curve of the first order, or right line; and the original equation represents this pair of lines, viz. the pair of lines is considered as a quadric curve. But it is an *improper* quadric curve; and in speaking of curves of the second or any other given order, we frequently imply that the curve is a proper curve represented by an equation which does not break up.

The intersections of two curves are obtained by combining their equations; viz. the elimination from the two equations of y (or x) gives for x (or y) an equation of a certain order, say the resultant equation; and then to each value of x (or y) satisfying this equation there corresponds in general a single value of y (or x), and consequently a single point of intersection; the number of intersections is thus equal to the order of the resultant equation in x (or y).

Supposing that the two curves are of the orders m, n, respectively, then the order of the resultant equation is in general and at most $=mn$; in particular, if the curve of the order n is an arbitrary line $(n=1)$, then the order of the resultant equation is $=m$; and the curve of the order m meets therefore the line in m points. But the resultant equation may have all or any of its roots imaginary, and it is thus not always that there are m real intersections.

The notion of imaginary intersections, thus presenting itself, through algebra, in geometry, must be accepted in geometry—and it in fact plays an all-important part in modern geometry. As in algebra we say that an equation of the mth order has m roots, viz. we state this generally without in the first instance, or it may be without ever, distinguishing whether these are real or imaginary; so in geometry we say that a curve of the mth order is met by an arbitrary line in m points, or rather we thus, through algebra, obtain the proper geometrical definition of a curve of the mth order, as a curve which is met by an arbitrary line in m points (that is, of course, in m, and not more than m, points).

The theorem of the m intersections has been stated in regard to an *arbitrary* line; in fact, for particular lines the resultant equation may be or appear to be of an order less than m; for instance, taking $m=2$, if the hyperbola $xy-1=0$ be cut by the line $y=\beta$, the resultant equation in x is $\beta x-1=0$, and there is apparently only the intersection $\left(x=\frac{1}{\beta}, y=\beta\right)$; but the theorem is, in fact, true for every line whatever: a curve of the order m meets every line whatever in precisely m points. We have, in the case just referred to, to take account of a point at infinity on the line $y=\beta$; the two intersections are the point $\left(x=\frac{1}{\beta}, y=\beta\right)$, and the point at infinity on the line $y=\beta$.

It is moreover to be noticed that the points at infinity may be all or any of them imaginary, and that the points of intersection, whether finite or at infinity, real or imaginary, may coincide two or more of them together, and have to be counted accordingly; to support the theorem in its universality, it is necessary to take account of these various circumstances.

The foregoing notion of a point at infinity is a very important one in modern geometry; and we have also to consider the paradoxical statement that in plane geometry, or say as regards the plane, infinity is a right line. This admits of an easy illustration in solid geometry. If with a given centre of projection, by drawing from it lines to every point of a given line, we project the given line on a given plane, the projection is a line, i.e., this projection is the intersection of the given plane with the plane through the centre and the given line. Say the projection is *always* a line, then if the figure is such that the two planes are parallel, the projection is the intersection of the given plane by a parallel plane, or it is the system of points at infinity on the given plane, that is, these points at infinity are regarded as situate on a given line, the line infinity of the given plane*.

Reverting to the purely plane theory, infinity is a line, related like any other right line to the curve, and thus intersecting it in m points, real or imaginary, distinct or coincident.

Descartes in the *Géométrie* defined and considered the remarkable curves called after him ovals of Descartes, or simply Cartesians, which will be again referred to. The next important work, founded on the *Géométrie*, was Sir Isaac Newton's *Enumeratio linearum tertii ordinis* (1706), establishing a classification of cubic curves founded chiefly on the nature of their infinite branches, which was in some details completed by Stirling, Murdoch, and Cramer; the work contains also the remarkable theorem (to be again referred to), that there are five kinds of cubic curves giving by their projections every cubic curve whatever.

Various properties of curves in general, and of cubic curves, are established in Maclaurin's memoir, "De linearum geometricarum proprietatibus generalibus Tractatus" (posthumous, say 1746, published in the 6th edition of his *Algebra*). We have in it a particular kind of *correspondence* of two points on a cubic curve, viz. two points correspond to each other when the tangents at the two points again meet the cubic in the same point.

* More generally, in solid geometry infinity is a plane,—its intersection with any given plane being the right line which is the infinity of this given plane.

The *Géométrie Descriptive* by Monge was written in the year 1794 or 1795 (7th edition, Paris, 1847), and in it we find stated, *in plano* with regard to the circle, and in three dimensions with regard to a surface of the second order, the fundamental theorem of reciprocal polars, viz. "Given a surface of the second order and a circumscribed conic surface which touches it .... then if the conic surface moves so that its summit is always in the same plane, the plane of the curve of contact passes always through the same point." The theorem is here referred to partly on account of its bearing on the theory of imaginaries in geometry. It is, in Brianchon's memoir "Sur les surfaces du second degré" (*Jour. Polyt.*, t. VI., 1806), shown how for any given position of the summit the plane of contact is determined, or reciprocally; say the plane XY is determined when the point P is given, or reciprocally; and it is noticed that when P is situate in the interior of the surface the plane XY does not cut the surface; that is, we have a real plane XY intersecting the surface in the imaginary curve of contact of the imaginary circumscribed cone having for its summit a given real point P inside the surface.

Stating the theorem in regard to a conic, we have a real point P (called the pole) and a real line XY (called the polar), the line joining the two (real or imaginary) points of contact of the (real or imaginary) tangents drawn from the point to the conic; and the theorem is that when the point describes a line the line passes through a point, this line and point being polar and pole to each other. The term "pole" was first used by Servois, and "polar" by Gergonne (*Gerg.*, t. I. and III., 1810—13); and from the theorem we have the method of reciprocal polars for the transformation of geometrical theorems, used already by Brianchon (in the memoir above referred to) for the demonstration of the theorem called by his name, and in a similar manner by various writers in the earlier volumes of Gergonne. We are here concerned with the method less in itself than as leading to the general notion of duality. And, bearing in a somewhat similar manner also on the theory of imaginaries in geometry (but the notion presents itself in a more explicit form), there is the memoir by Gaultier, on the graphical construction of circles and spheres (*Jour. Polyt.*, t. IX., 1813). The well-known theorem as to radical axes may be stated as follows. Consider two circles partially drawn so that it does not appear whether the circles, if completed, would or would not intersect in real points, say two arcs of circles; then we can, by means of a third circle drawn so as to intersect in two real points each of the two arcs, determine a right line, which, if the complete circles intersect in two real points, passes through the points, and which is on this account regarded as a line passing through two (real or imaginary) points of intersection of the two circles. The construction in fact is, join the two points in which the third circle meets the first arc, and join also the two points in which the third circle meets the second arc, and from the point of intersection of the two joining lines, let fall a perpendicular on the line joining the centre of the two circles; this perpendicular (considered as an indefinite line) is what Gaultier terms the "radical axis of the two circles"; it is a line determined by a real construction and itself always real; and by what precedes it is the line joining two (real or imaginary, as the case may be) intersections of the given circles.

The intersections which lie on the radical axis are two out of the four intersections of the two circles. The question as to the remaining two intersections did not present itself to Gaultier, but it is answered in Poncelet's *Traité des propriétés projectives* (1822), where we find (p. 49) the statement, "deux circles placés arbitrairement sur un plan...ont idéalement deux points imaginaires communs à l'infini"; that is, a circle *qua* curve of the second order is met by the line infinity in two points; but, more than this, they are the same two points for any circle whatever. The points in question have since been called (it is believed first by Dr Salmon) the circular points at infinity, or they may be called the circular points; these are also frequently spoken of as the points I, J; and we have thus the circle characterized as a conic which passes through the two circular points at infinity; the number of conditions thus imposed upon the conic is $= 2$, and there remain three arbitrary constants, which is the right number for the circle. Poncelet throughout his work makes continual use of the foregoing theories of imaginaries and infinity, and also of the before-mentioned theory of reciprocal polars.

Poncelet's two memoirs "Sur les centres des moyennes harmoniques," and "Sur la théorie générale des polaires réciproques," although presented to the Paris Academy in 1824 were only published (*Crelle*, t. III. and IV., 1828, 1829), subsequent to the memoir by Gergonne, "Considérations philosophiques sur les élémens de la science de l'étendue" (*Gerg.*, t. XVI., 1825—26). In this memoir by Gergonne, the theory of duality is very clearly and explicitly stated; for instance, we find "dans la géométrie plane, à chaque théorème il en répond nécessairement un autre qui s'en déduit en échangeant simplement entre eux les deux mots *points* et *droites*; tandis que dans la géométrie de l'espace ce sont les mots *points* et *plans* qu'il faut échanger entre eux pour passer d'un théorème à son corrélatif"; and the plan is introduced of printing correlative theorems, opposite to each other, in two columns. There was a reclamation as to priority by Poncelet in the *Bulletin Universel* reprinted with remarks by Gergonne (*Gerg.*, t. XIX., 1827), and followed by a short paper by Gergonne, "Rectifications de quelques théorèmes, &c.," which is important as first introducing the word *class*. We find in it explicitly the two correlative definitions:—"a plane curve is said to be of the mth degree (order) when it has with a line m real or ideal intersections," and "a plane curve is said to be of the mth class when from any point of its plane there can be drawn to it m real or ideal tangents."

It may be remarked that in Poncelet's memoir on reciprocal polars, above referred to, we have the theorem that the number of tangents from a point to a curve of the order m, or say the class of the curve, is in general and at most $= m(m-1)$, and that he mentions that this number is subject to reduction when the curve has double points or cusps.

The theorem of duality as regards plane figures may be thus stated:—two figures may correspond to each other in such manner that to each point and line in either figure there corresponds in the other figure a line and point respectively. It is to be understood that the theorem extends to all points or lines, drawn or not drawn; thus if in the first figure there are any number of points on a line drawn or not drawn, the corresponding lines in the second figure, produced if necessary, must meet in a point. And we thus see how the theorem extends to curves, their points and tangents: if there is in the first figure a curve of the order m, any line meets it in m points; and hence from the corresponding point in the second figure there must be to the corresponding curve m tangents; that is, the corresponding curve must be of the class m.

Trilinear coordinates (to be again referred to) were first used by Bobillier in the memoir, "Essai sur un nouveau mode de recherche des propriétés de l'étendue" (*Gerg.*, t. XVIII., 1827—28). It is convenient to use these rather than Cartesian coordinates. We represent a curve of the order m by an equation $(*\between x, y, z)^m = 0$, the function on the left-hand being a homogeneous rational and integral function of the order m of the three coordinates (x, y, z); clearly the number of constants is the same as for the equation $(*\between x, y, 1)^m = 0$ in Cartesian coordinates.

The theory of duality is considered and developed, but chiefly in regard to its metrical applications, by Chasles in the "Mémoire de géométrie sur deux principes généraux de la science, la dualité et l'homographie," which forms a sequel to the "Aperçu historique sur l'origine et le développement des méthodes en géométrie" (*Mem. de Brux.*, t. XI., 1837).

We now come to Plücker; his "six equations" were given in a short memoir in *Crelle* (1842) preceding his great work, the *Theorie der algebraischen Curven* (1844).

Plücker first gave a scientific dual definition of a curve, viz. "A curve is a locus generated by a point, and enveloped by a line,—the point moving continuously along the line, while the line rotates continuously about the point"; the point is a point (ineunt) of the curve, the line is a tangent of the curve.

And, assuming the above theory of geometrical imaginaries, a curve such that m of its points are situate in an arbitrary line is said to be of the order m; a curve such that n of its tangents pass through an arbitrary point is said to be of the class n; as already appearing, this notion of the order and the class of a curve is, however, due to Gergonne. Thus the line is a curve of the order 1 and the class 0; and corresponding dually thereto, we have the point as a curve of the order 0 and the class 1.

Plücker moreover imagined a system of line-coordinates (tangential coordinates). The Cartesian coordinates (x, y) and trilinear coordinates (x, y, z) are point-coordinates for determining the position of a point; the new coordinates, say (ξ, η, ζ), are line-coordinates for determining the position of a line. It is possible, and (not so much for any application thereof as in order to more fully establish the analogy between the two kinds of coordinates) important, to give independent quantitative definitions of the two kinds of coordinates; but we may also derive the notion of line-coordinates from that of point-coordinates; viz. taking $\xi x + \eta y + \zeta z = 0$ to be the equation of a line, we say that (ξ, η, ζ) are the line-coordinates of this line. A linear relation $a\xi + b\eta + c\zeta = 0$ between these coordinates determines a point, viz. the point whose point-coordinates are (a, b, c); in fact, the equation in question $a\xi + b\eta + c\zeta = 0$ expresses that the equation $\xi x + \eta y + \zeta z = 0$, where (x, y, z) are current point-coordinates, is satisfied on writing therein $x, y, z = a, b, c$; or that the line in question passes through the point (a, b, c). Thus (ξ, η, ζ) are the line-coordinates of any line whatever; but when these, instead of being absolutely arbitrary, are subject to the restriction $a\xi + b\eta + c\zeta = 0$, this obliges the line to pass through a point (a, b, c); and the last-mentioned equation $a\xi + b\eta + c\zeta = 0$ is considered as the line-equation of this point.

A line has only a point-equation, and a point has only a line-equation; but any other curve has a point-equation and also a line-equation; the point-equation $(*\between x, y, z)^m = 0$ is the relation which is satisfied by the point-coordinates (x, y, z) of each point of the curve; and similarly the line-equation $(*\between \xi, \eta, \zeta)^n = 0$ is the relation which is satisfied by the line-coordinates (ξ, η, ζ) of each line (tangent) of the curve.

There is in analytical geometry little occasion for any explicit use of line-coordinates; but the theory is very important; it serves to show that, in demonstrating by point-coordinates any purely descriptive theorem whatever, we demonstrate the correlative theorem; that is, we do not demonstrate the one theorem, and then (as by the method of reciprocal polars) deduce from it the other, but we do at one and the same time demonstrate the two theorems; our (x, y, z) instead of meaning point-coordinates may mean line-coordinates, and the demonstration is then in every step of it a demonstration of the correlative theorem.

The above dual generation explains the nature of the singularities of a plane curve. The ordinary singularities, arranged according to a cross division, are

	Proper.	*Improper.*
Point-singularities—	1. The stationary point, cusp, or spinode;	2. The double point, or node;
Line-singularities—	3. The stationary tangent, or inflexion;	4. The double tangent:—

arising as follows:—

1. The cusp: the point as it travels along the line may come to rest, and then reverse the direction of its motion.

3. The stationary tangent: the line may in the course of its rotation come to rest, and then reverse the direction of its rotation.

2. The node: the point may in the course of its motion come to coincide with a former position of the point, the two positions of the line not in general coinciding.

4. The double tangent: the line may in the course of its motion come to coincide with a former position of the line, the two positions of the point not in general coinciding.

It may be remarked that we cannot with a real point and line obtain the node with two imaginary tangents (conjugate or isolated point, or acnode), nor again the real double tangent with two imaginary points of contact; but this is of little consequence, since in the general theory the distinction between real and imaginary is not attended to.

The singularities (1) and (3) have been termed proper singularities, and (2) and (4) improper; in each of the first-mentioned cases there is a real singularity, or peculiarity in the motion; in the other two cases there is not; in (2) there is not when the point is first at the node, or when it is secondly at the node, any peculiarity in the motion; the singularity consists in the point coming twice into the same position; and so in (4) the singularity is in the line coming twice into the same position. Moreover (1) and (2) are, the former a proper singularity, and the latter an improper singularity, *as regards the motion of the point;* and similarly (3) and (4) are, the former a proper singularity, and the latter an improper singularity, *as regards the motion of the line.*

But as regards the representation of a curve by an equation, the case is very different.

First, if the equation be in point-coordinates, (3) and (4) are in a sense not singularities at all. The curve $(*\between x, y, z)^m = 0$, or general curve of the order m, has double tangents and inflexions; (2) presents itself as a singularity, for the equations $d_x(*\between x, y, z)^m = 0$, $d_y(*\between x, y, z)^m = 0$, $d_z(*\between x, y, z)^m = 0$, implying $(*\between x, y, z)^m = 0$, are not in general satisfied by any values (a, b, c) whatever of (x, y, z), but if such values exist, then the point (a, b, c) is a node or double point; and (1) presents itself as a further singularity or sub-case of (2), a cusp being a double point for which the two tangents become coincident.

In line-coordinates all is reversed:—(1) and (2) are not singularities; (3) presents itself as a sub-case of (4).

The theory of compound singularities will be referred to further on.

In regard to the ordinary singularities, we have

m, the order,

n „ class,

δ „ number of double points,

ι „ „ cusps,

τ „ „ double tangents,

κ „ „ inflexions;

and this being so, Plücker's "six equations" are

(1) $n = m(m-1) - 2\delta - 3\kappa,$

(2) $\iota = 3m(m-2) - 6\delta - 8\kappa,$

(3) $\tau = \frac{1}{2}m(m-2)(m^2-9) - (m^2-m-6)(2\delta+3\kappa) + 2\delta(\delta-1) + 6\delta\kappa + \frac{9}{2}\kappa(\kappa-1),$

(4) $m = n(n-1) - 2\tau - 3\iota,$

(5) $\kappa = 3n(n-2) - 6\tau - 8\iota,$

(6) $\delta = \frac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2\tau+3\iota) + 2\tau(\tau-1) + 6\tau\iota + \frac{9}{2}\iota(\iota-1).$

It is easy to derive the further forms—

$$\begin{aligned}
&(7) && \iota - \kappa && = 3(n-m),\\
&(8) && 2(\tau - \delta) && = (n-m)(n+m-9),\\
&(9) && \tfrac{1}{2}m(m+3) - \delta - 2\kappa && = \tfrac{1}{2}n(n+3) - \tau - 2\iota,\\
&(10) && \tfrac{1}{2}(m-1)(m-2) - \delta - \kappa && = \tfrac{1}{2}(n-1)(n-2) - \tau - \iota,\\
&(11,\ 12) && m^2 - 2\delta - 3\kappa && = n^2 - 2\tau - 3\iota, = m+n,
\end{aligned}$$

the whole system being equivalent to three equations only: and it may be added that, using α to denote the equal quantities $3m+\iota$ and $3n+\kappa$, everything may be expressed in terms of m, n, α. We have

$$\begin{aligned}
\kappa &= \alpha - 3n,\\
\iota &= \alpha - 3m,\\
2\delta &= m^2 - m + 8n - 3\alpha,\\
2\tau &= n^2 - n + 8m - 3\alpha.
\end{aligned}$$

It is implied in Plücker's theorem that, m, n, δ, κ, τ, ι signifying as above in regard to any curve, then in regard to the reciprocal curve n, m, τ, ι, δ, κ will have the same significations, viz. for the reciprocal curve these letters denote respectively the order, class, number of nodes, cusps, double tangents, and inflexions.

The expression $\frac{1}{2}m(m+3) - \delta - 2\kappa$ is that of the number of the disposable constants in a curve of the order m with δ nodes and κ cusps (in fact that there shall be a node is 1 condition, a cusp 2 conditions): and the equation (9) thus expresses that the curve and its reciprocal contain each of them the same number of disposable constants.

For a curve of the order m, the expression $\frac{1}{2}m(m-1) - \delta - \kappa$ is termed the "deficiency" (as to this more hereafter); the equation (10) expresses therefore that the curve and its reciprocal have each of them the same deficiency.

The relations $m^2 - 2\delta - 3\kappa = n^2 - 2\tau - 3\iota, = m+n$, present themselves in the theory of envelopes, as will appear further on.

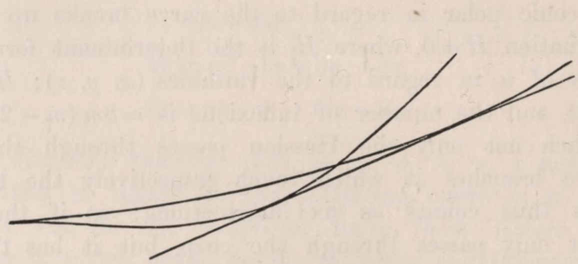
With regard to the demonstration of Plücker's equations it is to be remarked that we are not able to write down the equation in point-coordinates of a curve of the order m, having the given numbers δ and κ of nodes and cusps. We can only use the general equation $(*\between x, y, z)^m = 0$, say for shortness $u=0$, of a curve of the mth order, which equation, so long as the coefficients remain arbitrary, represents a curve without nodes or cusps. Seeking then, for this curve, the values n, ι, τ of the class, number of inflexions, and number of double tangents,—first, as regards the class, this is equal to the number of tangents which can be drawn to the curve from an arbitrary point, or what is the same thing, it is equal to the number of the points of contact of these tangents. The points of contact are found as the intersections of the curve $u=0$ by a curve depending on the position of the arbitrary point, and called the "first polar" of this point; the order of the first polar is $=m-1$, and the number of intersections is thus $= m(m-1)$. But it can be shown, analytically or geometrically, that if the given curve has a node, the first polar passes through this node, which therefore counts as two intersections: and that if the curve has a cusp, the first polar passes through the cusp, touching the curve there, and hence the cusp counts as three intersections. But, as is evident, the node or cusp is not a point of contact of a proper tangent from the arbitrary point; we have, therefore, for a node a diminution 2, and for a cusp a diminution 3, in the number of the intersections; and thus, for a curve with δ nodes and κ cusps, there is a diminution $2\delta + 3\kappa$, and the value of n is $n = m(m-1) - 2\delta - 3\kappa$.

Secondly, as to the inflexions, the process is a similar one; it can be shown that the inflexions are the intersections of the curve by a derivative curve called (after Hesse, who first considered it) the Hessian, defined geometrically as the locus of a point such that its conic polar in regard to the curve breaks up into a pair of lines, and which has an equation $H = 0$, where H is the determinant formed with the second differential coefficients of u in regard to the variables (x, y, z); $H = 0$ is thus a curve of the order $3(m-2)$, and the number of inflexions is $= 3m(m-2)$. But if the given curve has a node, then not only the Hessian passes through the node, but it has there a node the two branches at which touch respectively the two branches of the curve, and the node thus counts as six intersections; so if the curve has a cusp, then the Hessian not only passes through the cusp, but it has there a cusp through which it again passes, that is, there is a cuspidal branch touching the cuspidal branch of the curve, and besides a simple branch passing through the cusp, and hence the cusp counts as eight intersections. The node or cusp is not an inflexion, and we have thus for a node a diminution 6, and for a cusp a diminution 8, in the number of the intersections; hence for a curve with δ nodes and κ cusps, the diminution is $= 6\delta + 8\kappa$, and the number of inflexions is $\iota = 3m(m-2) - 6\delta - 8\kappa$.

Thirdly, for the double tangents; the points of contact of these are obtained as the intersections of the curve by a curve $\Pi = 0$, which has not as yet been geometrically defined, but which is found analytically to be of the order $(m-2)(m^2-9)$; the number of intersections is thus $= m(m-2)(m^2-9)$; but if the given curve has a node then there is a diminution $= 4(m^2 - m - 6)$, and if it has a cusp then there is a diminution $= 6(m^2 - m - 6)$, where, however, it is to be noticed that the factor $(m^2 - m - 6)$ is in the case of a curve having only a node or only a cusp the number of the tangents which can be drawn from the node or cusp to the curve, and is used as denoting the number of these tangents, and ceases to be the correct expression if the number of nodes and cusps is greater than unity. Hence, in the case of a curve which has δ nodes and κ cusps, the apparent diminution $2(m^2 - m - 6)(2\delta + 3\kappa)$ is too great, and it has in fact to be diminished by $2\{2\delta(\delta - 1) + 6\delta\kappa + \frac{9}{2}\kappa(\kappa - 1)\}$, or the half thereof is 4 for each pair of nodes, 6 for each combination of a node and cusp, and 9 for each pair of cusps. We have thus finally an expression for 2τ, $= m(m-2)(m^2-9) -$ &c.; or dividing the whole by 2, we have the expression for τ given by the third of Plücker's equations.

It is obvious that we cannot by consideration of the equation $u = 0$ in point-coordinates obtain the remaining three of Plücker's equations; they might be obtained in a precisely analogous manner by means of the equation $v=0$ in line-coordinates, but they follow at once from the principle of duality, viz. they are obtained by the mere interchange of m, δ, κ with n, τ, ι respectively.

To complete Plücker's theory it is necessary to take account of compound singularities; it might be possible, but it is at any rate difficult, to effect this by considering the curve as in course of description by the point moving along the rotating line; and it seems easier to consider the compound singularity as arising from the variation of an actually described curve with ordinary singularities. The most simple case is when three double points come into coincidence, thereby giving rise to a triple point; and a somewhat more complicated one is when we have a cusp of the second kind, or node-cusp arising from the coincidence of a node, a cusp, an inflexion, and a double tangent, as shown in the annexed figure, which represents the singularities as on the



point of coalescing. The general conclusion (see Cayley, *Quart. Math. Jour.* t. VII., 1866, [374], "On the higher singularities of a plane curve") is that every singularity whatever may be considered as compounded of ordinary singularities, say we have a singularity $=\delta'$ nodes, κ' cusps, τ' double tangents, and ι' inflexions. So that, in fact, Plücker's equations properly understood apply to a curve with any singularities whatever.

By means of Plücker's equations we may form a table—

m	n	δ	κ	τ	ι
0	1	—	—	0	0
1	0	0	0	—	—
2	2	0	0	0	0
3	6	0	0	0	9
"	4	1	0	0	3
"	3	0	1	0	1
4	12	0	0	28	24
"	10	1	0	16	18
"	9	0	1	10	16
"	8	2	0	8	12
"	7	1	1	4	10
"	6	0	2	1	8
"	6	3	0	4	6
"	5	2	1	2	4
"	4	1	2	1	2
"	3	0	3	1	0

The table is arranged according to the value of m; and we have $m=0$, $n=1$, the point; $m=1$, $n=0$, the line; $m=2$, $n=2$, the conic; of $m=3$, the cubic, there are three cases, the class being 6, 4, or 3, according as the curve is without singularities, or as it has 1 node, or 1 cusp; and so of $m=4$, the quartic, there are nine cases, where observe that in *two* of them the class is $=6$,—the reduction of class arising from two cusps or else from three nodes. The nine cases may be also grouped together into four, according as the number of nodes and cusps $(\delta+\kappa)$ is $=0$, 1, 2, or 3.

The cases may be divided into sub-cases, by the consideration of compound singularities; thus when $m=4$, $n=6$, $=3$, the three nodes may be all distinct, which is the general case, or two of them may unite together into the singularity called a tacnode, or all three may unite together into a triple point, or else into an oscnode.

We may further consider the inflexions and double tangents, as well in general as in regard to cubic and quartic curves.

The expression for the number of inflexions $3m(m-2)$ for a curve of the order m was obtained analytically by Plücker, but the theory was first given in a complete form by Hesse in the two papers "Ueber die Elimination, u.s.w.," and "Ueber die Wendepuncte der Curven dritter Ordnung" (*Crelle*, t. XXVIII., 1844); in the latter of these the points of inflexion are obtained as the intersections of the curve $u=0$ with the Hessian, or curve $\Delta=0$, where Δ is the determinant formed with the second derived functions of u. We have in the Hessian the first instance of a covariant of a ternary form. The whole theory of the inflexions of a cubic curve is discussed in a very interesting manner by means of the canonical form of the equation $x^3+y^3+z^3+6lxyz=0$; and in particular a proof is given of Plücker's theorem that the nine points of inflexion of a cubic curve lie by threes in twelve lines.

It may be noticed that the nine inflexions of a cubic curve are three real, six imaginary; the three real inflexions lie in a line, as was known to Newton and Maclaurin. For an acnodal cubic the six imaginary inflexions disappear, and there remain three real inflexions lying in a line. For a crunodal cubic, the six inflexions which disappear are two of them real, the other four imaginary, and there remain two imaginary inflexions and one real inflexion. For a cuspidal cubic the six imaginary inflexions and two of the real inflexions disappear, and there remains one real inflexion.

A quartic curve has 24 inflexions; it was conjectured by Salmon, and has been verified recently by Zeuthen, that at most 8 of these are real.

The expression $\frac{1}{2}m(m-2)(m^2-9)$ for the number of double tangents of a curve of the order m was obtained by Plücker only as a consequence of his first, second, fourth, and fifth equations. An investigation by means of the curve $\Pi=0$, which by its intersections with the given curve determines the points of contact of the double tangents, is indicated by Cayley, "Recherches sur l'élimination et la théorie des courbes", (*Crelle*, t. XXXIV., 1847), [53]: and in part carried out by Hesse in the memoir "Ueber Curven dritter Ordnung" (*Crelle*, t. XXXVI., 1848). A better process was indicated by Salmon in the "Note on the double tangents to plane curves," *Phil. Mag.* 1858; considering the $m-2$ points in which any tangent to the curve again meets the curve, he showed how to form the equation of a curve of the order $(m-2)$, giving by its intersection with the tangent the points in question; making the tangent touch this curve of the order $(m-2)$, it will be a double tangent of the original curve. See Cayley, "On the Double Tangents of a Plane Curve", (*Phil. Trans.* t. CXLVIII., 1859), [260], and Dersch (*Math. Ann.* t. VII., 1874). The solution is still in so far incomplete that we have no properties of the curve $\Pi=0$, to distinguish one such curve from the several other curves which pass through the points of contact of the double tangents.

A quartic curve has 28 double tangents, their points of contact determined as the intersections of the curve by a curve $\Pi=0$ of the order 14, the equation of which in a very elegant form was first obtained by Hesse (1849). Investigations in regard to them are given by Plücker in the *Theorie der algebraischen Curven,* and in two memoirs by Hesse and Steiner (*Crelle*, t. XLV., 1855), in respect to the triads of double tangents which have their points of contact on a conic, and other like relations. It was assumed by Plücker that the number of real double tangents might be 28, 16, 8, 4, or 0, but Zeuthen has recently found that the last case does not exist.

The Hessian Δ has just been spoken of as a covariant of the form u; the notion of invariants and covariants belongs rather to the form u than to the curve $u=0$ represented by means of this form; and the theory may be very briefly referred to. A curve $u=0$ may have some invariantive property, viz. a property independent of the particular axes of coordinates used in the representation of the curve by its equation; for instance, the curve may have a node, and in order to this, a relation, say $A=0$, must exist between the coefficients of the equation; supposing the axes of coordinates altered, so that the equation becomes $u'=0$, and writing $A'=0$ for the relation between the new coefficients, then the relations $A=0$, $A'=0$, as two different expressions of the same geometrical property, must each of them imply the other; this can only be the case when A, A' are functions differing only by a constant factor, or say, when A is an invariant of u. If, however, the geometrical property requires two or more relations between the coefficients, say $A=0$, $B=0$, &c., then we must have between the new coefficients the like relations, $A'=0$, $B'=0$, &c., and the two systems of equations must each of them imply the other; when this is so, the system of equations, $A=0$, $B=0$, &c., is said to be invariantive, but it does not follow that A, B, &c., are of necessity invariants of u. Similarly, if we have a curve $U=0$ derived from the curve $u=0$ in a manner independent of the particular axes of coordinates, then from the transformed equation $u'=0$ deriving in like manner the curve $U'=0$, the two equations $U=0$, $U'=0$ must each of them imply the other; and when this is so, U will be a covariant of u. The case is less frequent, but it may arise, that there are covariant systems $U=0$, $V=0$, &c., and $U'=0$, $V'=0$, &c., each implying the other, but where the functions U, V, &c., are not of necessity covariants of u.

The theory of the invariants and covariants of a ternary cubic function u has been studied in detail, and brought into connexion with the cubic curve $u=0$; but the theory of the invariants and covariants for the next succeeding case, the ternary quartic function, is still very incomplete.

In further illustration of the Plückerian dual generation of a curve, we may consider the question of the *envelope* of a variable curve. The notion is very probably older, but it is at any rate to be found in Lagrange's *Théorie des fonctions analytiques* (1798); it is there remarked that the equation obtained by the elimination of the parameter a from an equation $f(x, y, a) = 0$ and the derived equation in respect to a is a curve, the envelope of the series of curves represented by the equation $f(x, y, a) = 0$ in question. To develope the theory, consider the curve corresponding to any particular value of the parameter; this has with the consecutive curve (or curve belonging to the consecutive value of the parameter) a certain number of intersections, and of common tangents, which may be considered as the tangents at the intersections; and the so-called envelope is the curve which is at the same time generated by the points of intersection and enveloped by the common tangents; we have thus a dual generation. But the question needs to be further examined. Suppose that in general the variable curve is of the order m with δ nodes and κ cusps, and therefore of the class n with τ double tangents and ι inflexions, m, n, δ, κ, τ, ι being connected by the Plückerian equations,—the number of nodes or cusps may be greater for particular values of the parameter, but this is a speciality which may be here disregarded. Considering the variable curve corresponding to a given value of the parameter, or say simply the variable curve, the consecutive curve has then also δ and κ nodes and cusps, consecutive to those of the variable curve; and it is easy to see that among the intersections of the two curves we have the nodes each counting twice, and the cusps each counting three times; the number of the remaining intersections is $= m^2 - 2\delta - 3\kappa$. Similarly among the common tangents of the two curves we have the double tangents each counting twice, and the stationary tangents each counting three times, and the number of the remaining common tangents is $= n^2 - 2\tau - 3\iota$ $(= m^2 - 2\delta - 3\kappa$, inasmuch as each of these numbers is as was seen $= m + n)$. At any one of the $m^2 - 2\delta - 3\kappa$ points the variable curve and the consecutive curve have tangents distinct from yet infinitesimally near to each other, and each of these two tangents is also infinitesimally near to one of the $n^2 - 2\tau - 3\iota$ common tangents of the two curves; whence, attending only to the variable curve, and considering the consecutive curve as coming into actual coincidence with it, the $n^2 - 2\tau - 3\iota$ common tangents are the tangents to the variable curve at the $m^2 - 2\delta - 3\kappa$ points respectively, and the envelope is at the same time generated by the $m^2 - 2\delta - 3\kappa$ points, and enveloped by the $n^2 - 2\tau - 3\iota$ tangents; we have thus a dual generation of the envelope, which only differs from Plücker's dual generation, in that in place of a single point and tangent we have the group of $m^2 - 2\delta - 3\kappa$ points and $n^2 - 2\tau - 3\iota$ tangents.

The parameter which determines the variable curve may be given as a point upon a given curve, or say as a parametric point; that is, to the different positions of the parametric point on the given curve correspond the different variable curves, and the nature of the envelope will thus depend on that of the given curve; we have thus the envelope as a derivative curve of the given curve. Many well-known derivative curves present themselves in this manner; thus the variable curve may be the normal (or line at right angles to the tangent) at any point of the given curve; the intersection of the consecutive normals is the centre of curvature; and we have the evolute as at once the locus of the centre of curvature and the envelope of the normal. It may be added that the given curve is one of a series of curves, each cutting the several normals at right angles. Any one of these is a "parallel" of the given curve; and it can be obtained as the envelope of a circle of constant radius having its centre on the given curve. We have in like manner, as derivatives of a given curve, the caustic, catacaustic, or diacaustic, as the case may be, and the secondary caustic, or curve cutting at right angles the reflected or refracted rays.

We have in much that precedes disregarded, or at least been indifferent to, reality; it is only thus that the conception of a curve of the mth order, as one which is met by every right line in m points, is arrived at; and the curve itself, and the line which cuts it, although both are tacitly assumed to be real, may perfectly well be imaginary. For real figures we have the general theorem that imaginary intersections, &c., present themselves in conjugate pairs: hence, in particular, that a curve of an even order is met by a line in an even number (which may be $=0$) of points; a curve of an odd order in an odd number of points, hence in one point at least; it will be seen further on that the theorem may be generalized in a remarkable manner. Again, when there is in question only one pair of points or lines, these, if coincident, must be real; thus, a line meets a cubic curve in three points, one of them real, the other two real or imaginary; but if two of the intersections coincide they must be real, and we have a line cutting a cubic in one real point and touching it in another real point. It may be remarked that this is a limit separating the two cases where the intersections are all real, and where they are one real, two imaginary.

Considering always real curves, we obtain the notion of a branch; any portion capable of description by the continuous motion of a point is a branch; and a curve consists of one or more branches. Thus the curve of the first order or right line consists of one branch; but in curves of the second order, or conics, the ellipse and the parabola consist each of one branch, the hyperbola of two branches. A branch is either re-entrant, or it extends both ways to infinity, and in this case, we may regard it as consisting of two legs (*crura*, Newton), each extending one way to infinity, but without any definite separation. The branch, whether re-entrant or infinite, may have a cusp or cusps, or it may cut itself or another branch, thus having or giving rise to crunodes; an acnode is a branch by itself,—it may be considered as an indefinitely small re-entrant branch. A branch may have inflexions and double tangents, or there may be double tangents which touch two distinct branches; there are also double tangents with imaginary points of contact, which are thus lines having no visible connexion with the curve. A re-entrant branch not cutting itself may be everywhere convex, and it is then properly said to be an oval; but the term oval may be used more generally for any re-entrant branch not cutting itself; and we may thus speak of a once indented, twice indented oval, &c., or even of a cuspidate oval. Other descriptive names for ovals and re-entrant branches cutting themselves may be used when required; thus, in the last-mentioned case a simple form is that of a figure of eight; such a form may break up into two ovals, or into a doubly indented oval or hour-glass. A form which presents itself is when two ovals, one inside the other, unite, so as to give rise to a crunode—in default of a better name this may be called, after the curve of that name, a limaçon. Names may also be used for the different forms of infinite branches, but we have first to consider the distinction of hyperbolic and parabolic. The leg of an infinite branch may have at the extremity a tangent; this is an asymptote of the curve, and the leg is then hyperbolic; or the leg may tend to a fixed direction, but so that the tangent goes further and further off to infinity, and the leg is then parabolic; a branch may thus be hyperbolic or parabolic as to its two legs; or it may be hyperbolic as to one leg, and parabolic as to the other. The epithets hyperbolic and parabolic are of course derived from the conics hyperbola and parabola respectively. The nature of the two kinds of branches is best understood by considering them as projections, in the same way as we in effect consider the hyperbola and the parabola as projections of the ellipse. If a line Ω cut an arc aa', so that the two segments ab, ba' lie on opposite sides of the line, then projecting the figure so that the line Ω goes off to infinity, the tangent at b is projected into the asymptote, and the arc ab is projected into a hyperbolic leg touching the asymptote at one extremity; the arc ba' will at the same time be projected into a hyperbolic leg touching the same asymptote at the other extremity (and on the opposite side), but so that the two hyperbolic legs may or may not belong to one and the same branch. And we thus see that the two hyperbolic legs belong to a simple intersection of the curve by the line infinity. Next, if the line Ω touch at b the arc aa' so that the two portions ab', ba lie on the same side of the line Ω, then projecting the figure as before, the tangent at b, that is, the line Ω itself, is projected to infinity; the arc ab is projected into a parabolic leg, and at the same time the arc ba' is projected into a parabolic leg, having at infinity the same direction as the other leg, but so that the two legs may or may not belong to the same branch. And we thus see that the two parabolic legs represent a contact of the line infinity with the curve,—the point of contact being of course the point at infinity determined by the common direction of the two legs. It will readily be understood how the like considerations apply to other cases,—for instance, if the line Ω is a tangent at an inflexion, passes through a crunode, or touches one of the branches of a crunode, &c.; thus, if the line Ω passes through a crunode we have pairs of hyperbolic legs belonging to two parallel asymptotes. The foregoing considerations also show (what is very important) how different branches are connected together at infinity, and lead to the notion of a complete branch, or circuit.

The two legs of a hyperbolic branch may belong to different asymptotes, and in this case we have the forms which Newton calls inscribed, circumscribed, ambigene, &c.; or they may belong to the same asymptote, and in this case we have the serpentine form, where the branch cuts the asymptote, so as to touch it at its two extremities on opposite sides, or the conchoidal form, where it touches the asymptote on the same side. The two legs of a parabolic branch may converge to ultimate parallelism, as in the conic parabola, or diverge to ultimate parallelism, as in the semi-cubical parabola $y^2=x^3$, and the branch is said to be convergent, or divergent, accordingly; or they may tend to parallelism in opposite senses, as in the cubical parabola $y=x^3$. As mentioned with regard to a branch generally, an infinite branch of any kind may have cusps, or, by cutting itself or another branch, may have or give rise to a crunode, &c.

We may now consider the various forms of cubic curves, as appearing by Newton's *Enumeratio*, and by the figures belonging thereto. The species are reckoned as 72, which are numbered accordingly 1 to 72; but to these should be added 10^a, 13^a, 22^a, and 22^b. It is not intended here to consider the division into species, nor even completely that into genera, but only to explain the principle of classification. It may be remarked generally that there are at most three infinite branches, and that there may besides be a re-entrant branch or oval.

The genera may be arranged as follows:—

1, 2, 3, 4	redundant hyperbolas,
5, 6	defective hyperbolas,
7, 8	parabolic hyperbolas,
9	hyperbolisms of hyperbola,
10	„ „ ellipse,
11	„ „ parabola,
12	trident curve,
13	divergent parabolas,
14	cubic parabola;

and, thus arranged, they correspond to the different relations of the line infinity to the curve. First, if the three intersections by the line infinity are all distinct, we have the hyperbolas; if the points are real, the redundant hyperbolas, with three hyperbolic branches; but if only one of them is real, the defective hyperbolas, with one hyperbolic branch. Secondly, if two of the intersections coincide, say if the line infinity meets the curve in a onefold point and a twofold point, both of them real, then there is always one asymptote: the line infinity may at the twofold point touch the curve, and we have the parabolic hyperbolas; or the twofold point may be a singular point,—viz. a crunode giving the hyperbolisms of the hyperbola; an acnode, giving the hyperbolisms of the ellipse; or a cusp, giving the hyperbolisms of the parabola. As regards the so-called hyperbolisms, observe that (besides the single asymptote) we have in the case of those of the hyperbola two parallel asymptotes; in the case of those of the ellipse the two parallel asymptotes become imaginary, that is, they disappear, and in the case of those of the parabola they become coincident, that is, there is here an ordinary asymptote, and a special asymptote answering to a cusp at infinity. Thirdly, the three intersections by the line infinity may be coincident and real; or say we have a threefold point: this may be an inflexion, a crunode, or a cusp, that is, the line infinity may be a tangent at an inflexion, and we have the divergent parabolas; a tangent at a crunode to one branch, and we have the trident curve; or lastly, a tangent at a cusp, and we have the cubical parabola.

It is to be remarked that the classification mixes together non-singular and singular curves, in fact, the five kinds presently referred to: thus the hyperbolas and the divergent parabolas include curves of every kind, the separation being made in the species; the hyperbolisms of the hyperbola and ellipse, and the trident curve, are nodal; the hyperbolisms of the parabola, and the cubical parabola, are cuspidal. The divergent parabolas are of five species which respectively belong to and determine the five kinds of cubic curves; Newton gives (in two short paragraphs without any development) the remarkable theorem that the five divergent parabolas by their shadows generate and exhibit all the cubic curves.

The five divergent parabolas are curves each of them symmetrical with regard to an axis. There are two non-singular kinds, the one with, the other without, an oval, but each of them has an infinite (as Newton describes it) *campaniform* branch; this cuts the axis at right angles, being at first convex, but ultimately concave, towards the axis, the two legs continually tending to become at right angles to the axis. The oval may unite itself with the infinite branch, or it may dwindle into a point, and we have the crunodal and the acnodal forms respectively; or if simultaneously the oval dwindles into a point and unites itself to the infinite branch, we have the cuspidal form. Drawing a line to cut any one of these curves and projecting the line to infinity, it would not be difficult to show how the line should be drawn in order to obtain a curve of any given species. We have herein a better principle of classification; considering cubic curves, in the first instance, according to singularities, the curves are non-singular, nodal (viz. crunodal or acnodal), or cuspidal; and we see further that there are two kinds of non-singular curves, the complex and the simplex. There is thus a complete division into the five kinds, the complex, simplex, crunodal, acnodal, and cuspidal. Each singular kind presents itself as a limit separating two kinds of inferior singularity; the cuspidal separates the crunodal and the acnodal, and these last separate from each other the complex and the simplex.

The whole question is discussed very fully and ably by Möbius in the memoir "Ueber die Grundformen der Linien dritter Ordnung" (*Abh. der K. Sachs. Ges. zu Leipzig*, t. I., 1852; *Ges. Werke*, t. I.). The author considers not only plane curves, but also cones, or, what is almost the same thing, the spherical curves which are their sections by a concentric sphere. Stated in regard to the cone, we have there the fundamental theorem that there are two different kinds of sheets: viz. the single sheet, not separated into two parts by the vertex (an instance is afforded by the plane considered as a cone of the first order generated by the motion of a line about a point), and the double or twin-pair sheet, separated into two parts by the vertex (as in the cone of the second order). And it then appears that there are two kinds of non-singular cubic cones, viz. the simplex, consisting of a single sheet, and the complex, consisting of a single sheet and a twin-pair sheet; and we thence obtain (as for cubic curves) the crunodal, the acnodal, and the cuspidal kinds of cubic cones. It may be mentioned that the single sheet is a sort of wavy form, having upon it three lines of inflexion, and which is met by any plane through the vertex in one or in three lines; the twin-pair sheet has no lines of inflexion, and resembles in its form a cone on an oval base.

In general a cone consists of one or more single or twin-pair sheets, and if we consider the section of the cone by a plane, the curve consists of one or more complete branches, or say circuits, each of them the section of one sheet of the cone;

thus, a cone of the second order is one twin-pair sheet, and any section of it is one circuit composed, it may be, of two branches. But although we thus arrive by projection at the notion of a circuit, it is not necessary to go out of the plane, and we may (with Zeuthen, using the shorter term *circuit* for his *complete branch*) define a circuit as any portion (of a curve) capable of description by the continuous motion of a point, it being understood that a passage through infinity is permitted. And we then say that a curve consists of one or more circuits; thus the right line, or curve of the first order, consists of one circuit; a curve of the second order consists of one circuit; a cubic curve consists of one circuit or else of two circuits.

A circuit is met by any right line always in an even number, or always in an odd number, of points, and it is said to be an even circuit or an odd circuit accordingly; the right line is an odd circuit, the conic an even circuit. And we have then the theorem, two odd circuits intersect in an odd number of points; an odd and an even circuit, or two even circuits, in an even number of points. An even circuit not cutting itself divides the plane into two parts, the one called the internal part, incapable of containing any odd circuit, the other called the external part, capable of containing an odd circuit.

We may now state in a more convenient form the fundamental distinction of the kinds of cubic curve. A non-singular cubic is simplex, consisting of one odd circuit, or it is complex, consisting of one odd circuit and one even circuit. It may be added that there are on the odd circuit three inflexions, but on the even circuit no inflexion; it hence also appears that from any point of the odd circuit there can be drawn to the odd circuit two tangents, and to the even circuit (if any) two tangents, but that from a point of the even circuit there cannot be drawn (either to the odd or the even circuit) any real tangent; consequently, in a simplex curve the number of tangents from any point is two; but in a complex curve the number is four, or none,—four if the point is on the odd circuit, none if it is on the even circuit. It at once appears from inspection of the figure of a non-singular cubic curve, which is the odd and which the even circuit. The singular kinds arise as before; in the crunodal and the cuspidal kinds the whole curve is an odd circuit, but in the acnodal kind the acnode must be regarded as an even circuit.

The analogous question of the classification of quartics (in particular non-singular quartics and nodal quartics) is considered in Zeuthen's memoir "Sur les différentes formes des courbes planes du quatrième ordre" (*Math. Ann.* t. VII., 1874). A non-singular quartic has only even circuits; it has at most four circuits external to each other, or two circuits one internal to the other, and in this last case the internal circuit has no double tangents or inflexions. A very remarkable theorem is established as to the double tangents of such a quartic:—distinguishing as a double tangent of the first kind a real double tangent which either twice touches the same circuit, or else touches the curve in two imaginary points, the number of the double tangents of the first kind of a non-singular quartic is $=4$; it follows that the quartic has at most 8 real inflexions. The forms of the non-singular quartics are very numerous, but it is not necessary to go further into the question.

We may consider in relation to a curve, not only the line infinity, but also the circular points at infinity; assuming the curve to be real, these present themselves always conjointly; thus a circle is a conic passing through the two circular points, and is thereby distinguished from other conics. Similarly a cubic through the two circular points is termed a circular cubic; a quartic through the two points is termed a circular quartic, and if it passes twice through each of them, that is, has each of them for a node, it is termed a bicircular quartic. Such a quartic is of course binodal $(m=4,\ \delta=2,\ \kappa=0)$; it has not in general, but it may have, a third node, or a cusp. Or again, we may have a quartic curve having a cusp at each of the circular points: such a curve is a "Cartesian," it being a complete definition of the Cartesian to say that it is a bicuspidal quartic curve $(m=4,\ \delta=0,\ \kappa=2)$, having a cusp at each of the circular points. The circular cubic and the bicircular quartic, together with the Cartesian (being in one point of view a particular case thereof), are interesting curves which have been much studied, generally, and in reference to their *focal* properties.

The points called *foci* presented themselves in the theory of the conic, and were well known to the Greek geometers, but the general notion of a focus was first established by Plücker, in the memoir "Ueber solche Puncte die bei Curven einer höheren Ordnung den Brennpuncten der Kegelschnitte entsprechen," (*Crelle*, t. x., 1833). We may from each of the circular points draw tangents to a given curve; the intersection of two such tangents (belonging of course to the two circular points respectively) is a focus. There will be from each circular point λ tangents (λ, a number depending on the class of the curve and its relation to the line infinity and the circular points, $=2$ for the general conic, 1 for the parabola, 2 for a circular cubic or a bicircular quartic, &c.); the λ tangents from the one circular point and those from the other circular point intersect in λ real foci (viz. each of these is the only real point on each of the tangents through it), and in $\lambda^2-\lambda$ imaginary foci; each pair of real foci determines a pair of imaginary foci (the so-called antipoints of the two real foci), and the $\frac{1}{2}\lambda(\lambda-1)$ pairs of real foci thus determine the $\lambda^2-\lambda$ imaginary foci. There are in some cases points termed centres, or singular or multiple foci (the nomenclature is unsettled), which are the intersections of improper tangents from the two circular points respectively; thus, in the circular cubic, the tangents to the curve at the two circular points respectively (or two imaginary asymptotes of the curve) meet in a centre.

The notions of *distance* and of lines *at right angles* are connected with the circular points; and almost every construction of a curve by means of lines of a determinate length, or at right angles to each other, and (as such) mechanical constructions by means of linkwork, give rise to curves passing the same definite number of times through the two circular points respectively, or say to circular curves, and in which the fixed centres of the construction present themselves as ordinary, or as singular, foci. Thus the general curve of three-bar motion (or locus of the vertex of a triangle, the other two vertices whereof move on fixed circles) is a tricircular sextic, having besides three nodes $(m=6,\ \delta=3+3+3,\ =9)$, and having the centres of the fixed circles each for a singular focus; there is a third singular focus, and we have thus the remarkable theorem (due to Mr S. Roberts) of the triple generation of the curve by means of the three several pairs of singular foci.

Again, the normal, *qua* line at right angles to the tangent, is connected with the circular points, and these accordingly present themselves in the before-mentioned theories of evolutes and parallel curves.

We have several recent theories which depend on the notion of *correspondence:* two points whether in the same plane or in different planes, or on the same curve or in different curves, may determine each other in such wise that to any given position of the first point there correspond α' positions of the second point, and to any given position of the second point α positions of the first point; the two points have then an (α, α') correspondence; and if α, α' are each $=1$, then the two points have a (1, 1) or rational correspondence. Connecting with each theory the author's name, the theories in question are—Riemann, the rational transformation of a plane curve; Cremona, the rational transformation of a plane; and Chasles, correspondence of points on the same curve, and united points. The theory first referred to, with the resulting notion of Geschlecht, or *deficiency*, is more than the other two an essential part of the theory of curves, but they will all be considered.

Riemann's results are contained in the memoirs on "Theorie der Abel'schen Functionen," (*Crelle*, t. LIV., 1857); and we have next Clebsch, "Ueber die Singularitäten algebraischer Curven," (*Crelle*, t. LXV., 1865), and Cayley, "On the Transformation of Plane Curves," (*Proc. Lond. Math. Soc.* t. I., 1865, [384]). The fundamental notion of the rational transformation is as follows:—

Taking u, X, Y, Z to be rational and integral functions (X, Y, Z all of the same order) of the coordinates (x, y, z), and u', X', Y', Z' rational and integral functions (X', Y', Z' all of the same order) of the coordinates (x', y', z'), we transform a given curve $u=0$, by the equations $x' : y' : z' = X : Y : Z$, thereby obtaining a transformed curve $u'=0$, and a converse set of equations $x : y : z = X' : Y' : Z'$; viz. assuming that this is so, the point (x, y, z) on the curve $u=0$ and the point (x', y', z') on the curve $u'=0$ will be points having a (1, 1) correspondence. To show how this is, observe that to a given point (x, y, z) on the curve $u=0$ there corresponds a single point (x', y', z') determined by the equations $x' : y' : z' = X : Y : Z$; from these equations and the equation $u=0$ eliminating x, y, z we obtain the equation $u'=0$ of the transformed curve. To a given point (x', y', z') not on the curve $u'=0$ there corresponds, not a single point, but the system of points (x, y, z) given by the equations $x' : y' : z' = X : Y : Z$, viz. regarding x', y', z' as constants (and to fix the ideas, assuming that the curves $X=0$, $Y=0$, $Z=0$ have no common intersections), these are the points of intersection of the curves $X : Y : Z = x' : y' : z'$, but no one of these points is situate on the curve $u=0$. If, however, the point (x', y', z') is situate on the curve $u'=0$, then one point of the system of points in question is situate on the curve $u=0$, that is, to a given point of the curve $u'=0$ there corresponds a single point of the curve $u=0$; and hence also this point must be given by a system of equations such as $x : y : z = X' : Y' : Z'$.

It is an old and easily proved theorem that, for a curve of the order m, the number $\delta+\kappa$ of nodes and cusps is at most $=\frac{1}{2}(m-1)(m-2)$; for a given curve the deficiency of the actual number of nodes and cusps below this maximum number, viz.

$\frac{1}{2}(m-1)(m-2)-\delta-\kappa$, is the "Geschlecht," or "deficiency," of the curve, say this is $=D$. When $D=0$, the curve is said to be unicursal, when $=1$, bicursal, and so on.

The general theorem is that two curves corresponding rationally to each other have the same deficiency. In particular, a curve and its reciprocal have this rational or (1, 1) correspondence, and it has been already seen that a curve and its reciprocal have the same deficiency.

A curve of a given order can in general be rationally transformed into a curve of a lower order; thus a curve of any order for which $D=0$, that is, a unicursal curve, can be transformed into a line; a curve of any order having the deficiency 1 or 2 can be rationally transformed into a curve of the order $D+2$, deficiency D; and a curve of any order deficiency = or > 3 can be rationally transformed into a curve of the order $D+3$, deficiency D.

Taking x', y', z' as coordinates of a point of the transformed curve, and in its equation writing $x' : y' : z' = 1 : \theta : \phi$ we have ϕ a certain irrational function of θ, and the theorem is that the coordinates x, y, z of any point of the given curve can be expressed as proportional to rational and integral functions of θ, ϕ, that is, of θ and a certain irrational function of θ.

In particular, if $D=0$, that is, if the given curve be unicursal, the transformed curve is a line, ϕ is a mere linear function of θ, and the theorem is that the coordinates x, y, z of a point of the unicursal curve can be expressed as proportional to rational and integral functions of θ; it is easy to see that for a given curve of the order m, these functions of θ must be of the same order m.

If $D=1$, then the transformed curve is a cubic; it can be shown that in a cubic, the axes of coordinates being properly chosen, ϕ can be expressed as the square root of a quartic function of θ; and the theorem is that the coordinates x, y, z of a point of the bicursal curve can be expressed as proportional to rational and integral functions of θ, and of the square root of a quartic function of θ.

And so if $D=2$, then the transformed curve is a nodal quartic; ϕ can be expressed as the square root of a sextic function of θ, and the theorem is, that the coordinates x, y, z of a point of the tricursal curve can be expressed as proportional to rational and integral functions of θ, and of the square root of a sextic function of θ. But when $D=3$, we have no longer the like law, viz. ϕ is not expressible as the square root of an octic function of θ.

Observe that the radical, square root of a quartic function, is connected with the theory of elliptic functions, and the radical, square root of a sextic function, with that of the first kind of Abelian functions, but that the next kind of Abelian functions does not depend on the radical, square root of an octic function.

It is a form of the theorem for the case $D=1$, that the coordinates x, y, z of a point of the bicursal curve, or in particular the coordinates of a point of the cubic, can be expressed as proportional to rational and integral functions of the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$; in fact, taking the radical to be $\sqrt{1-\theta^2 . 1-k^2\theta^2}$, and writing

$\theta = \operatorname{sn} u$, the radical becomes $= \operatorname{cn} u \,.\, \operatorname{dn} u$; and we have expressions of the form in question.

It will be observed that the equations $x' : y' : z' = X : Y : Z$ before-mentioned do not of themselves lead to the other system of equations $x : y : z = X' : Y' : Z'$, and thus that the theory does not in anywise establish a (1, 1) correspondence between the points (x, y, z) and (x', y', z') of two planes or of the same plane; this is the correspondence of Cremona's theory.

In this theory, given in the memoirs "Sulle trasformazioni geometriche delle figure piane," *Mem. di Bologna*, t. II. (1863), and t. V. (1865), we have a system of equations $x' : y' : z' = X : Y : Z$ which *does* lead to a system $x : y : z = X' : Y' : Z'$, where, as before, X, Y, Z denote rational and integral functions, all of the same order, of the coordinates x, y, z, and X', Y', Z' rational and integral functions, all of the same order, of the coordinates x', y', z', and there is thus a (1, 1) correspondence given by these equations between the two points (x, y, z) and (x', y', z'). To explain this, observe that starting from the equations $x' : y' : z' = X : Y : Z$, to a given point (x, y, z) there corresponds one point (x', y', z'), but that if n be the order of the functions X, Y, Z, then to a given point x', y', z' there would, if the curves $X = 0$, $Y = 0$, $Z = 0$ had no common intersections, correspond n^2 points (x, y, z). If, however, the functions are such that the curves $X = 0$, $Y = 0$, $Z = 0$ have k common intersections, then among the n^2 points are included these k points, which are fixed points independent of the point (x', y', z'); so that, disregarding these fixed points, the number of points (x, y, z) corresponding to the given point (x', y', z') is $= n^2 - k$; and in particular if $k = n^2 - 1$, then we have one corresponding point; and hence the original system of equations $x' : y' : z' = X : Y : Z$ must lead to the equivalent system $x : y : z = X' : Y' : Z'$; and in this system by the like reasoning the functions must be such that the curves $X' = 0$, $Y' = 0$, $Z' = 0$ have $n'^2 - 1$ common intersections. The most simple example is in the two systems of equations $x' : y' : z' = yz : zx : xy$ and $x : y : z = y'z' : z'x' : x'y'$; where $yz = 0$, $zx = 0$, $xy = 0$ are conics (pairs of lines) having three common intersections, and where obviously either system of equations leads to the other system. In the case where X, Y, Z are of an order exceeding 2, the required number $n^2 - 1$ of common intersections can only occur by reason of common multiple points on the three curves; and assuming that the curves $X = 0$, $Y = 0$, $Z = 0$ have $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1}$ common intersections, where the α_1 points are ordinary points, the α_2 points are double points, the α_3 points are triple points, &c., on each curve, we have the condition

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 + \ldots + (n-1)^2 \alpha_{n-1} = n^2 - 1\,;$$

but to this must be joined the condition

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 + \ldots + \tfrac{1}{2}(n-1)(n-2)\,\alpha_{n-1} = \tfrac{1}{2}n(n+3) - 2,$$

(without which the transformation would be illusory); and the conclusion is that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ may be any numbers satisfying these two equations. It may be added that the two equations together give

$$\alpha_2 + 3\alpha_3 + \ldots + \tfrac{1}{2}(n-1)(n-2)\,\alpha_{n-1} = \tfrac{1}{2}(n-1)(n-2),$$

which expresses that the curves $X=0$, $Y=0$, $Z=0$ are unicursal. The transformation may be applied to any curve $u=0$, which is thus rationally transformed into a curve $u'=0$, by a rational transformation such as is considered in Riemann's theory; hence the two curves have the same deficiency.

Coming next to Chasles, the principle of correspondence is established and used by him in a series of memoirs relating to the conics which satisfy given conditions, and to other geometrical questions, contained in the *Comptes Rendus*, t. LVIII. *et seq.* (1864 to the present time). The theorem of united points in regard to points in a right line was given in a paper, June—July 1864, and it was extended to unicursal curves in a paper of the same series (March 1866), "Sur les courbes planes ou à double courbure dont les points peuvent se déterminer individuellement—application du principe de correspondance dans la théorie de ces courbes."

The theorem is as follows: if in a unicursal curve two points have an (α, β) correspondence, then the number of united points (or points each corresponding to itself) is $=\alpha+\beta$. In fact, in a unicursal curve the coordinates of a point are given as proportional to rational and integral functions of a parameter, so that any point of the curve is determined uniquely by means of this parameter; that is, to each point of the curve corresponds one value of the parameter, and to each value of the parameter one point on the curve; and the (α, β) correspondence between the two points is given by an equation of the form $(*\between\theta, 1)^\alpha(\phi, 1)^\beta=0$ between their parameters θ and ϕ; at a united point $\phi=\theta$, and the value of θ is given by an equation of the order $\alpha+\beta$. The extension to curves of any given deficiency D was made in the memoir of Cayley, "On the correspondence of two points on a curve,"—*Proc. Lond. Math. Soc.* t. I. (1866), [385],—viz. taking P, P' as the corresponding points in an (α, α') correspondence on a curve of deficiency D, and supposing that when P is given the corresponding points P' are found as the intersections of the curve by a curve Θ containing the coordinates of P as parameters, and having with the given curve k intersections at the point P, then the number of united points is $a=\alpha+\alpha'+2kD$; and more generally, if the curve Θ intersect the given curve in a set of points P' each p times, a set of points Q' each q times, &c., in such manner that the points (P, P'), the points (P, Q'), &c., are pairs of points corresponding to each other according to distinct laws; then if (P, P') are points having an (α, α') correspondence with a number $=a$ of united points, (P, Q') points having a (β, β') correspondence with a number $=b$ of united points, and so on, the theorem is that we have

$$p(a-\alpha-\alpha')+q(b-\beta-\beta')+\ldots=2kD.$$

The principle of correspondence, or say rather the theorem of united points, is a most powerful instrument of investigation, which may be used in place of analysis for the determination of the number of solutions of almost every geometrical problem. We can by means of it investigate the class of a curve, number of inflexions, &c.,—in fact, Plücker's equations; but it is necessary to take account of special solutions; thus, in one of the most simple instances, in finding the class of a curve, the cusps present themselves as special solutions.

Imagine a curve of order m, deficiency D, and let the corresponding points P, P' be such that the line joining them passes through a given point O; this is an $(m-1,\ m-1)$ correspondence, and the value of k is $=1$, hence the number of united points is $=2m-2+2D$; the united points are the points of contact of the tangents from O and (as special solutions) the cusps, and we have thus the relation $n+\kappa=2m-2+2D$; or, writing $D=\frac{1}{2}(m-1)(m-2)-\delta-\kappa$, this is $n=m(m-1)-2\delta-3\kappa$, which is right.

The principle in its original form as applying to a right line was used throughout by Chasles in the investigations on the number of the conics which satisfy given conditions, and on the number of solutions of very many other geometrical problems.

There is one application of the theory of the $(\alpha,\ \alpha')$ correspondence between two planes which it is proper to notice.

Imagine a curve, real or imaginary, represented by an equation (involving, it may be, imaginary coefficients) between the Cartesian coordinates u, u'; then, writing $u=x+iy$, $u'=x'+iy'$, the equation determines real values of $(x,\ y)$, and of $(x',\ y')$, corresponding to any given real values of $(x',\ y')$ and $(x,\ y)$ respectively; that is, it establishes a real correspondence (not of course a rational one) between the points $(x,\ y)$ and $(x',\ y')$; for example in the imaginary circle $u^2+u'^2=(a+bi)^2$, the correspondence is given by the two equations $x^2-y^2+x'^2-y'^2=a^2-b^2$, $xy+x'y'=ab$. We have thus a means of geometrical representation for the portions, as well imaginary as real, of any real or imaginary curve. Considerations such as these have been used for determining the series of values of the independent variable, and the irrational functions thereof in the theory of Abelian integrals, but the theory seems to be worthy of further investigation.

The researches of Chasles (*Comptes Rendus*, t. LVIII., 1864, *et seq.*) refer to the conics which satisfy given conditions. There is an earlier paper by De Jonquières, "Théorèmes généraux concernant les courbes géométriques planes d'un ordre quelconque," *Liouv.* t. VI. (1861), which establishes the notion of a system of curves (of any order) of the index N, viz. considering the curves of the order n which satisfy $\frac{1}{2}n(n+3)-1$ conditions, then the index N is the number of these curves which pass through a given arbitrary point. But Chasles in the first of his papers (February 1864), considering the conics which satisfy four conditions, establishes the notion of the two characteristics $(\mu,\ \nu)$ of such a system of conics, viz. μ is the number of the conics which pass through a given arbitrary point, and ν is the number of the conics which touch a given arbitrary line. And he gives the theorem, a system of conics satisfying four conditions, and having the characteristics $(\mu,\ \nu)$ contains $2\nu-\mu$ line-pairs (that is, conics, each of them a pair of lines), and $2\mu-\nu$ point-pairs (that is, conics, each of them a pair of points,—coniques infiniment aplaties), which is a fundamental one in the theory. The characteristics of the system can be determined when it is known how many there are of these two kinds of degenerate conics in the system, and how often each is to be counted. It was thus that Zeuthen (in the paper *Nyt Bydrag*, "Contribution to the Theory of Systems of Conics which satisfy four Conditions," Copenhagen, 1865, translated with an addition in the *Nouvelles Annales*) solved the question of finding the characteristics of the systems of conics which satisfy four conditions of contact with a given curve or curves; and this led to the solution of the further problem of finding the number of the conics which satisfy five conditions of contact with a given curve or curves (Cayley, *Comptes Rendus*, t. LXIII., 1866, [377]), and "On the Curves which satisfy given Conditions" (*Phil. Trans.* t. CLVIII., 1868, [406]).

It may be remarked that although, as a process of investigation, it is very convenient to seek for the characteristics of a system of conics satisfying 4 conditions, yet what is really determined is in every case the number of the conics which satisfy 5 conditions; the characteristics of the system ($4p$) of the conics which pass through $4p$ points are ($5p$), ($4p$, $1l$), the number of the conics which pass through 5 points, and which pass through 4 points and touch 1 line: and so in other cases. Similarly as regards cubics, or curves of any other order: a cubic depends on 9 constants, and the elementary problems are to find the number of the cubics ($9p$), ($8p$, $1l$), &c., which pass through 9 points, pass through 8 points and touch 1 line, &c.; but it is in the investigation convenient to seek for the characteristics of the systems of cubics ($8p$), &c., which satisfy 8 instead of 9 conditions.

The elementary problems in regard to cubics are solved very completely by Maillard in his *Thèse, Recherche des caractéristiques des systèmes élémentaires des courbes planes du troisième ordre* (Paris, 1871). Thus, considering the several cases of a cubic

		No. of consts.
1.	With a given cusp	5,
2.	„ cusp on given line	6,
3.	„ cusp	7,
4.	„ a given node	6,
5.	„ node on given line	7,
6.	„ node	8,
7.	non-singular	9,

he determines in every case the characteristics (μ, ν) of the corresponding systems of cubics ($4p$), ($3p$, $1l$), &c. The same problems, or most of them, and also the elementary problems in regard to quartics are solved by Zeuthen, who in the elaborate memoir "Almindelige Egenskaber, &c.," *Danish Academy*, t. X. (1873), considers the problem in reference to curves of any order, and applies his results to cubic and quartic curves.

The methods of Maillard and Zeuthen are substantially identical; in each case the question considered is that of finding the characteristics (μ, ν) of a system of curves by consideration of the special or degenerate forms of the curves included in the system. The quantities which have to be considered are very numerous. Zeuthen in the case of curves of any given order establishes between the characteristics μ, ν', and 18 other quantities, in all 20 quantities, a set of 24 equations (equivalent to 23 independent equations), involving (besides the 20 quantities) other quantities relating to the various forms of the degenerate curves, which supplementary terms he determines, partially for curves of any order, but completely only for quartic curves. It is in the discussion and complete enumeration of the special or degenerate forms of the curves, and of the supplementary terms to which they give rise, that the great difficulty of the question seems to consist; it would appear that the 24 equations are a complete system, and that (subject to a proper determination of the supplementary terms) they contain the solution of the general problem.

The remarks which follow have reference to the analytical theory of the degenerate curves which present themselves in the foregoing problem of the curves which satisfy given conditions.

A curve represented by an equation in point-coordinates may break up: thus if P_1, P_2,... be rational and integral functions of the coordinates (x, y, z) of the orders m_1, m_2,... respectively, we have the curve $P_1^{\alpha_1}P_2^{\alpha_2}\ldots = 0$, of the order $m, = \alpha_1 m_1 + \alpha_2 m_2 + \ldots$, composed of the curve $P_1 = 0$ taken α_1 times, the curve $P_2 = 0$ taken α_2 times, &c.

Instead of the equation $P_1^{\alpha_1}P_2^{\alpha_2}\ldots = 0$, we may start with an equation $u = 0$, where u is a function of the order m containing a parameter θ, and for a particular value say $\theta = 0$, of the parameter reducing itself to $P_1^{\alpha_1}P_2^{\alpha_2}\ldots$. Supposing θ indefinitely small, we have what may be called the penultimate curve, and when $\theta = 0$ the ultimate curve. Regarding the ultimate curve as derived from a given penultimate curve, we connect with the ultimate curve, and consider as belonging to it, certain points called "summits" on the component curves $P_1 = 0$, $P_2 = 0$, respectively; a summit Σ is a point such that, drawing from an arbitrary point O the tangents to the penultimate curve, we have $O\Sigma$ as the limit of one of these tangents. The ultimate curve together with its summits may be regarded as a degenerate form of the curve $u = 0$. Observe that the positions of the summits depend on the penultimate curve $u = 0$, viz. on the values of the coefficients in the terms multiplied by θ, $\theta^2, \ldots$; they are thus in some measure arbitrary points as regards the ultimate curve $P_1^{\alpha_1}P_2^{\alpha_2}\ldots = 0$.

It may be added that we have summits only on the component curves $P_1 = 0$, of a multiplicity $\alpha_1 > 1$; the number of summits on such a curve is in general $= (\alpha_1^2 - \alpha_1)\, m_1^2$. Thus assuming that the penultimate curve is without nodes or cusps, the number of the tangents to it is $= m^2 - m, = (\alpha_1 m_1 + \alpha_2 m_2 + \ldots)^2 - (\alpha_1 m_1 + \alpha_2 m_2 + \ldots)$, taking $P_1 = 0$ to have δ_1 nodes and κ_1 cusps, and therefore its class n_1 to be $= m_1^2 - m_1 - 2\delta_1 - 3\kappa_1$, &c., the expression for the number of tangents to the penultimate curve is

$$= (\alpha_1^2 - \alpha_1)\, m_1^2 + (\alpha_2^2 - \alpha_2)\, m_2^2 + \ldots + 2\alpha_1\alpha_2 m_1 m_2 + \ldots + \alpha_1 (n_1 + 2\delta_1 + 3\kappa_1) + \alpha_2 (n_2 + 2\delta_2 + 3\kappa_2) + \ldots$$

where a term $2\alpha_1\alpha_2 m_1 m_2$ indicates tangents which are in the limit the lines drawn to the intersections of the curves $P_1 = 0$, $P_2 = 0$ each line $2\alpha_1\alpha_2$ times; a term $\alpha_1(n_1 + 2\delta_1 + 3\kappa_1)$ tangents which are in the limit the proper tangents to $P_1 = 0$ each α_1 times, the lines to its nodes each $2\alpha_1$ times, and the lines to its cusps each $3\alpha_1$ times; the remaining terms $(\alpha_1^2 - \alpha_1)\, m_1^2 + (\alpha_2^2 - \alpha_2)\, m_2^2 + \ldots$ indicate tangents which are in the limit the lines drawn to the several summits, that is, we have $(\alpha_1^2 - \alpha_1)\, m_1^2$ summits on the curve $P_1 = 0$, &c.

There is of course a precisely similar theory as regards line-coordinates; taking Π_1, Π_2, &c., to be rational and integral functions of the coordinates (ξ, η, ζ), we connect with the ultimate curve $\Pi_1^{\alpha_1}\Pi_2^{\alpha_2}\ldots = 0$, and consider as belonging to it certain lines, which for the moment may be called "axes," tangents to the component curves

$\Pi_1 = 0$, $\Pi_2 = 0$ respectively. Considering an equation in point-coordinates, we may have among the component curves right lines; and, if in order to put these in evidence, we take the equation to be $L_1^{\gamma_1} \ldots P_1^{\alpha_1} \ldots = 0$, where $L_1 = 0$ is a right line, $P_1 = 0$ a curve of the second or any higher order, then the curve will contain as part of itself summits not exhibited in this equation, but the corresponding line-equation will be $\Lambda_1^{\delta_1} \ldots \Pi_1^{\alpha_1} \ldots = 0$, where $\Lambda_1 = 0, \ldots$ are the equations of the summits in question, $\Pi_1 = 0$, &c., are the line-equations corresponding to the several point-equations $P_1 = 0$, &c.; and this curve will contain as part of itself axes not exhibited by this equation, but which are the lines $L_1 = 0, \ldots$ of the equation in point-coordinates.

In conclusion a little may be said as to curves of double curvature, otherwise twisted curves, or curves in space. The analytical theory by Cartesian coordinates was first considered by Clairaut, *Recherches sur les courbes à double courbure* (Paris, 1731). Such a curve may be considered as described by a point, moving in a line which at the same time rotates about the point in a plane which at the same time rotates about the line; the point is a point, the line a tangent, and the plane an osculating plane, of the curve; moreover the line is a generating line, and the plane a tangent plane, of a developable surface or torse, having the curve for its edge of regression. Analogous to the order and class of a plane curve we have the order, rank, and class, of the system (assumed to be a geometrical one), viz. if an arbitrary plane contains m points, an arbitrary line meets r lines, and an arbitrary point lies in n planes, of the system, then m, r, n are the order, rank, and class respectively. The system has singularities, and there exist between m, r, n and the numbers of the several singularities equations analogous to Plücker's equations for a plane curve.

It is a leading point in the theory that a curve in space cannot in general be represented by means of two equations $U = 0$, $V = 0$; the two equations represent surfaces, intersecting in a curve; but there are curves which are not the complete intersection of any two surfaces; thus we have the cubic in space, or skew cubic, which is the residual intersection of two quadric surfaces which have a line in common; the equations $U = 0$, $V = 0$ of the two quadric surfaces represent the cubic curve, not by itself, but together with the line.