777.

A SOLVABLE CASE OF THE QUINTIC EQUATION.

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THE roots of the general quintic equation

$$(a,\ b,\ c,\ d,\ e,\ f \between x,\ 1)^5 = 0$$

may be taken to be

$$\begin{array}{l} -\dfrac{b}{a} + \quad B + \quad C + \quad D + \quad E \\ - \ ,, + \omega^4 \ ,, + \omega^3 \ ,, + \omega^2 \ ,, + \omega \ \ ,, \\ - \ ,, + \omega^3 \ ,, + \omega \ \ ,, + \omega^4 \ ,, + \omega^2 \ ,, \\ - \ ,, + \omega^2 \ ,, + \omega^4 \ ,, + \omega \ \ ,, + \omega^3 \ ,, \\ - \ ,, + \omega \ \ ,, + \omega^2 \ ,, + \omega^3 \ ,, + \omega^4 \ ,,, \end{array}$$

where ω is an imaginary fifth root of unity; and if one of the four functions B, C, D, E is $=0$, say if $E=0$ (this implies of course a single relation between the coefficients), then the equation is solvable.

Writing $x = \xi - \dfrac{b}{a}$, we have

$$(a,\ b,\ c,\ d,\ e,\ f)\left(\xi - \frac{b}{a},\ 1\right)^5 = (a',\ 0,\ c',\ d',\ e',\ f' \between \xi,\ 1)^5,$$

where

$$\begin{aligned} a' &= a, \\ ac' &= ac - b^2, \\ a^2d' &= a^2d - 3abc + 2b^3, \\ a^3e' &= a^3e - 4a^2bd + 6ab^2c - 3b^4, \\ a^4f' &= a^4f - 5a^3be + 10ab^2d - 10ab^2c + 4b^5, \end{aligned}$$

and the roots of the new equation

$$(a',\ 0,\ c',\ d',\ e',\ f' \between \xi,\ 1)^5 = 0$$

have the above-mentioned values, omitting therefrom the terms $-\frac{b}{a}$; we find without difficulty

$$2\frac{c'}{a'} = -BE - CD,$$
$$2\frac{d'}{a'} = -B^2D - BC^2 - CE^2 - D^2E,$$
$$\frac{e'}{a'} = -B^3C - B^2E^2 + BCDE + BD^3 + C^3E + C^2D^2 - DE^2,$$
$$\frac{f'}{a'} = -B^5 + 5B^3DE - 5B^2C^2E - 5B^2CD^2 + 5BC^3D + 5BCE^3$$
$$-5BD^2E^2 - C^5 + 5CD^3E - 5CD^2E^2 - D^5 - E^5,$$

and hence, when $E = 0$, we have

$$2\frac{c'}{a'} = -CD,$$
$$2\frac{d'}{a'} = -B^2D - BC^2,$$
$$\frac{e'}{a'} = -B^3C - BD^3 - C^2D^2,$$
$$\frac{f'}{a'} = -B^5 - 5B^2CD^2 + 5BC^3D - C^5 - D^5,$$

or, as these may be written,

$$-2\frac{c'}{a'} = CD,$$
$$-2\frac{d'}{a'} = B^2D + BC^2,$$
$$-\frac{e'}{a'} - 4\frac{c'^2}{a'^2} = B^3C - BD^3,$$
$$-\frac{f'}{a'} = B^5 + C^5 + D^5 - 10\frac{c'}{a'}(B^2D - BC^2),$$

equations which imply a single relation between the coefficients a', c', d', e', f'. Supposing this satisfied, we may attend only to the first three equations; or, writing for convenience,

$$\gamma = -2\frac{c'}{a'}, \qquad = -\frac{2}{a^2}(ac - b^2),$$
$$\delta = -2\frac{d'}{a'}, \qquad = -\frac{2}{a^3}(a^2d - 3abc + 2b^3),$$
$$\theta = -\frac{e'}{a'} - 4\frac{c'^2}{a'^2}, \quad = -\frac{1}{a^4}\{a^2(ae - 4bd + 3c^2) + (ac - b^2)^2\},$$

the equations are

$$\gamma = CD,$$
$$\delta = B(BD + C^2),$$
$$\theta = B(D^3 - B^2C).$$

The first equation gives $C=\frac{\gamma}{D}$, and substituting this value in the other two equations, we have

$$B^2D^3+B\gamma^4-\delta D^2=0,$$
$$B^3\gamma+BD^4+\theta D=0.$$

Eliminating B, the result is obtained in the form Det. = 0. where in the last column of the determinant each term is divisible by D; and omitting this factor, the result is

$$\begin{vmatrix} & & D^3, & \gamma^2, & -\delta D \\ & D^3, & \gamma^2, & -\delta D^2, & \\ D^3, & \gamma^2, & -\delta D^2, & & \\ & \gamma, & 0, & -D^4, & \theta \\ \gamma, & 0, & -D^4, & \theta D, & \end{vmatrix}=0.$$

If, in order to develope the determinant, we consider it as a sum of products, each first factor being a minor composed out of columns 1 and 2, and the second factor being the complementary minor composed out of columns 3, 4, 5 (the several products being of course taken each with its proper sign), the expansion presents itself in the form

$$\begin{aligned} & D^3\gamma(-\theta\delta\gamma^2D^2+\delta^2D^7),\\ & -D^6(-\theta\gamma^2D^4+\delta D^9-\theta^2D^4)\\ & -\gamma D^3.-\delta D^2(\delta D^5-\theta\gamma^2)\\ & +\gamma^3(\gamma^2\delta D^5-\theta\delta D^5-\theta\gamma^4)\\ & -\gamma^2\delta^3D^5. \end{aligned}$$

Hence, collecting, and changing the sign of the whole expression, we obtain

$$\delta D^{15}-(2\gamma\delta^2+\gamma^2\theta+\theta^2)D^{10}+(-\gamma^3\delta+3\gamma\delta\theta+\delta^3)\gamma^2D^5+\gamma^7\theta=0,$$

a cubic equation for D^5. We have then as above $C=\frac{\gamma}{D}$, and B is given rationally as the common root of the foregoing quadric and cubic equations satisfied by B.

Substituting for γ, δ, θ their values in terms of the original coefficients, the equation for D^5 becomes

$$\begin{aligned} & 2(a^2d-3abc+2b^3)(aD)^{15}\\ & +\left\{\begin{array}{l} a^4(ae-4bd+3c^2)^2\\ +a^2(ac-b^2)^2(ae-4bd+3c^2)\\ -16(ac-b^2)(a^2d-3abc+2b^3)^2 \end{array}\right\}(aD)^{10}\\ & +4(ac-b^2)^2\left\{\begin{array}{l} 28\quad(ac-b^2)^3(a^2d-3abc+2b^3)\\ +12a^2(ac-b^2)(a^2d-3abc+2b^3)(ae-4bd+3c^2)\\ +\;8\qquad(a^2d-3abc+2b^3)^3 \end{array}\right\}(aD)^5\\ & -128(ac-b^2)^7\{a^2(ae-4bd+3c^2)+(ac^2-b^2)^2\}=0, \end{aligned}$$

and the solution of the given quintic equation thus ultimately depends upon that of this cubic equation.