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ON THE ANALYTICAL FORMS CALLED TREES.

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In a tree of N knots, selecting any knot at pleasure as a root, the tree may be regarded as springing from this root, and it is then called a root-tree. The same tree thus presents itself in various forms as a root-tree; and if we consider the different root-trees with N knots, these are not all of them distinct trees. We have thus the two questions, to find the number of root-trees with N knots; and, to find the number of distinct trees with N knots.

I have in my paper "On the Theory of the Analytical Forms called Trees," *Phil. Mag.*, t. XIII. (1857), pp. 172—176, [203] given the solution of the first question; viz. if ϕ_N denotes the number of the root-trees with N knots, then the successive numbers ϕ_1, ϕ_2, ϕ_3, etc., are given by the formula

$$\phi_1 + x\phi_2 + x^2\phi_3 + \ldots = (1-x)^{-\phi_1}(1-x^2)^{-\phi_2}(1-x^3)^{-\phi_3}\ldots,$$

viz. we thus find

suffix of ϕ	1	2	3	4	5	6	7	8	9	10	11	12	13
$\phi =$	1	1	2	4	9	20	48	115	286	719	1842	4766	12486.

And I have, in the paper "On the analytical forms called Trees, with application to the theory of chemical combinations," *Brit. Assoc. Report*, 1875, pp. 257—305, [610] also shown how by the consideration of the centre or bicentre "of length" we can obtain formulæ for the number of central and bicentral trees, that is, for the number of distinct trees, with N knots: the numerical result obtained for the total number of distinct trees with N knots is given as follows:

No. of Knots	1	2	3	4	5	6	7	8	9	10	11	12	13
No. of Central Trees	1	0	1	1	2	3	7	12	27	55	127	284	682
„ Bicentral „	0	1	0	1	1	3	4	11	20	51	108	267	619
Total	1	1	1	2	3	6	11	23	47	106	235	551	1301.

But a more simple solution is obtained by the consideration of the centre or bicentre "of number." A tree of an odd number N of knots has a centre of number, and a tree of an even number N of knots has a centre or else a bicentre of number. To explain this notion (due to M. Camille Jordan) we consider the branches which proceed from any knot, and (excluding always this knot itself) we count the number of the knots upon the several branches; say these numbers are α, β, γ, δ, ϵ, etc., where of course $\alpha+\beta+\gamma+\delta+\epsilon+\text{etc.}=N-1$. If N is even we may have, say $\alpha=\frac{1}{2}N$; and then $\beta+\gamma+\delta+\epsilon+\text{etc.}=\frac{1}{2}N-1$, viz. α is larger by unity than the sum of the remaining numbers: the branch with α knots, or the number α, is said to be "merely dominant." If N be odd, we cannot of course have $\alpha=\frac{1}{2}N$, but we may have $\alpha>\frac{1}{2}N$; here α exceeds by 2 at least the sum of the other numbers; and the branch with α knots, or the number α, is said to be "predominant." In every other case, viz. in the case where each number α is less than $\frac{1}{2}N$, (and where consequently the largest number α does not exceed the sum of the remaining numbers), the several branches, or the numbers α, β, γ, etc., are said to be subequal. And we have the theorem. First, when N is odd, there is always one knot (and only one knot) for which the branches are subequal: such knot is called the centre of number. Secondly, when N is even, either there is one knot (and only one knot) for which the branches are subequal: and such knot is then called the centre of number; or else there is no such knot, but there are two adjacent knots (and no other knot) each having a merely-dominant branch: such two knots are called the bicentre of number, and each of them separately is a half-centre.

Considering now the trees with N knots as springing from a centre or a bicentre of number, and writing ψ_N for the whole number of distinct trees with N knots, we readily obtain these in terms of the foregoing numbers ϕ_1, ϕ_2, ϕ_3, etc., viz. we have

$$\psi_1 = 1,$$

$$\psi_2 = \tfrac{1}{2}\phi_1(\phi_1+1),$$

$$\psi_3 = \text{coeff. } x^2 \text{ in } (1-x)^{-\phi_1},$$

$$\psi_4 = \tfrac{1}{2}\phi_2(\phi_2+1) + \text{coeff. } x^3 \text{ in } (1-x)^{-\phi_1},$$

$$\psi_5 = \text{coeff. } x^4 \text{ in } (1-x)^{-\phi_1}(1-x^2)^{-\phi_2},$$

$$\psi_6 = \tfrac{1}{2}\phi_3(\phi_3+1) + \text{coeff. } x^5 \text{ in } (1-x)^{-\phi_1}(1-x^2)^{-\phi_2},$$

$$\psi_7 = \text{coeff. } x^6 \text{ in } (1-x)^{-\phi_1}(1-x^2)^{-\phi_2}(1-x^3)^{-\phi_3},$$

and so on, the law being obvious. And the formulæ are at once seen to be true. Thus for $N=6$, the formula is

$$\psi_6 = \tfrac{1}{2}\phi_3(\phi_3+1) + \tfrac{1}{2}\phi_2(\phi_2+1)\,.\,\phi_1 + \phi_2\,.\,\tfrac{1}{6}\phi_1(\phi_1+1)(\phi_1+2) + \tfrac{1}{120}\phi_1(\phi_1+1)(\phi_1+2)(\phi_1+3)(\phi_1+4).$$

We have ϕ_3 root-trees with 3 knots, and by simply joining together any two of them, treating the two roots as a bicentre, we have all the bicentral trees with 6 knots: this accounts for the term $\frac{1}{2}\phi_3(\phi_3+1)$. Again, we have ϕ_1 root-trees with 1 knot, ϕ_2 root-trees with 2 knots; and with a given knot as centre, and the partitions (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1) successively, we build up the central trees of 6 knots, viz. 1° we take as branches any two ϕ_2's and any one ϕ_1; 2° any one ϕ_2 and any three ϕ_1's; 3° any five ϕ_1's; the partitions in question being all the partitions of 5 with no part greater than 2, that is, all the partitions with sub-equal parts. We easily obtain

suffix of ψ	1	2	3	4	5	6	7	8	9	10	11	12	13
$\psi=$	1	1	1	2	3	6	11	23	47	106	235	551	1301

agreeing with the results obtained by the much more complicated formulæ of the paper of 1875.