A MIXED FINITE ELEMENT FORMULATION FOR FINITE ELASTICITY WITH STIFF TWO FIBRE REINFORCEMENT

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1. Introduction

A frequent source of anisotropy of elastic materials is the presence of reinforcing fibres which display strong stiffening properties accompanying their stretching. This phenomenon appears as near inextensibility and it is in some way similar to near incompressibility which is observed for the ruber-like materials. Numerical modeling of a nearly inextensible material by the Finite Element Method (FEM) may cause similar difficulties as approximation of a nearly incompressible solid body: unstable or oscilatory solutions. The remedy for incompressible mechanics is the well known splitting of the description of kinematics into the volumetric part (dilatation) θ and the unimodular deformation gradient. The mixed formulation with adequate approximation in the Q^p and P^{p-1} finite element spaces for the displacements $\boldsymbol{u}$ and the auxiliary variables θ and pressure p, respectively, allows one for effective modeling of the nearly incompressible solid [1]. In this work we propose analogous approach to approximation of elastic materials with two fibres reinforcement. One fibre case was studied in [2].

2. Description of kinematics and stresses

In this section we briefly present the main principles of constructing the mixed formulation for the materials reinforced with two families of fibres. We assume that the two preferred directions of reinforcement are given by two distinct fields of unit vectors $\boldsymbol{G}_A, A = 1, 2$ in the reference configuration. We augment them with the third direction $\boldsymbol{G}_3 := \boldsymbol{G}_1 \times \boldsymbol{G}_2/|\boldsymbol{G}_1 \times \boldsymbol{G}_2|$. We consider $\boldsymbol{G}_A$ a basis of the curvilinear system of coordinates corresponding to some parametrization $\boldsymbol{X} = \boldsymbol{X}(\xi^A), A = 1, 2, 3$, i.e. $\boldsymbol{G}_A = \partial \boldsymbol{X}/\partial \xi^A$. We consider also the basis $\underline{\boldsymbol{G}}^A$ of the adjoint space (of linear functionals) which is dual to $\boldsymbol{G}_A$: $\langle \underline{\boldsymbol{G}}^A, \boldsymbol{G}_B \rangle = \delta^A_B$. We use convective spatial coordinates, i.e. the spatial basis vectors are generated by the parametrization $\boldsymbol{x} = \boldsymbol{x}(\zeta^a)$, $\boldsymbol{g}_a = \partial \boldsymbol{x}/\partial \zeta^a$ for wich $\boldsymbol{\zeta} = \boldsymbol{\xi}$. It is known that the deformation gradient $\boldsymbol{F}$, its adjoint $\boldsymbol{F}^*$ and their inverses $\boldsymbol{F}^{-1}$ and $\boldsymbol{F}^{-*}$ take the form:

$$\boldsymbol{F} = \delta^a_A \boldsymbol{g}_a \otimes \underline{\boldsymbol{G}}^A, \quad \boldsymbol{F}^{-1} = \delta^A_a \boldsymbol{G}_A \otimes \underline{\boldsymbol{g}}^a, \quad \boldsymbol{F}^* = \delta^a_A \underline{\boldsymbol{G}}^A \otimes \boldsymbol{g}_a, \quad \boldsymbol{F}^{-*} = \delta^A_a \underline{\boldsymbol{g}}^a \otimes \boldsymbol{G}_A, \tag{2.1}$$

where $\underline{\boldsymbol{g}}^a$ denotes the adjoint basis dual to $\boldsymbol{g}_a$, i.e. satisfying the condition $\langle \underline{\boldsymbol{g}}^a, \boldsymbol{g}_b \rangle = \delta^a_b$. We also introduce the material and spatial metric tensors:

$$\boldsymbol{G} = G_{AB}\, \underline{\boldsymbol{G}}^A \otimes \underline{\boldsymbol{G}}^B \quad \text{and} \quad \boldsymbol{g} = g_{ab}\, \underline{\boldsymbol{g}}^a \otimes \underline{\boldsymbol{g}}^b, \tag{2.2}$$

where $G_{AB} := \boldsymbol{G}_A \cdot \boldsymbol{G}_B$ and $g_{ab} := \boldsymbol{g}_a \cdot \boldsymbol{g}_b$. The right Cauchy-Green deformation tensor takes the form:

$$\boldsymbol{C} = \boldsymbol{F}^* \boldsymbol{g} \boldsymbol{F}, \quad \boldsymbol{C} = \delta^a_A \delta^b_B g_{ab}\, \underline{\boldsymbol{G}}^A \otimes \underline{\boldsymbol{G}}^B. \tag{2.3}$$

We also introduce the structural tensors corresponding to the preferred directions of fibres:

$$\boldsymbol{A}_F := \boldsymbol{G}_F \otimes \boldsymbol{G}_F \quad \text{(no sum)}, \quad F = 1, 2. \tag{2.4}$$

The stretches λ_F of the directions $\boldsymbol{G}_F$ and the cosines between their images $\boldsymbol{g}_F$ can be found as follows:

$$\lambda_F = \langle \boldsymbol{C}, \boldsymbol{A}_F \rangle^{1/2}, \quad \alpha_{FG} = \boldsymbol{g}_F \cdot \boldsymbol{g}_G / (\lambda_F \lambda_G). \tag{2.5}$$

We express the Cauchy-Green deformation tensor in terms of these new variables applying the extra substitution $\lambda_F(\boldsymbol{C}) = \tilde{\lambda}_F,\ F = 1, 2$ reflecting the anticipated procedure of separate numerical approximation of stretches along fibres:

$$\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{C}}(\boldsymbol{C}, \tilde{\lambda}_1, \tilde{\lambda}_2) = \begin{bmatrix} \tilde{\lambda}_1^2 & \alpha_{12}\tilde{\lambda}_1\tilde{\lambda}_2 & \alpha_{13}\tilde{\lambda}_1\lambda_3 \\ \bullet & \tilde{\lambda}_2^2 & \alpha_{23}\tilde{\lambda}_2\lambda_3 \\ \bullet & \bullet & \lambda_3^2 \end{bmatrix}_{AB} \underline{\boldsymbol{G}}^A \otimes \underline{\boldsymbol{G}}^B. \tag{2.6}$$

With these notions in mind we propose the ansatz for the strain energy function $\Psi = \Psi(\boldsymbol{C}; \boldsymbol{A}_1, \boldsymbol{A}_2)$ in the form taking into account that $\tilde{\boldsymbol{C}}$ is expressed by the dependent on deformation tensor $\boldsymbol{C}$ and separately approximated stretches $\tilde{\lambda}_F$:

$$\tilde{\Psi}(\tilde{\boldsymbol{C}}, \tilde{\lambda}_1, \tilde{\lambda}_2; \boldsymbol{A}_1, \boldsymbol{A}_2) = \Psi(\boldsymbol{C}; \boldsymbol{A}_1, \boldsymbol{A}_2). \tag{2.7}$$

Selecting separate approximation of stretches $\tilde{\lambda}_F$ suggests assuming the following augmented strain energy ansatz with the Lagrange multipliers $\tilde{\rho}^F$ corresponding to the constraints $\tilde{\lambda}_F = \lambda_F(\boldsymbol{C})$, F=1,2:

$$\Psi = \tilde{\Psi}(\tilde{\boldsymbol{C}}, \tilde{\lambda}_1, \tilde{\lambda}_2; \boldsymbol{A}_1, \boldsymbol{A}_2) - \textstyle\sum_{F=1}^2 \tilde{\rho}^F [\tilde{\lambda}_F - \lambda_F(\boldsymbol{C})]. \tag{2.8}$$

The assumptions above and the Clausius-Plank inequality lead to the following constitutive equations for the 2nd Piola-Kirchhoff stress:

$$\begin{cases} \boldsymbol{S} = \sum_{F=1}^2 \tilde{\rho}^F \lambda_F^{-1} \boldsymbol{A}_F + \tilde{\boldsymbol{S}}, \\ \tilde{\boldsymbol{S}} = \tilde{\mathbb{P}} \left[2 \dfrac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{C}}} \right], \ \text{ with } \tilde{\mathbb{P}} := \left[\dfrac{\partial \tilde{\boldsymbol{C}}}{\partial \boldsymbol{C}} \right]^*, \\ \tilde{\rho}^F = \partial \tilde{\Psi} / \partial \tilde{\lambda}_F, \end{cases} \tag{2.9}$$

The mixed formulation for $(\boldsymbol{u}, \tilde{\rho}^F, \tilde{\lambda}_F)$ involves the principle of virtual work (expressing equilibrium), the identification $\tilde{\lambda}_F = \lambda_F(\boldsymbol{C})$ and the constitutive relation for $\tilde{\rho}^F$, and it takes the form: find $(\boldsymbol{u}, \tilde{\rho}^F, \tilde{\lambda}_F) \in (V + \boldsymbol{u}_0) \times Q^4$ such that:

$$\begin{cases} \int_\Omega \langle D_u \boldsymbol{E}(\boldsymbol{u})[\delta \boldsymbol{u}], \boldsymbol{S} \rangle dV = \int_\Omega \langle \boldsymbol{G}\delta\boldsymbol{u}, \bar{\boldsymbol{B}} \rangle dV + \int_{\Gamma_N} \langle \boldsymbol{G}\delta\boldsymbol{u}, \bar{\boldsymbol{P}} \rangle dA, \\ \int_\Omega \delta\tilde{\rho}^F \{ \lambda_F(\boldsymbol{C}) - \tilde{\lambda}_F \} dV = 0, \\ \int_\Omega \delta\tilde{\lambda}_F \{ \partial\tilde{\Psi}/\partial\tilde{\lambda}_F - \tilde{\rho}^F \} dV = 0, \end{cases} \tag{2.10}$$

for all $\delta\boldsymbol{u} \in V, \delta\tilde{\lambda}_F, \delta\tilde{\rho}^F \in Q$, $V = \{ \boldsymbol{v} \in H^1(\Omega) : \boldsymbol{v} = 0, \text{ on } \Gamma_D \}, Q = L^2(\Omega)$. In (2.10) $\bar{\boldsymbol{B}}$ denotes the volume forces, $\bar{\boldsymbol{P}}$ and $\boldsymbol{u}_0$ are the Neumann and Dirichlet data on Γ_N and Γ_D. In addition $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$ and $D_u\boldsymbol{E}(\boldsymbol{u})[\delta\boldsymbol{u}] = \frac{1}{2}(\boldsymbol{F}^* \boldsymbol{\nabla} \delta\boldsymbol{u} + \boldsymbol{\nabla}^* \delta\boldsymbol{u} \boldsymbol{F})$. The FE approximation of (2.10) results in a system of nonlinear equations which is solved using the Newton-Raphson algorithm applied to linearization of (2.10).

Numerical tests confirm effectivity of the proposed formulation for strongly anisotropic materials.

References

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