ANALYZING DEFORMATIONS OF 2D SOLID STRUCTURES USING ELASTIC MULTIPOLES

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1. Introduction

Recently we have seen an expansion of research on mechanical metamaterials, where the geometry of highly deformable structures is responsible for their unusual mechanical properties, such as negative Poisson's ratio [1], mechanical cloaking and tuneable phononic band gaps [2]. Understanding how such structures deform in response to applied external stresses/loads is crucial for designing novel mechanical metamaterials. Here we present a method for predicting deformations of 2D solid structures with holes and inclusions in the linear response regime by employing analogies with electrostatics.

Just like external electric field induces polarization (dipoles, quadrupoles, etc.) of conductive objects, external stress induces "elastic multipoles" inside holes. In structures with many holes, interactions between induced elastic multipoles are responsible for complex deformation patterns observed in experiments and finite element simulations. We demonstrate that our method can successfully predict deformation patterns in periodic as well as aperiodic structures with holes and inclusions of varying sizes.

2. Elastic multipole method

The elastic multipole method employs analogies between electrostatics and 2D linear elasticity to predict deformations in 2D elastic materials. Electrostatics problems can be formulated in terms of a scalar field, electric potential, whose gradient is the electric field. Similarly, in 2D elasticity a scalar field, named Airy stress function χ, can be defined such that spatial derivatives are normal and shear stresses. From mechanical equilibrium and compatibility conditions in 2D linear elasticity we obtain the governing equation $\Delta\Delta\chi = Y\rho$, where Y is the 2D Young's modulus and ρ is the elastic charge density. This equation is similar to Gauss's Law $\nabla^2 U = -\rho / \varepsilon$ in electrostatics and just like in electrostatics, we can define point charges/monopoles, dipoles and quadrupoles in 2D elasticity.

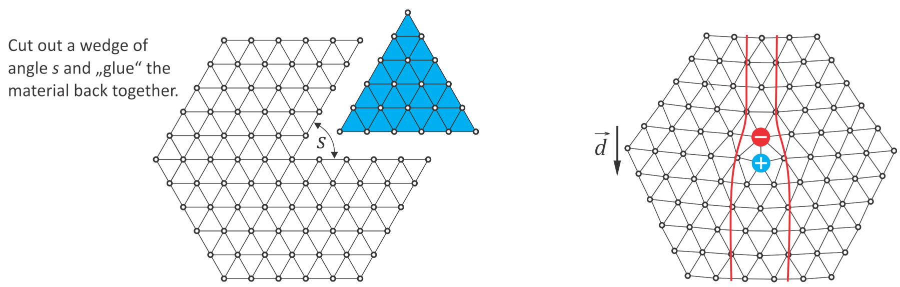

Figure 1: disclination defect (left) and a dislocation defect combining two opposite-sign disclinations.

When a wedge from a 2D elastic material is cut out and the two newly created boundaries of the remaining portion are glued back together, a disclination defect is formed, Fig. 1. The Airy stress function for a disclination defect (monopole), which is similar to a point charge in electrostatics, located at the origin of the coordinate system is given by $\Delta\Delta\chi = Ys\delta(\vec{x})$, which upon solving analytically gives:

$$\chi = s\chi_m = \frac{Ys}{8\pi}|\vec{x}|^2\left(\ln|\vec{x}| - 1/2\right).$$

Here the magnitude of the elastic "charge" s corresponds to the angle of the removed wedge. Two opposite disclinations that are adjacent to each other are referred to as dislocation/dipole, Fig. 1. The Airy stress function in this case is $\chi_d = -\vec{d} \cdot \nabla \chi_m$. Similarly, the Airy stress function for a quadrupole can be derived. The general solution to the equation $\Delta\Delta\chi = 0$, except at the origin, in polar coordinates (r, φ) is given by an infinite series also known as the Michell solution [3]. All the elastic multipoles discussed above are included in this solution along with higher order multipoles. Some of these multipoles have to be omitted because their energy diverges with increasing size of the elastic material or they produce zero stress. The only terms from the Michell solution that we therefore need to consider are:

$$\chi = A_0 \ln r + A_1 r^{-1} \cos\varphi + C_1 r^{-1} \sin\varphi + \sum_{n=2}^{\infty} \left(A_n r^{-n} + B_n r^{-n+2}\right) \cos n\varphi + \sum_{n=2}^{\infty} \left(C_n r^{-n} + D_n r^{-n+2}\right) \sin n\varphi.$$

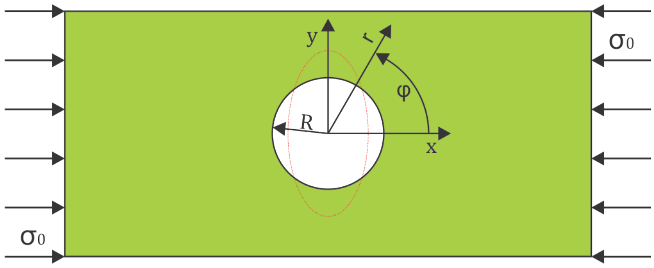

Figure 2: infinite sheet with a circular hole subjected to uniaxial stress.

As an example, we can look at the problem of an infinite 2D sheet with a circular hole of radius R at the center, Fig. 2. When this sheet is subjected to an external stress ($\sigma_{xx,ext} = -\sigma_0$), the resultant Airy stress function can be calculated analytically. From the expression given below in polar coordinates we can see that the external stress induces two types of quadrupoles and a higher order multipole inside the hole:

$$\chi = -\frac{\sigma_0 r^2}{4}\left(1 - \cos 2\varphi\right) + \frac{\sigma_0 R^2}{2} \ln r - \frac{\sigma_0 r^2}{2} \cos 2\varphi + \frac{\sigma_0 R^4}{4r^2} \cos 2\varphi.$$

External stress ↓ Quadrupole P ↓ Quadrupole Q ↓ Higher order multipole ↓

When more than one hole/inclusion is present in the elastic material, the external stress induces elastic multipoles in each hole, which then interact with each other. The strength of the elastic multipoles can be obtained by satisfying the appropriate boundary conditions at the boundary of each hole.

Results calculated using the elastic multipole method and linear FEM for the case of a hole placed near the edge of an infinite half-space, are presented below in Fig. 3. Finite element calculations were performed using commercial Finite Element code Ansys.

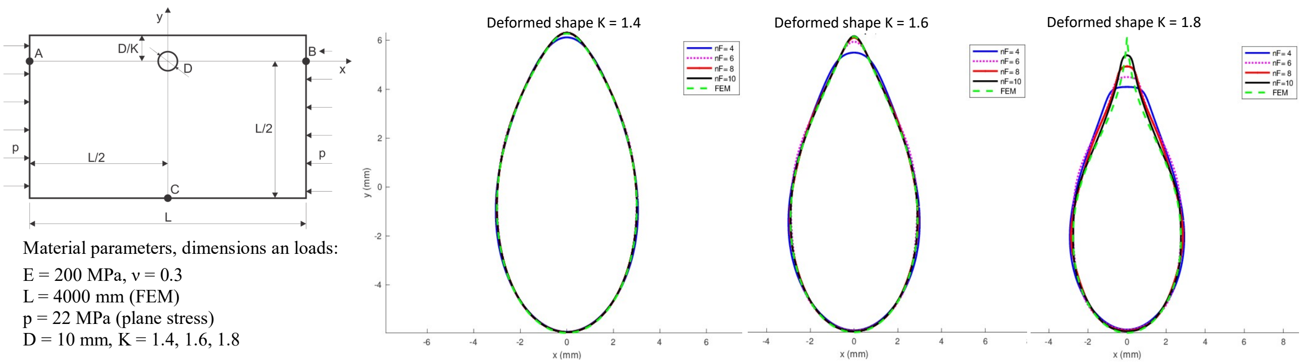

Figure 3: hole placed near the edge of a half-space, FEM and elastic multipole results.

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