958.

ON THE SURFACE OF THE ORDER n WHICH PASSES THROUGH A GIVEN CUBIC CURVE.

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It is natural to assume that, taking A, B, C to denote the general functions $(x, y, z, w)^{n-2}$ of the order $n-2$, the general surface of the order n which passes through the curve

$$\begin{Bmatrix} x, & y, & z \\ y, & z, & w \end{Bmatrix} = 0,$$

(or, what is the same thing, the curve $x : y : z : w = 1 : \theta : \theta^2 : \theta^3$), has for its equation

$$\begin{vmatrix} A, & B, & C \\ x\,, & y\,, & z \\ y\,, & z\,, & w \end{vmatrix} = 0;$$

but the formal proof is not immediate. Writing the equation in the form

$$U = Sax^\alpha y^\beta z^\gamma w^\delta, \ = 0, \ \alpha + \beta + \gamma + \delta = n,$$

then U must vanish on writing therein $x : y : z : w = 1 : \theta : \theta^2 : \theta^3$; a term $ax^\alpha y^\beta z^\gamma w^\delta$ becomes $= a\theta^p$, where $p = \beta + 2\gamma + 3\delta$ is the weight of the term reckoning the weights of x, y, z, w as 0, 1, 2, 3 respectively; and hence the condition is that, for each given weight p, the sum Sa of the coefficients of the several terms of this weight shall be $= 0$. Using any such equation to determine one of the coefficients thereof in terms of the others, the function U is reduced to a sum of duads $a(x^\alpha y^\beta z^\gamma w^\delta - x^{\alpha'} y^{\beta'} z^{\gamma'} w^{\delta'})$, where in each duad the two terms are of the same degree and of the same weight, and where a is an arbitrary coefficient; it ought therefore to be true that each such duad $x^\alpha y^\beta z^\gamma w^\delta - x^{\alpha'} y^{\beta'} z^{\gamma'} w^{\delta'}$ has the property in question—or writing P, Q, $R = yw - z^2$, $zy - xw$, $xz - y^2$, say that each such duad is of the form $AP + BQ + CR$.

Suppose for a moment that α' is greater than α, but that β', γ', δ' are each less than β, γ, δ respectively: the duad is $x^{\alpha'} y^\beta z^\gamma w^\delta (x^\lambda - y^\mu z^\nu w^\rho)$, where λ, μ, ν, ρ are each positive, and hence $x^\lambda - y^\mu z^\nu w^\rho$ is a duad having the property in question, or changing the notation say $x^\alpha - y^\beta z^\gamma w^\delta$ has the property in question; and in like manner, by considering the several cases that may happen, we have to show that each of the duads

$$x^\alpha - y^\beta z^\gamma w^\delta,\ \ y^\beta - x^\alpha z^\gamma w^\delta,\ \ z^\gamma - x^\alpha y^\beta w^\delta,\ \ w^\delta - x^\alpha y^\beta z^\gamma,$$

$$x^\alpha y^\beta - z^\gamma w^\delta,\ \ x^\alpha z^\gamma - y^\beta w^\delta,\ \ x^\alpha w^\delta - y^\beta z^\gamma,$$

has the property in question; it being of course understood that, in each of these duads, the two terms have the same degree and the same weight. The first form cannot exist; for we must have therein $\alpha = \beta + \gamma + \delta$ and $0 = \beta + 2\gamma + 3\delta$, which is inconsistent with α, β, γ, δ each of them positive. For the second form $\beta = \alpha + \gamma + \delta$, $\beta = 2\gamma + 3\delta$: this is $\alpha = \gamma + 2\delta$ or the duad is $y^{2\gamma+3\delta} - x^{\gamma+2\delta} z^\gamma w^\delta$, $= (y^2)^\gamma\, y^{3\delta} - (xz)^\gamma\, (x^2w)^\delta$. Writing $y^2 = xz - R$, we have terms containing the factor R, and a residual term $(xz)^\gamma\, \{y^{3\delta} - (x^2w)^\delta\}$, and writing herein

$$xw = yz - Q \ \text{ or } \ x^2w = xyz - Q,$$

we have terms containing Q as a factor and a residual term

$$(xz)^\gamma\, \{y^{3\delta} - (xyz)^\delta\},\ = (xz)^\gamma\, y^\delta\, \{(y^2)^\delta - (xz)^\delta\},$$

and again writing herein $y^2 = xz - R$, we see that this term contains the factor R: hence the duad in question consists of terms having the factor R or the factor Q. Similarly for the other cases: either α, β, γ, δ can be expressed as positive numbers, and then the duad consists of terms each divisible by P, Q, or R; or else α, β, γ, δ cannot be expressed as positive numbers, and then the duad does not exist: thus for the third form $z^\gamma - x^\alpha y^\beta w^\delta$, here $\gamma = \alpha + \beta + \delta$, $2\gamma = \beta + 3\delta$, or say $\gamma = 3\alpha + 2\beta$, $\delta = 2\alpha + \beta$, and the duad is $z^{3\alpha+2\beta} - x^\alpha y^\beta w^{2\alpha+\beta}$, $= z^{3\alpha}(z^2)^{2\beta} - (xw^2)^\alpha\, (yw)^\beta$, which can be reduced to the required form. But for the duad $x^\alpha y^\beta - z^\gamma w^\delta$, we have $\alpha + \beta = \gamma + \delta$, $\beta = 2\gamma + 3\delta$, which cannot be satisfied by positive values of α, β, γ, δ, and thus the duad does not exist.

A surface of the order n which passes through $3n+1$ points of a cubic curve contains the curve: hence the number of constants, or say the capacity of a surface of the order n, through the curve $P=0$, $Q=0$, $R=0$, is

$$\tfrac{1}{6}(n+1)(n+2)(n+3) - 1 - (3n+1),\ = \tfrac{1}{6}(n^3 + 6n^2 - 7n - 6).$$

Primâ facie the capacity of the surface $AP + BQ + CR = 0$, A, B, C being general functions of the order $n-2$, is

$$3 \,.\, \tfrac{1}{6}(n-1)\,n\,(n+1) - 1,\ = \tfrac{1}{2}(n^3 - n - 2),$$

but there is a reduction on account of the identical equations

$$xP + yQ + zR = 0,\quad yP + zQ + wR = 0,$$

which connect the functions P, Q, R: for $n=2$, the formulæ give each of them as it should do, Capacity $= 2$; viz. the quadric surface through the curve is

$$aP + bQ + cR = 0.$$