# 964.

## ON THE NINE-POINTS CIRCLE OF A SPHERICAL TRIANGLE.

THE definition is in effect given in Hart's paper, "Extension of Terquem's theorem respecting the circle which bisects three sides of a triangle," *Quarterly Mathematical Journal*, t. IV. (1861), pp. 260, 261, viz. if we have a spherical triangle $ABC$, then we have a circle (i.e. a small circle of the sphere), say the nine-points circle, meeting the sides $BC$, $CA$, $AB$ in the points $F$, $L$; $G$, $M$; $H$, $N$ respectively, where

$$\tan \tfrac{1}{2} BF = \frac{\cos \tfrac{1}{2} b - \cos \tfrac{1}{2} c \cos \tfrac{1}{2} a}{\cos \tfrac{1}{2} c \sin \tfrac{1}{2} a},$$

$$\tan \tfrac{1}{2} CF = \frac{\cos \tfrac{1}{2} c - \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b}{\cos \tfrac{1}{2} b \sin \tfrac{1}{2} a},$$

(which equations agree with $BF + FC = BC$), and

$$\tan \tfrac{1}{2} BL = \frac{\cos \tfrac{1}{2} c \sin \tfrac{1}{2} a}{\cos \tfrac{1}{2} b + \cos \tfrac{1}{2} c \cos \tfrac{1}{2} a},$$

$$\tan \tfrac{1}{2} CL = \frac{\cos \tfrac{1}{2} b \sin \tfrac{1}{2} a}{\cos \tfrac{1}{2} c + \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b},$$

(which equations agree with $BL + CL = BC$); and with the like formulæ for the points $G$, $M$: and $H$, $N$: respectively.

If, as usual, the sides of the triangle are called $a$, $b$, $c$, and for shortness we write

$$(\cos \tfrac{1}{2} a, \ \cos \tfrac{1}{2} b, \ \cos \tfrac{1}{2} c; \ \sin \tfrac{1}{2} a, \ \sin \tfrac{1}{2} b, \ \sin \tfrac{1}{2} c) = (p, \ q, \ r; \ p_1, \ q_1, \ r_1),$$

then the formulæ are

$$\tan \tfrac{1}{2} BF = \frac{q - rp}{rp_1}, \quad \tan \tfrac{1}{2} CF = \frac{r - pq}{qp_1},$$

$$\tan \tfrac{1}{2} CG = \frac{r - pq}{pq_1}, \quad \tan \tfrac{1}{2} AG = \frac{p - qr}{rq_1},$$

$$\tan \tfrac{1}{2} AH = \frac{p - qr}{qr_1}, \quad \tan \tfrac{1}{2} BH = \frac{q - rp}{pr_1},$$

and

$$\tan \tfrac{1}{2} BL = \frac{rp_1}{q + rp}, \quad \tan \tfrac{1}{2} CL = \frac{qp_1}{r + pq},$$

$$\tan \tfrac{1}{2} CM = \frac{pq_1}{r + pq}, \quad \tan \tfrac{1}{2} AM = \frac{rq_1}{p + qr},$$

$$\tan \tfrac{1}{2} AN = \frac{qr_1}{p + qr}, \quad \tan \tfrac{1}{2} BN = \frac{pr_1}{q + rp}.$$

Before going further, it may be remarked that for $a$, $b$, $c$, each of them small, we have

$$\tfrac{1}{2} BF = \frac{(1 - \tfrac{1}{8} b^2) - (1 - \tfrac{1}{8} c^2)(1 - \tfrac{1}{8} a^2)}{\tfrac{1}{2} a},$$

that is, $BF = \dfrac{a^2 + c^2 - b^2}{2a} = c \cos B$, and similarly $CF = a \cos C$, if $A$, $B$, $C$ are the angles of the plane triangle; that is, in the plane triangle $F$, $G$, $H$ are the feet of the perpendiculars let fall from the angles on the opposite sides. Moreover, $\tan \tfrac{1}{2} BL = \dfrac{\tfrac{1}{2} a}{1 + 1 \cdot 1}$, that is, $BL = \tfrac{1}{2} a$, and similarly $CL = \tfrac{1}{2} a$; that is, $L$, $M$, $N$ are the median points of the three sides respectively.

In the general case of the spherical triangle $ABC$, the construction is effected by means of a triangle $A'B'C'$, the sides whereof are respectively the halves of those
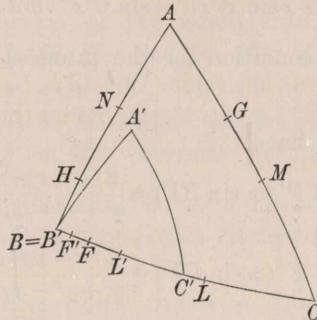
of the original triangle: viz. for this triangle $A'B'C'$, we construct the points $F'$, $G'$, $H'$ and $L'$, $M'$, $N'$, and then on the sides of the triangle $ABC$ taking $BF = 2BF'$, $BL = 2BL'$, &c., we have the points $F$, $G$, $H$, $L$, $M$, $N$.

Thus $p$, $q$, $r$, $p_1$, $q_1$, $r_1$ denoting as above the cosines and sines of the half-sides of the triangle $ABC$, that is, the cosines and sines of the sides of the triangle $A'B'C'$, we have

$$\tan B'F' = \frac{q - rp}{rp_1}, \quad \tan C'F' = \frac{r - pq}{qp_1},$$
$$\vdots$$
$$\tan B'L' = \frac{rp_1}{q + rp}, \quad \tan C'L' = \frac{qp_1}{r + pq}.$$
$$\vdots$$

First, for the points $F'$, $G'$, $H'$, we have $A'F'$, $B'G'$, $C'H'$, the perpendiculars from the angles on the opposite sides, meeting in a point $O'$, the orthocentre of the triangle $A'B'C'$: in fact, from the triangle $A'B'F'$, right-angled at $F'$, we have

$$\tan B'F' = \tan A'B' \cos B' = \tan \tfrac{1}{2}c \, \frac{\cos \tfrac{1}{2}b - \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a}{\sin \tfrac{1}{2}c \sin \tfrac{1}{2}a},$$

$$= \frac{r_1}{r} \frac{q - rp}{r_1 p_1} = \frac{q - rp}{rp_1},$$

as above, and similarly for the points $G'$ and $H'$.

I notice that we have

$$\sin B'F' = \frac{q - rp}{\sqrt{\{(q - rp)^2 + r^2 p_1^2\}}}, \quad = \frac{q - rp}{\sqrt{(q^2 + r^2 - 2pqr)}},$$

and thus

$$\sin B'F' = \frac{q - rp}{\sqrt{(q^2 + r^2 - 2pqr)}}, \quad \sin C'F' = \frac{r - pq}{\sqrt{(q^2 + r^2 - 2pqr)}},$$

$$\sin C'G' = \frac{r - pq}{\sqrt{(r^2 + p^2 - 2pqr)}}, \quad \sin A'G' = \frac{p - qr}{\sqrt{(r^2 + p^2 - 2pqr)}},$$

$$\sin A'H' = \frac{p - qr}{\sqrt{(p^2 + q^2 - 2pqr)}}, \quad \sin B'H' = \frac{q - rp}{\sqrt{(p^2 + q^2 - 2pqr)}};$$

hence

$$\sin B'F' \cdot \sin C'G' \cdot \sin A'H' = \sin C'F' \cdot \sin A'G' \cdot \sin B'H',$$

which (as is well known) is the condition for the intersection of the arcs $A'F'$, $B'G'$, $C'H'$ in the orthocentre $O'$.

But I say further that we have

$$\sin 2B'F' \,(= \sin BF) = \frac{2\,(q - rp)\,rp_1}{q^2 + r^2 - 2pqr},$$

$$\sin 2C'F' \,(= \sin CF) = \frac{2\,(r - pq)\,qp_1}{q^2 + r^2 - 2pqr},$$

and thence

$$\sin BF \cdot \sin CG \cdot \sin AH = \sin CF \cdot \sin AG \cdot \sin BH,$$

and thus the arcs $AF$, $BG$, $CH$ meet in a point which is obviously *not* the orthocentre of the triangle $ABC$.

Secondly, for the points $L'$, $M'$, $N'$, we have

$$\sin B'L' = \frac{rp_1}{\sqrt{(q^2 + r^2 + 2pqr)}}, \quad \sin C'L' = \frac{qp_1}{\sqrt{(q^2 + r^2 + 2pqr)}},$$

that is,

$$\sin B'L' \ : \ \sin C'L' = r : q, \ = \cos B'A' : \cos C'A';$$

and similarly

$$\sin C'M' : \sin A'M' = p : r, \ = \cos C'B' : \cos A'B,$$

$$\sin A'N' : \sin B'N' = q : p, \ = \cos A'C' : \cos B'C',$$

viz. the sides $B'C'$, $C'A'$, $A'B'$ are by the points $L'$, $M'$, $N'$ divided each into two parts such that for any side the sines of the two parts are proportional to the cosines of the other two sides. We have

$$\sin B'L' . \sin C'M' . \sin A'N' = \sin C'L' . \sin A'M' . \sin B'N',$$

viz. the arcs $A'L'$, $B'M'$, $C'N'$ meet in a point $K'$ which may be called the cos-centre of the triangle $A'B'C'$ (where observe that, for $a$, $b$, $c$ indefinitely small, i.e. for a plane triangle, the points $L'$, $M'$, $N'$ are the mid-points of the sides, and the centre $K'$ is the C. G. or median point of the triangle).

But further, we have

$$\sin 2B'L' (= \sin BL) = \frac{2rp_1 (q + rp)}{q^2 + r^2 + 2pqr},$$
$$\vdots$$
$$\sin 2C'L' (= \sin CL) = \frac{2qp_1 (r + pq)}{q^2 + r^2 + 2pqr},$$
$$\vdots$$

and thence

$$\sin BL . \sin CM . \sin AN = \sin CL . \sin AM . \sin BN,$$

viz. the arcs $AL$, $BM$, $CN$ meet in a point, which is obviously *not* the cos-centre of the triangle $ABC$.

We have thus the construction of the nine-points circle as a six-points circle, by means of the points $F$, $G$, $H$, $L$, $M$, $N$; and by way of recapitulation we may say that the nine-points circle meets the sides $BC$, $CA$, $AB$ in the points $F$, $L$; $G$, $M$; $H$, $N$ respectively, where the points $F$, $G$, $H$ depend on the ortho-centre of the semi-triangle, and the points $L$, $M$, $N$ depend on the cos-centre of the semi-triangle.

The triangle $ABC$ has an inscribed circle and three escribed circles, and we have (as is known) the theorem that the nine-points circle touches each of these four circles. The circles $BC$, $CA$, $AB$ and the nine-points circle form a tetrad of circles, and the inscribed circle and the three escribed circles a tetrad of circles, or say the eight circles form a bitetrad, such that each circle of the one tetrad touches each circle of the other tetrad.