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COORDINATES VERSUS QUATERNIONS.

[From the *Proceedings of the Royal Society of Edinburgh*, vol. XX. (1895), pp. 271—275. Read July 2, 1894.]

IT is contended that Quaternions (as a method) are more comprehensive and less artificial than—and, in fact, in every way far superior to—Coordinates. Thus Professor Tait, in the Preface to his *Elementary Treatise on Quaternions* (1867), reproduced in the second and third editions (1873 and 1890), writes—"It must always be remembered that Cartesian methods are mere particular cases of quaternions where most of the distinctive features have disappeared; and that when, in the treatment of any particular question, scalars have to be adopted, the quaternion solution becomes identical with the Cartesian one. Nothing, therefore, is ever lost, though much is generally gained, by employing quaternions in place of ordinary methods. In fact, even when quaternions degrade to scalars, they give the solution of the most general statement of the problem they are applied to, quite independent of any limitations as to choice of particular coordinate axes." And he goes on to speak of "such elegant trifles as trilinear coordinates," and would, I presume, think as lightly of quadriplanar coordinates. It is right to notice that the claims of quaternions are chiefly insisted upon in regard to their applications to the physical sciences; and I would here refer to his paper, "On the Importance of Quaternions in Physics" (*Phil. Mag.*, Jan. 1890), being an abstract of an address to the Physical Society of the University of Edinburgh, Nov. 1889; but these claims certainly extend to and include the science of geometry.

I wish to examine into these claims on behalf of quaternions. My own view is that quaternions are merely a particular method, or say a theory, in coordinates. I have the highest admiration for the notion of a quaternion; but (I am not sure whether I did or did not use the illustration many years ago in conversation with Professor Tait), as I consider the full moon far more beautiful than any moonlit view, so I regard the notion of a quaternion as far more beautiful than any of its applications. As another illustration which I gave him, I compare a quaternion formula to a pocket-map—a capital thing to put in one's pocket, but which for use must be unfolded: the formula, to be understood, must be translated into coordinates.

I remark that the imaginary of ordinary algebra—for distinction call this θ—has no relation whatever to the quaternion symbols i, j, k; in fact, in the general point of view, all the quantities which present themselves are, or may be, complex values $a+\theta b$, or, in other words, say that a scalar quantity is in general of the form $a+\theta b$. Thus quaternions do not properly present themselves in plane or two-dimensional geometry at all—although, as will presently appear, we may use them in plane geometry; but they belong essentially to solid or three-dimensional geometry, and they are most naturally applicable to the class of problems which in coordinates are dealt with by means of the three rectangular coordinates x, y, z.

In plane geometry, considering an origin O, and through it two rectangular axes Ox, Oy, then in coordinates we determine the position of a point by means of its coordinates x, y; or, writing x, y, z to denote given linear functions of the original rectangular coordinates x, y, we may, if we please, determine it by trilinear coordinates, or say by the ratios $x : y : z$. The advantage is, that we thereby deal with the line infinity as with any other line, whereas with the rectangular coordinates x, y the line infinity presents itself as a line *sui generis*, and that we thereby bring the theory into connexion with that of the homogeneous functions $(*\between x, y, z)^n$.

In quaternions, the position of a point is determined in reference to the fixed point O, by its vector α, which is in fact $=ix+jy$, where i, j are the quaternion imaginaries ($i^2=-1$, $j^2=-1$, $ij=-ji$), but the idea is to use as little as possible the foregoing equation $\alpha=ix+jy$, and thus to conduct the investigations independently, as far as may be, of the particular positions of the axes Ox, Oy.

As the most simple example, take the theorem that the lines joining the extremities of equal and parallel lines in a plane are themselves equal and parallel, viz. (writing $\sim$ to denote equal and parallel), if $AB \sim CD$, then $AC \sim BD$.

Coordinates.

A, B, C, D are determined by their coordinates

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4).$$

$$AB \sim CD$$

gives

$$\left.\begin{aligned} x_2 - x_1 &= x_4 - x_3 \\ y_2 - y_1 &= y_4 - y_3 \end{aligned}\right\},$$

whence

$$\left.\begin{aligned} x_3 - x_1 &= x_4 - x_2 \\ y_3 - y_1 &= y_4 - y_2 \end{aligned}\right\},$$

that is,

$$AC \sim BD.$$

Quaternions.

AB, CD are determined by their vectors α, β, and then writing γ for the vector AD,

$$AB \sim CD$$

gives

$$\alpha = \beta,$$

whence

$$\gamma - \beta = -\alpha + \gamma,$$

that is,

$$AC \sim BD.$$

And for the comparison of the two solutions, we have

$$\alpha = i(x_2 - x_1) + j(y_2 - y_1), \quad \beta = i(x_4 - x_3) + j(y_4 - y_3).$$

But this example of a plane theorem is a trivial one, given only for the sake of completeness.

Passing to solid geometry, we have—

Coordinates.—Considering a fixed point O, and through it the rectangular axes Ox, Oy, Oz, the position of a point is determined by its coordinates x, y, z. But we may, in place of these, consider the quadriplanar coordinates (x, y, z, w) linear functions of the original rectangular coordinates x, y, z.

Quaternions.—The position of a point in reference to the fixed origin O is determined by its vector α, which is in fact $= ix + jy + kz$, where i, j, k are the Hamiltonian symbols $(i^2 = j^2 = k^2 = -1,\ jk = -kj = i,\ ki = -ik = j,\ i = -ji = k)$; but the idea is to use as little as possible the foregoing equation $\alpha = ix + jy + kz$, and thus to conduct the investigations independently, as far as may be, of the particular positions of the axes Ox, Oy, Oz.

I consider the problem to determine the line OC at right angles to the plane of the lines OA, OB.

Coordinates.

Taking O as origin, the coordinates of A, B, C are taken to be

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x, y, z)$$

respectively. Then

$$xx_1 + yy_1 + zz_1 = 0,$$

$$xx_2 + yy_2 + zz_2 = 0;$$

whence

$$x : y : z = y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1.$$

Quaternions.

Points A, B, C are determined by their vectors α, β, γ. Then

$$S\alpha\gamma = 0, \quad S\beta\gamma = 0;$$

whence

$$m\gamma = V\alpha\beta,$$

m being an arbitrary scalar.

Here to compare the two solutions, observe that the two equations $S\alpha\gamma = 0$, $S\beta\gamma = 0$ are in fact the equations $xx_1 + yy_1 + zz_1 = 0$, $xx_2 + yy_2 + zz_2 = 0$; and so also $m\gamma = V\alpha\beta$ denotes the relations $x : y : z = y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1$. But a quaternionist says that $m\gamma = V\alpha\beta$ *is* the compendious and elegant solution of the problem as opposed to the artificial and clumsy one $x : y : z = y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1$. And it is upon this that I join issue; $m\gamma = V\alpha\beta$ is a very pretty formula, like the folded-up pocket-map, but, to be intelligible, I consider that it requires to be developed into the other form. Of course, the example is as simple a one as could have been selected; and, in the case of a more complicated example, the mere abbreviation of the quaternion formula would be very much greater, but just for this reason there is the more occasion for the developed coordinate formula. To take another example, the condition, in order that the vectors α, β, γ may be coplanar, is $S\alpha\beta\gamma = 0$, and Professor Tait contrasts this with the prolixity of the corresponding coordinate formula

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} = 0.$$

I remark that, when all the components of a determinant have to be expressed, nothing can be shorter than this, the ordinary determinant notation, which simply expresses the several components in their line-and-column relation to each other. But as a mere abbreviation, it would be allowable to write Δ, $=(ABC)$, to denote the determinant formed by the coordinates of the three points.

In conclusion, I would say that while coordinates are applicable to the whole science of geometry, and are the natural and appropriate basis and method in the science, quaternions seem to me a particular and very artificial method for treating such parts of the science of three-dimensional geometry as are most naturally discussed by means of the rectangular coordinates x, y, z.