# 950.

## ON THE SEXTIC RESOLVENT EQUATIONS OF JACOBI AND KRONECKER.

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THE equations referred to are: the first of them, that given by Jacobi in the paper "Observatiunculae ad theoriam aequationum pertinentes," *Crelle*, t. XIII. (1835), pp. 340—352, [*Ges. Werke*, t. III., pp. 269—284], under the heading "Observatio de aequatione sexti gradus ad quam aequationes sexti gradus revocari possunt," and the second, that of Kronecker in the note "Sur la résolution de l'équation du cinquième degré," *Comptes Rendus*, t. XLVI. (1858), pp. 1150—1152. Jacobi's equation is closely connected with that obtained by Malfatti in 1771, see Brioschi's paper "Sulla resolvente di Malfatti per l'equazione del quinto grado," *Mem. R. Ist. Lomb.*, t. IX. (1863); but the characteristic property first presents itself in Jacobi's form, and I think the equation is properly described as Jacobi's resolvent equation. The other equation has been always known as Kronecker's resolvent equation; it belongs to the class of equations for the multiplier of an elliptic function considered by Jacobi in the paper "Suite des notices sur les fonctions elliptiques," *Crelle*, t. III. (1828), pp. 303—310, see p. 308, [*Ges. Werke*, t. I., pp. 255—263, see p. 261]: say Kronecker's equation belongs to the class of Jacobi's Multiplier Equations. We have in regard to it the paper by Brioschi, "Sul metodo di Kronecker per la resoluzione delle equazioni di quinto grado," *Atti Ist. Lomb.*, t. I. (1858), pp. 275—282, and see also the "Appendice terza" to his translation of my *Elliptic Functions* (Milan, 1880): it seems to me however that the theory of Kronecker's equation has not hitherto been exhibited in the clearest form.

I consider the forms

| 12345 | 13524 |
|---|---|
| 13254 | 12435 |
| 24315 | 23541 |
| 35421 | 34152 |
| 41532 | 45213 |
| 52143 | 51324 |

which, if in the first instance the figures are regarded as points, represent the twelve pentagons which can be formed with the points 1, 2, 3, 4, 5; each form in the right-hand column is derived from the corresponding form in the left-hand column by *stellation*, say we have $13524 = S12345$, and so in other cases.

A pentagon is in general reversible, but we sometimes consider it as irreversible (viz. we distinguish between the pentagons 12345 and 15432); when this is so, we write $15432 = R12345$, and we have thus twelve new forms, in all twenty-four forms. The symbols $R$, $S$ are such that $R^2 = 1$, $S^2 = R$, $RS = SR$, $S^4 = 1$. But for a reversible pentagon, there is no occasion to use the symbol $R$, and we have simply $S^2 = 1$.

In a somewhat different point of view, we may for an irreversible pentagon write

12345 to denote any one of the forms 12345, 23451, 34512, 45123, 51234,

$R$12345 „ „ „ „ 15432, 54321, 43215, 32154, 21543,

and for a reversible pentagon 12345 to denote any one of these same ten forms.

Each pentagon gives thus ten forms, viz. we have in all 120 forms which all are the different arrangements of the five figures. But further regarding the arrangement 12345 as positive, then the forms in the left-hand column are each of them positive, and the ten forms derived from any one of these are each of them positive, that is, the forms in the left-hand column give all the 60 positive arrangements of the five figures: and similarly the forms in the right-hand column give all the 60 negative arrangements of the five figures.

Taking 1, 2, 3, 4, 5 to denote any quantities, or say the five roots $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ of a quintic equation, we regard 12345, ..., as denoting functions of these roots: in particular, 12345 may denote a cyclic reversible function, the analogue of the reversible pentagon, or it may denote a cyclic irreversible function, the analogue of the irreversible pentagon.

*Jacobi's resolvent equation.*

The most simple course is to take 12345 a cyclic reversible function of the roots $x$; a root of the resolvent equation is then $12345 - 13524$, $= (1 - S)\,12345$, and the six roots are

$$z_1 = (1 - S)\,12345,$$
$$z_2 = (1 - S)\,13254,$$
$$z_3 = (1 - S)\,24315,$$
$$z_4 = (1 - S)\,35421,$$
$$z_5 = (1 - S)\,41532,$$
$$z_6 = (1 - S)\,52143.$$

Here effecting on the roots $x$ any positive substitution whatever, we permute *inter se* the roots $z$; but effecting on the roots $x$ any negative substitution whatever, then reversing the signs of all the roots $z$, we permute *inter se* these reversed values. Thus effecting on the root $z_1$ the negative substitution 12, it becomes $(1-S)\,21345$, which is

$$=(1-S)\,S23514=(S-S^2)\ (\text{that is, } S-1 \text{ or } -(1-S))\ 23514=-(1-S)\,41532=-z_5;$$

and similarly for the effect of the same substitution 12 upon any other of the roots $z$.

It follows that any rational symmetrical function of the roots $z$ is a two-valued function of the coefficients of the quintic equation, viz. it is a function of the form $P+Q\sqrt{\Delta}$, where $\Delta$ is the discriminant and $P$, $Q$ are rational functions of the coefficients of the quintic equation.

In particular, if 12345, ..., are rational and integral functions of the roots $x$, then for any rational and integral function of the roots $z$, we have $P$ and $Q$ rational and integral functions of the coefficients, and any rational and integral homogeneous function of the roots $z$, according as it is of an even or an odd degree in these roots, will be of the form $P$ or of the form $Q\sqrt{\Delta}$; the resolvent equation is thus of the form

$$(1,\ B\sqrt{\Delta},\ C,\ D\sqrt{\Delta},\ E,\ F\sqrt{\Delta},\ G\between z,\ 1)^6=0,$$

where $\Delta$ is the discriminant, and $B$, $C$, $D$, $E$, $F$, $G$ are rational and integral functions of the coefficients of the quintic equation.

The most simple form of the function 12345 is that employed by Jacobi and for which $12345=12+23+34+45+51$, where 12, ..., denote $x_1x_2$, ..., respectively.

For comparison with Kronecker's equation, it is proper to take 12345 a cyclic *irreversible* function of the roots $x$; we have then

$$12345+15432=(1+R)\,12345,$$

a cyclic reversible function, and the roots of Jacobi's equation will be in the first (or since Kronecker writes $x_0$, $x_1$, $x_2$, $x_3$, $x_4$ instead of $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, in the second) of the following two forms, say

| | |
|---|---|
| $z_1=(1+R)(1-S)\,12345,$ | 01234, |
| $z_2=(1+R)(1-S)\,13254,$ | 02143, |
| $z_3=(1+R)(1-S)\,24315,$ | 13204, |
| $z_4=(1+R)(1-S)\,35421,$ | 24310, |
| $z_5=(1+R)(1-S)\,41532,$ | 30421, |
| $z_6=(1+R)(1-S)\,52143;$ | 41032, |

viz. in the first form the terms are 12345, ..., and in the second they are 01234, ..., but the theory is in no wise altered by this change of form.

*Kronecker's resolvent equation.*

Kronecker writes $x_m$ to denote $x_0$, $x_1$, $x_2$, $x_3$, $x_4$ according as the residue of $m$ (mod. 5) is $=0$, 1, 2, 3 or 4: then putting

$$x_m x^2_{m+n} x^2_{m+2n} + v x^3_m x_{m+n} x_{m+2n} = m\,.\,m+n\,.\,m+2n,$$

a root is

$$\begin{aligned} f = \; & (012+123+234+340+412)\sin\frac{2\pi}{5} \\ & +(024+130+241+302+413)\sin\frac{4\pi}{5} \\ & +(031+142+203+314+420)\sin\frac{6\pi}{5} \\ & +(043+104+210+321+432)\sin\frac{8\pi}{5}, \end{aligned}$$

and the other roots are deduced from this by changing 01234 into 03412, 14023, 20134, 31240, 42301 respectively.

Taking $\epsilon$ an imaginary fifth root of unity, say $\epsilon = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, so that $\sin\frac{2\pi}{5}$, $\sin\frac{4\pi}{5}$, $\sin\frac{6\pi}{5}$, $\sin\frac{8\pi}{5}$ are as $\epsilon-\epsilon^4$, $\epsilon^2-\epsilon^3$, $\epsilon^3-\epsilon^2$, $\epsilon^4-\epsilon$; also writing

$$01234 = 012+123+234+340+412, \ldots,$$

so that 01234, ..., are cyclic irreversible functions of the roots $x$, then the expression for the root $f$ is

$$\begin{aligned} f = \; & (\epsilon-\epsilon^4)\,01234 + (\epsilon^2-\epsilon^3)\,02413 \\ & -(\epsilon-\epsilon^4)\,04321 - (\epsilon^2-\epsilon^3)\,03142, \end{aligned}$$

or, as this may be written,

$$f = \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,01234,$$

and the expressions for the six roots are

$$\begin{aligned} f &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,01234, \\ f_0 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,03412, \\ f_1 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,14023, \\ f_2 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,20134, \\ f_3 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,31240, \\ f_4 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,42301. \end{aligned}$$

I write down the analogous functions

$$\begin{aligned} F &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,03142, \quad (=RS\,01234), \\ F_0 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,01324, \quad (=RS\,03412), \\ F_1 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,12430, \quad (=RS\,14023), \\ F_2 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,23041, \quad (=RS\,20134), \\ F_3 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,34102, \quad (=RS\,31240), \\ F_4 &= \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,40213, \quad (=RS\,42301). \end{aligned}$$

This being so, I say that any positive substitution on the roots $x$ permutes *inter se* the roots $f$, reversing in some cases the signs; and that any negative substitution on the roots $x$ changes the roots $f$ into the roots $F$, permuting these roots *inter se* and reversing in some cases the signs. And similarly as to the effect on the roots $F$ of a positive substitution and a negative substitution respectively.

Thus the positive substitution 123 changes

$$\begin{array}{lll} f \text{ into} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,02314 & = f_1, \\ f_0 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,01423 & = f_4, \\ f_1 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,24031 & = f_3, \\ f_2 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,30214 = R03412 & = -f_0, \\ f_3 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,12340 & = f, \\ f_4 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,43102 = R20134 & = -f_2; \end{array}$$

viz. $f, f_0, f_1, f_2, f_3, f_4$ are changed into $f_1, f_4, f_3, -f_0, f, -f_2$.

And similarly the negative substitution 12 changes

$$\begin{array}{lll} f \text{ into} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,02134 & = F_4, \\ f_0 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,03421 = R12430 & = -F_1, \\ f_1 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,24013 & = F_0, \\ f_2 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,10234 & = F_3, \\ f_3 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,32140 = R23041 & = -F_2, \\ f_4 \;\;,, & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,41302 = R03142 & = -F, \end{array}$$

viz. $f, f_0, f_1, f_2, f_3, f_4$ are changed into $F_4, -F_1, F_0, F_3, -F_2, -F$.

Hence considering the equation the roots of which are $f^2, f_0^2, f_1^2, f_2^2, f_3^2, f_4^2$, this is an equation of the form

$$(1, b, c, d, e, f, g \between f^2, 1)^6 = 0,$$

and similarly the equation the roots of which are $F^2, F_0^2, F_1^2, F_2^2, F_3^2, F_4^2$ is an equation of the form

$$(1, B, C, D, E, F, G \between F^2, 1)^6 = 0,$$

where $b$ and $B$ are conjugate values $\beta+\beta'\sqrt{\Delta}$, $\beta-\beta'\sqrt{\Delta}$, $c$ and $C$ are conjugate values $\gamma+\gamma'\sqrt{\Delta}$, $\gamma-\gamma'\sqrt{\Delta}$, ..., of two-valued functions of the form $P \pm Q\sqrt{\Delta}$, $\Delta$ being the discriminant equation, and $P$ and $Q$ rational functions of the coefficients of the quintic equation.

Each term $x_0x_1^2x_2^2$ and $x_0^3x_1x_2$ of Kronecker's function $x_0x_1^2x_2^2 + vx_0^3x_1x_2$ is of the form $0.12$ $(=0.21)$, a function of $x_0$ multiplied by a symmetrical function of $x_1, x_2$; and it is by reason hereof that the roots $f, f_0, f_1, f_2, f_3, f_4$ are connected by linear relations such that the equation belongs to the class of Jacobi's multiplier equations. Thus in the expressions for these roots, attending first to the terms multiplied by $\epsilon - \epsilon^4$, we have

$$f\ = (\epsilon - \epsilon^4) . 0 (12-34) + 1 (23-04) + 2 (34-01) + 3 (04-12) + 4 (01-23),$$

$$f_0 = (\epsilon - \epsilon^4) . 0 (34-12) + 1 (02-34) + 2 (03-14) + 3 (14-02) + 4 (12-03),$$
$$f_1 = (\epsilon - \epsilon^4) . 0 (23-14) + 1 (04-23) + 2 (13-04) + 3 (14-02) + 4 (02-13),$$
$$f_2 = (\epsilon - \epsilon^4) . 0 (13-24) + 1 (34-02) + 2 (01-34) + 3 (24-01) + 4 (02-13),$$
$$f_3 = (\epsilon - \epsilon^4) . 0 (13-24) + 1 (24-03) + 2 (04-13) + 3 (12-04) + 4 (03-12),$$
$$f_4 = (\epsilon - \epsilon^4) . 0 (14-23) + 1 (24-03) + 2 (03-14) + 3 (01-24) + 4 (23-01);$$

and next to the terms multiplied by $\epsilon^2 - \epsilon^3$, we have

$$f\ = (\epsilon^2 - \epsilon^3) . 0 (24-13) + 1 (03-24) + 2 (14-03) + 3 (02-14) + 4 (13-02),$$

$$f_0 = (\epsilon^2 - \epsilon^3) . 0 (24-13) + 1 (04-23) + 2 (13-04) + 3 (01-24) + 4 (23-01),$$
$$f_1 = (\epsilon^2 - \epsilon^3) . 0 (34-12) + 1 (03-24) + 2 (01-34) + 3 (24-01) + 4 (12-03),$$
$$f_2 = (\epsilon^2 - \epsilon^3) . 0 (23-14) + 1 (04-23) + 2 (14-03) + 3 (12-04) + 4 (03-12),$$
$$f_3 = (\epsilon^2 - \epsilon^3) . 0 (14-23) + 1 (34-02) + 2 (01-34) + 3 (02-14) + 4 (23-01),$$
$$f_4 = (\epsilon^2 - \epsilon^3) . 0 (34-12) + 1 (02-34) + 2 (04-13) + 3 (12-04) + 4 (13-02);$$

we have thus

$$\begin{aligned} f_0 + f_1 + f_2 + f_3 + f_4 = (\epsilon - \epsilon^4) . 0 &\{34 - 12 + 2 (13-24)\} + (\epsilon^2 - \epsilon^3) . 0 \{24 - 13 + 2 (34-12)\} \\ + 1 &\{04 - 23 + 2 (24-03)\} + \quad 1 \{03 - 24 + 2 (04-23)\} \\ + 2 &\{01 - 34 + 2 (03-14)\} + \quad 2 \{14 - 03 + 2 (10-24)\} \\ + 3 &\{12 - 04 + 2 (14-02)\} + \quad 3 \{20 - 14 + 2 (12-04)\} \\ + 4 &\{23 - 01 + 2 (02-13)\} + \quad 4 \{13 - 02 + 2 (23-01)\} \\ = - \{(\epsilon - \epsilon^4) + 2 (\epsilon^2 - \epsilon^3)\} (f)_1 &- \{2 (\epsilon - \epsilon^4) - (\epsilon^2 - \epsilon^3)\} (f)_2, \end{aligned}$$

if for a moment $(f)_1$ denotes the terms of $f$ which are multiplied by $\epsilon - \epsilon^4$ and $(f)_2$ the terms of $f$ which are multiplied by $\epsilon^2 - \epsilon^3$.

But we have

$$\sqrt{5} = \epsilon + \epsilon^4 - \epsilon^2 - \epsilon^3,$$

whence

$$\sqrt{5} (\epsilon - \epsilon^4) = \epsilon - \epsilon^4 + 2 (\epsilon^2 - \epsilon^3),$$
$$\sqrt{5} (\epsilon^2 - \epsilon^3) = 2 (\epsilon - \epsilon^4) + \epsilon^2 - \epsilon^3;$$

and hence the equation just obtained is $f_0 + f_1 + f_2 + f_3 + f_4 = -\sqrt{5} f$, viz. this equation, being satisfied separately by the terms such as $x_0 x_1^2 x_2^2$ and $x_0^3 x_1 x_2$, will be satisfied for $x_0 x_1^2 x_2^2 + v x_0^3 x_1 x_2$: and so for the like equations which follow. (I notice that Brioschi has $+\sqrt{5} f$; the difference is quite immaterial, since the formulæ would coincide by reversing the sign of $f$, or those of $f_0$, $f_1$, $f_2$, $f_3$, $f_4$.)

We show further that $f_0 + \epsilon^2 f_1 + \epsilon^4 f_2 + \epsilon f_3 + \epsilon^3 f_4 = 0$; to verify this, observe that in this expression the terms multiplied by 0 $(= x_0)$ are

$$(\epsilon - \epsilon^4) \{34 - 12 + \epsilon^2 (23-14) + \epsilon^4 (13-24) + \epsilon (13-24) + \epsilon^3 (14-23)\}$$
$$+ (\epsilon^2 - \epsilon^3) \{24 - 13 + \epsilon^2 (34-12) + \epsilon^4 (23-14) + \epsilon (14-23) + \epsilon^3 (34-12)\},$$

where the terms containing 12, 13, 14, 23, 24, 34 respectively are each $=0$, viz. the coefficient of 12 is $(-\epsilon+\epsilon^4)+(-\epsilon^4+1)+(-1+\epsilon)=0$: that of 13 is

$$1-\epsilon^3+(\epsilon^2-1)+(-\epsilon^2+\epsilon^3)=0:$$

and so for the other coefficients. In like manner it appears that the terms multiplied by 1, 2, 3, 4 ($=x_1, x_2, x_3, x_4$) respectively are each $=0$, and thus the equation in question is verified. And in like manner it is shown that

$$f_0+\epsilon^3 f_1+\epsilon f_2+\epsilon^4 f_3+\epsilon^2 f_4=0.$$

The roots $f$ thus satisfy the relations

$$\begin{aligned} f_0+f_1+f_2+f_3+f_4&=-f\sqrt{5},\\ f_0+\epsilon^2 f_1+\epsilon^4 f_2+\epsilon f_3+\epsilon^3 f_4&=0,\\ f_0+\epsilon^3 f_1+\epsilon f_2+\epsilon^4 f_3+\epsilon^2 f_4&=0,\end{aligned}$$

or the equation for $f^2$ belongs to the class of Jacobi's multiplier equations. Hence (see Brioschi's "Appendice terza" before referred to) the form of the equation is

$$(f^2-a)^6-4a(f^2-a)^5+10b(f^2-a)^3-4c(f^2-a)+5b^2-4ac=0,$$

or determining the arbitrary coefficient $v$ so that $a$ may be $=0$, the form is

$$f^{12}+10bf^6-4cf^2+5b^2=0,$$

which is Kronecker's equation

$$f^{12}-10\phi f^6+5\psi^2=\psi f^2.$$

As to the meaning of the coefficients $a$, $b$, $c$, I recall that, in virtue of the foregoing linear relations between the roots, these may be expressed in terms of three arbitrary quantities $a_0$, $a_1$, $a_2$ as follows:

$$\begin{aligned} f&=a_0\sqrt{5},\\ f_0&=a_0+a_1+a_2,\\ f_1&=a_0+\epsilon a_1+\epsilon^4 a_2,\\ f_2&=a_0+\epsilon^2 a_1+\epsilon^3 a_2,\\ f_3&=a_0+\epsilon^3 a_1+\epsilon^2 a_2,\\ f_4&=a_0+\epsilon^4 a_1+\epsilon a_2,\end{aligned}$$

and $a$, $b$, $c$ are then determinate functions of $a_0$, $a_1$, $a_2$, viz. we have

$$\begin{aligned} a&=a_0^2+a_1a_2,\\ b&=8a_0^4a_1a_2-2a_0^2a_1^2a_2^2+a_1^3a_2^3-a_0(a_1^5+a_2^5),\\ c&=80a_0^6a_1^2a_2^2-40a_0^4a_1^3a_2^3+5a_0^2a_1^4a_2^4+a_1^5a_2^5\\ &\quad -a_0(32a_0^4-20a_0^2a_1a_2+5a_1^2a_2^2)(a_1^5+a_2^5)\\ &\quad +\tfrac{1}{4}(a_1^5+a_2^5)^2;\end{aligned}$$

so that, for $a=0$ and therefore $a_0=\sqrt{-a_1a_2}$, we have

$$\begin{aligned} b&=11a_1^3a_2^3-a_0(a_1^5+a_2^5),\\ c&=-44a_1^5a_2^5-57a_0a_1^2a_2^2(a_1^5+a_2^5)+\tfrac{1}{4}(a_1^5+a_2^5)^2,\end{aligned}$$

but I do not know that for Kronecker's form the actual values of $a_0$, $a_1$, $a_2$ in terms of the coefficients of the quintic equation have been calculated.