949.

ON HALPHEN'S CHARACTERISTIC n, IN THE THEORY OF CURVES IN SPACE.

[From *Crelle's Journal d. Mathematik,* t. CXI. (1893), pp. 347—352.]

IF we consider a curve in space without actual singularities, of the order (or degree) d, then this has a number h of apparent double points (adps.), viz. taking as vertex an arbitrary point in space, we have through the curve a cone of the order d, with h nodal lines; and Halphen denotes by n the order of the cone of lowest order which passes through these h lines. For a given value of d, h is at most $=\frac{1}{2}(d-1)(d-2)$, and as shown by Halphen it is at least $=[\frac{1}{4}(d-1)^2]$, if we denote in this manner the integer part of $\frac{1}{4}(d-1)^2$. For given values of d, h, it is easy to see that n must lie within certain limits, viz. if ν be the smallest number such that $\frac{1}{2}\nu(\nu+3)$ equal to or greater than h, then n is at most $=\nu$; and moreover n must have a value such that nd is at least $=2h$, or say we must have $nd=2h+\theta$, where θ is $=0$ or positive. For any given value of d, we thus have a finite number of forms (d, h, n), and we have thus *prima facie* curves in space of the several forms (d, h, n): but it may very well be, and in fact Halphen finds, that when $d=9$ or upwards, then for certain values of h, n as above, there is not any curve (d, h, n); thus $d=9$, $h=17$ the values of n are $n=4$, $n=5$, but there is not any curve $d=9$, $h=17$ for either of these values of n; or say the curves (9, 17, 4) and (9, 17, 5) are non-existent. And in the Notes and References to the papers 302, 305 in vol. V. of my *Collected Mathematical Papers,* 4to. Cambridge, 1892, see p. 615, I remarked that, in certain cases for which Halphen finds a curve (d, h, n), such curve does not exist except for special configurations of the h nodal lines not determined by the mere definition of n as the order of the cone of lowest order which passes through the h nodal lines; for instance $d=9$, $h=16$, $n=4$, for which Halphen gives a curve, I find that for the existence of the curve it is not enough that the 16 lines are situate upon a quartic cone, but they must be the 16 lines of intersection of two quartic cones.

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In fact, starting from an existing curve, say the complete intersection of two given surfaces of the orders μ, ν respectively, we have, as is known,

$$d = \mu\nu, \quad 2h = \mu\nu(\mu - 1)(\nu - 1):$$

we find also, as will appear, $n = (\mu - 1)(\nu - 1)$: hence also $nd = 2h$, viz. the cone of the order n through the h nodal lines meets the cone of the order d in these lines counting each twice, and in no other lines. I remark also that h is $= \frac{1}{2}n(n+3)$ if $\mu + \nu = 4$; viz. we have here two nodal lines lying in a plane; but if $\mu + \nu > 4$, then h is greater than $\frac{1}{2}n(n+3)$, viz. in this case the nodal lines are not h lines taken at pleasure, but they are lines subject to the condition of lying in a cone of the order n. But this is not all; the h nodal lines lie not only in this cone of lowest order n, but also in cones of the orders $n+1$, $n+2$, ..., $n + (\mu + \nu - 2)$ respectively: I do not for the moment attempt to determine the number of independent conditions which are hereby imposed upon the h lines.

The last-mentioned theorem constitutes in fact the geometrical interpretation of results contained in Jacobi's paper "De eliminatione variabilis e duabus aequationibus algebraicis," *Crelle*, t. XV. (1836), pp. 101—124, [*Ges. Werke*, t. III. pp. 295—320]. Consider the two equations

$$U = (*\between x, y, z, w)^{\mu} = 0; \quad V = (*\between x, y, z, w)^{\nu} = 0;$$

representing surfaces of the orders μ, ν respectively; since the form of the equations is quite arbitrary, we may without loss of generality assume that the point

$$(x, y, z) = (0, 0, 0)$$

is an arbitrary point in space; and this being so, we find the equation of the cone, vertex this point, which passes through the curve of intersection of the two surfaces by the mere elimination of w between the two equations. As the reasoning is exactly the same for a particular case, I write for convenience $\mu = 3$, $\nu = 4$, and consider the two equations

$$A_0w^3 + A_1w^2 + A_2w + A_3 = 0,$$

$$B_0w^4 + B_1w^3 + B_2w^2 + B_3w + B_4 = 0,$$

where the suffixes show the degrees in regard to (x, y, z), viz. A_0, B_0 are mere constants, A_1, B_1 are linear functions $(*\between x, y, z)^1$, A_2, B_2 quadric functions $(*\between x, y, z)^2$, and so on. Multiplying the first equation successively by 1, w, w^2, w^3, and the second equation successively by 1, w, w^2, we have 7 equations from which to eliminate 1, w, w^2, w^3, w^4, w^5, w^6, and the result is

$$\begin{vmatrix} & & & A_0, & A_1, & A_2, & A_3 \\ & & A_0, & A_1, & A_2, & A_3, & \\ & A_0, & A_1, & A_2, & A_3, & & \\ A_0, & A_1, & A_2, & A_3, & & & \\ & & B_0, & B_1, & B_2, & B_3, & B_4 \\ & B_0, & B_1, & B_2, & B_3, & B_4, & \\ B_0, & B_1, & B_2, & B_3, & B_4, & & \end{vmatrix} = 0,$$

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viz. this is an equation

$$(*\between x, y, z)^{12} = 0,$$

of the cone of the order $d = 12$, through the curve of intersection of the two surfaces: say this equation is $\Omega = 0$.

But if we only multiply the first equation by 1, w, w^2 successively and the second equation by 1, w successively, then we have 5 equations serving to determine the ratios of w^5, w^4, w^3, w^2, w, 1, viz. we have these quantities proportional to the six determinants which can be formed out of the matrix

$$\left|\begin{array}{cccccc} & & A_0, & A_1, & A_2, & A_3 \\ & A_0, & A_1, & A_2, & A_3, & \\ A_0, & A_1, & A_2, & A_3, & & \\ & B_0, & B_1, & B_2, & B_3, & B_4 \\ B_0, & B_1, & B_2, & B_3, & B_4, & \end{array}\right|;$$

say we have

$$\begin{aligned} & w^5 : w^4 : w^3 : w^2 : w : 1 \\ &= L : M : N : P : Q : R, \end{aligned}$$

where L, M, N, P, Q, R represent homogeneous functions $(*\between x, y, z)^\theta$, of the degrees 11, 10, 9, 8, 7, 6 respectively. We may if we please write

$$w = \frac{L}{M} = \frac{M}{N} = \frac{N}{P} = \frac{P}{Q} = \frac{Q}{R};$$

or eliminating w, we have the series of equations which may be written

$$\left|\begin{array}{ccccc} L, & M, & N, & P, & Q \\ M, & N, & P, & Q, & R \end{array}\right| = 0,$$

viz. we thus denote that the determinants formed with any two columns of this matrix are severally $= 0$. This of course implies that each of the determinants in question is the product of Ω and a factor which is a homogeneous function of the proper degree in (x, y, z), so that the several equations are all of them satisfied if only $\Omega = 0$. We have for instance $PR - Q^2 = A\Omega$, where A is a quadric function $(*\between x, y, z)^2$; similarly $NR - PQ = B\Omega$, where B is a cubic function $(*\between x, y, z)^3$; and the like as regards the other determinants.

If the ratios $x : y : z$ have any given values such that we have for these $\Omega = 0$, then w has a determinate value, that is, on each line of the cone $\Omega = 0$, there is a single point of the curve of intersection of the two surfaces: the only exceptions are when, for the given values of $x : y : z$, the expressions for w assume an indeterminate form, viz. w has then two values, and there are upon the line two points of the curve, or what is the same thing, the line is a nodal line of the cone: the conditions for a nodal line thus are $L = 0$, $M = 0$, $N = 0$, $P = 0$, $Q = 0$, $R = 0$, viz. each of these equations is that of a cone passing through the nodal lines of the cone $\Omega = 0$; the cone of lowest order is $R = 0$, a cone of the order 6 meeting the

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cone $\Omega=0$ of the order 12 in 36 lines each twice, which lines are consequently the nodal lines of the cone $\Omega=0$. The mere condition of the 36 lines lying upon a cone of the order 6 shows that the 36 lines are not arbitrary; and we have moreover, through the 36 lines, cones of the orders 7, 8, 9, 10 and 11 respectively. Obviously the foregoing reasoning is quite general, and for the surfaces of the orders μ, ν we have (as stated above) the cone Ω of the order $\mu\nu$, with $h=\frac{1}{2}\mu\nu(\mu-1)(\nu-1)$ nodal lines, the intersections (each counting twice) of this cone with a cone of the order $n=(\mu-1)(\nu-1)$; and moreover the h nodal lines lie also in cones of the orders $n+1$, $n+2$, ..., $n+\mu+\nu-2$ respectively.

To examine the meaning of the theorem, I form the table

μ, ν	d	h	n, ..., $n+\mu+\nu-2$
2, 2	4	2	1, 2, 3
2, 3	6	6	2, 3, 4, 5
2, 4	8	12	3, 4, 5, 6, 7
2, 5	10	20	4, 5, 6, 7, 8, 9
⋮			
3, 3	9	18	4, 5, 6, 7, 8
3, 4	12	36	6, 7, 8, 9, 10, 11
3, 5	15	60	8, 9, 10, 11, 12, 13, 14
⋮			
4, 4	16	72	9, 10, 11, 12, 13, 14, 15
4, 5	20	120	12, 13, 14, 15, 16, 17, 18, 19
⋮			

Here μ, $\nu=2$, 2, there are 2 nodal lines, which are arbitrary, and of course lie on cones of the orders 1, 2, 3 respectively. So μ, $\nu=2$, 3; there are 6 nodal lines, which are not arbitrary, inasmuch as they lie on a cone of the order 2; but regarding them as arbitrary lines on such a cone, we can through them draw a cone of the order 3 or any higher order, and it is thus no specialisation to say that they lie upon cones of the orders 3, 4, and 5. But going a step further $\mu=2$, $\nu=4$: here we have 12 nodal lines which, inasmuch as they lie on a cone of the order 3, are not arbitrary: and they are not arbitrary lines upon this cone, for they lie on a cone of the order 4, and such a cone can be drawn through at most 11 arbitrary lines on a cubic cone. In fact, upon a cone of the order θ, taking at pleasure N lines, the condition that it may be possible through these to draw a cone of the order $\theta+1$ is $\frac{1}{2}(\theta+1)(\theta+4)=N+3$ at least; for if this number were $=N+2$, then through the N points we have only the improper cone $(x+\beta y+\gamma z)\,U_\theta=0$, if $U_\theta=0$ is the cone of the order θ. It thus appears that the 12 nodal lines are not arbitrary lines on a cubic cone, but that they constitute the complete intersection of a cubic cone and

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a quartic cone. But through such 12 lines we may draw cones of the 5th and higher orders, and it is thus no further condition that the 12 lines lie on cones of the orders 5, 6 and 7 respectively.

So again $\mu = 3$, $\nu = 3$; we have here 18 nodal lines which, inasmuch as they lie on a cone of the order 4, are not arbitrary: and they are not arbitrary lines on this cone inasmuch as they lie also on a cone of the order 5, and such a cone can be drawn through at most 17 arbitrary lines on the quartic cone: it thus appears that the 18 nodal lines are 18 out of the 20 lines of intersection of a quartic cone and a quintic cone. But there is no further condition, for through such lines we can draw a cone of the order 6 or any higher order and thus the lines lie on cones of the orders 6, 7 and 8 respectively. It appears probable however that, for higher values of μ, ν, it would be necessary tc take account not only (as in these examples) of the cones of the orders n and $n+1$, but of those of higher orders $n+2$, &c.; and thus that it is *not* the true form of the theorem to say that the h nodal lines must be h out of the $n(n+1)$ lines of intersection of two cones of the orders n and $n+1$ respectively.

It appears, by what precedes, that the h, $=\frac{1}{2}\mu\nu(\mu-1)(\nu-1)$, lines which are the nodal lines of the cone of arbitrary vertex which passes through the curve of intersection of two surfaces of the orders μ, ν respectively, form a remarkable special system of lines, which well deserve further study. I remark also that, without having proved the negative, it seems to me clear that given the values of d, h, n it is only in the cases where the h lines form some such special system (and not in the general case where the h nodal lines are any lines whatever on a cone of the order n) that there exists a curve (d, h, n); and thus that the question for further investigation is, for given values of (d, h, n) to determine the conditions necessary for the existence of a curve in space with these characteristics (d, h, n).

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