947.

ON A SYSTEM OF TWO TETRADS OF CIRCLES; AND OTHER SYSTEMS OF TWO TETRADS.

[From the *Proceedings of the Cambridge Philosophical Society*, vol. VIII. (1893), pp. 54—59.]

THE investigations of the present paper were suggested to me by Mr Orr's paper, "The Contacts of certain Systems of Circles," *Proc. Camb. Phil. Soc.*, vol. VII.

1. It is possible to find *in plano* two tetrads of circles, or say four red circles and four blue circles, such that each red circle touches each blue circle: in fact, counting the constants, a circle depends upon 3 constants, or say it has a capacity $=3$; the capacity of the eight circles is thus $=24$; and the postulation or number of conditions to be satisfied is $=16$: the resulting capacity of the system is thus *primâ facie*, $16-24=8$. It will, however, appear that, in the system considered, the true value is $=9$.

2. The *primâ facie* value of the capacity being $=8$, we are not at liberty to assume at pleasure three circles of the system. And, in fact, assuming at pleasure say 3 red circles, then touching each of these we have 8 circles, forming $\frac{1}{24}8.7.6.5, =70$, tetrads of circles: taking at random any one of these tetrads for the blue circles, the remaining red circle has to be determined so as to touch each of the four blue circles, that is, by four instead of three conditions; and there is not in general any red circle satisfying these four conditions. But the 8 tangent circles do not stand to each other in a relation of symmetry, but form in fact four pairs of circles; and it is possible out of the 70 tetrads to select (and that in 6 ways) a tetrad of blue circles, such that there exists a fourth red circle touching each of these four blue circles. We have thus a system depending upon 3 arbitrary circles, and for which, therefore, the capacity is $=9$. It is (as is known) possible, in quite a

different manner, out of the 70 tetrads to select (and that in 8 ways) a tetrad of blue circles such that there exists a fourth red circle touching each of these four blue circles—but the present paper relates exclusively to the first-mentioned 6 tetrads and not to these 8 tetrads.

3. I consider, in the first instance, a particular case in which the three red circles are not all of them arbitrary, but have a capacity $9-1, =8$; and pass from this to the general case where the capacity is $=9$. Calling the red circles 1, 2, 3 and 4; I start with the circles 1, and 2 arbitrary, and 3 a circle equal to 2: the radical axis, or common chord, of the circles 2 and 3 is thus a line bisecting at right angles the line joining the centres of the circles 2 and 3, say this is the line Ω. We have then four circles, each having its centre in the line Ω and touching the circles 1 and 2: in fact, the locus of the centre of a circle touching the circles 1 and 2 is a pair of conics, each of them having for foci the centres of these circles: the line Ω meets each of these conics in two points, and there are thus on the line Ω four points, each of them the centre of a circle touching the circles 1 and 2. But the equal circles 2 and 3 are symmetrically situate in regard to the line Ω; and it is obvious that the four circles, having their centres on the line Ω, will each of them also touch the circle 3; we have thus the four blue circles, each of them with its centre on the line Ω and touching each of the red circles 1, 2 and 3. And it is moreover clear that, taking the red circle 4 equal to 1 and situate symmetrically therewith in regard to the line Ω, then this circle 4 will touch each of the blue circles: so that we have here the four blue circles, each of them touching the four red circles. As already mentioned, the blue circles have their centre on the line Ω, that is, the line Ω is a common orthotomic of the four blue circles.

4. By inverting in regard to an arbitrary circle we pass to the general case; the line Ω becomes thus a circle Ω, orthotomic to each of the blue circles.

Starting *ab initio*, we have here at pleasure the red circles 1, 2, 3: the circle Ω is a circle having for centre a centre of symmetry of the circles 2 and 3, and passing through the points of intersection (real or imaginary) of these two circles; the circles 2 and 3 are thus the inverses (or say the images) each of the other in regard to the circle Ω. We can then find 4 circles each of them orthotomic to Ω, and touching the circles 1 and 2: but a circle orthotomic to Ω is its own inverse or image in regard to Ω; and it will thus touch the circle 3 which is the image of 2 in regard to Ω. We have thus the four blue circles each of them touching the red circles 1, 2 and 3; and then, taking the red circle 4 as the inverse or image of 1 in regard to Ω, this circle 4 will also touch each of the blue circles. Thus starting with the arbitrary red circles 1, 2, 3, we find the four blue circles and the remaining red circle 4, such that each of the blue circles touches each of the red circles. Since in the construction we group together at pleasure the two circles 2, 3 (out of the three circles 1, 2, 3) and use at pleasure either of the two centres of symmetry, it appears that the number of ways in which the figure might have been completed is $=6$.

5. The blue circles have a common orthotomic circle Ω, that is, the radical axis or common chord of each two of the blue circles passes through one and the same point, the centre of the circle Ω. The figure is symmetrical in regard to the red and blue circles respectively, and thus the red circles have a common orthotomic circle Ω', that is, the radical axis or common chord of each two of the red circles passes through one and the same point, the centre of the circle Ω'.

6. Projecting stereographically on a spherical surface, the four red circles and the four blue circles become circles of the sphere; and then making the general homographic transformation, they become plane sections of a quadric surface; we have thus the theorem that on a given quadric surface it is possible to find four red sections and four blue sections such that each blue section touches each red section; and moreover the capacity of the system is $=9$; viz. 3 of the red sections may be assumed at pleasure. But (as is well known) the theory of the tangency of plane sections of a quadric surface is far more simple than that of the tangency of circles: the condition in order that two sections may touch each other is simply the condition that the line of intersection of the two planes shall touch the quadric surface. And we construct, as follows, the sections touching each of three given sections: say the given sections are 1, 2, 3; through the sections 1 and 2 we have two quadric cones having for vertices say the points d_{12} and i_{12} (direct and inverse centres of the two sections): similarly, through the sections 1 and 3 we have two quadric cones vertices d_{13} and i_{13} respectively, and through the sections 2 and 3 we have two quadric cones vertices d_{23} and i_{23} respectively; the points d_{12}, i_{12}, d_{13}, i_{13}, d_{23}, i_{23} lie three and three in four intersecting lines or axes, viz. these are $d_{23}d_{31}d_{12}$, $d_{23}i_{31}i_{12}$, $d_{31}i_{12}i_{23}$, $d_{12}i_{23}i_{31}$ respectively. Through any one of these axes, say $d_{23}d_{31}d_{12}$, we may draw to the quadric surface two tangent planes each touching the three cones which have their vertices in the points d_{23}, d_{31}, d_{12} respectively; and the section by either tangent plane is thus a section touching each of the three given sections 1, 2, 3; we have thus the eight tangent sections of these three sections.

7. Taking as three of the red sections the arbitrary sections 1, 2, 3; and grouping together two at pleasure of these sections, say 2 and 3; we may take for the blue sections the two sections through the axis $d_{23}d_{31}d_{12}$, and those through the axis $d_{23}i_{31}i_{12}$; we have thus the four blue sections touching each of the given red sections 1, 2, 3; and this being so, there exists a remaining red section 4 touching each of the blue sections; we have thus the four blue sections touching each of the red sections 1, 2, 3 and 4. This implies that the vertices or points d_{24} and d_{34} lie on the axis $d_{23}d_{31}d_{12}$, and that the vertices or points i_{24} and i_{34} lie on the axis $d_{23}i_{21}i_{31}$; or, what is the same thing, that the four sections 1, 2, 3, 4 have in common an axis $d_{23}d_{21}d_{31}d_{24}d_{34}$ and also an axis $d_{23}i_{21}i_{31}i_{24}i_{34}$.

8. If the quadric surface be a flat surface (*surface aplatie*) or conic, then the red sections become chords of the conic; the axes are lines in the plane of the conic, and thus the tangent planes through an axis each coincide with the plane of the conic, and it would at first sight appear that any theorem as to tangency becomes nugatory. But this is not so; comparing with the last preceding paragraph, we still have the theorem: on a given conic, taking at pleasure any three chords

1, 2, 3, it is possible to find a fourth chord 4, such that the four chords have in common an axis $d_{23}d_{21}d_{31}d_{24}d_{34}$ and also an axis $d_{23}i_{21}i_{31}i_{24}i_{34}$. And the analytical theory (although somewhat complex) is extremely interesting. Considering the conic $xz - y^2 = 0$, the coordinates of a point on the conic are given by $x : y : z = 1 : \theta : \theta^2$, or say any point of the conic is determined by its parameter θ; and this being so, considering any three chords 1, 2, 3, I take for the two extremities of 1 the values ϵ, ζ; for those of 2 the values α, β; and for those of 3 the values γ, δ; the remaining chord 4 is to be determined as above, and I take for its two extremities the values E, Z.

9. Starting with the chords 1, 2, 3, we have each of the points d_{23}, &c., as the intersection of two lines, viz. these are

$$\text{for } d_{23} \begin{cases} x\alpha\delta - y(\alpha+\delta) + z = 0, \\ x\beta\gamma - y(\beta+\gamma) + z = 0, \end{cases} \quad \text{for } i_{23} \begin{cases} x\alpha\gamma - y(\alpha+\gamma) + z = 0, \\ x\beta\delta - y(\beta+\delta) + z = 0, \end{cases}$$

$$\text{,, } d_{12} \begin{cases} x\alpha\zeta - y(\alpha+\zeta) + z = 0, \\ x\beta\epsilon - y(\beta+\epsilon) + z = 0, \end{cases} \quad \text{,, } i_{12} \begin{cases} x\alpha\epsilon - y(\alpha+\epsilon) + z = 0, \\ x\beta\zeta - y(\beta+\zeta) + z = 0, \end{cases}$$

$$\text{,, } d_{13} \begin{cases} x\gamma\zeta - y(\gamma+\zeta) + z = 0, \\ x\delta\epsilon - y(\delta+\epsilon) + z = 0, \end{cases} \quad \text{,, } i_{13} \begin{cases} x\gamma\epsilon - y(\gamma+\epsilon) + z = 0, \\ x\delta\zeta - y(\delta+\zeta) + z = 0, \end{cases}$$

and we thence find without difficulty for the axis $d_{23}d_{12}d_{13}$ the equation

$$\begin{aligned} & x\{\alpha\beta(\delta\epsilon - \gamma\zeta) + \gamma\delta(\zeta\alpha - \epsilon\beta) + \epsilon\zeta(\beta\gamma - \alpha\delta)\} \\ & + y\{(\beta-\alpha)(\gamma\delta - \epsilon\zeta) + (\delta-\gamma)(\epsilon\zeta - \alpha\beta) + (\zeta-\epsilon)(\alpha\beta - \gamma\delta)\} \\ & + z\{\ -(\delta\epsilon - \gamma\zeta) - (\zeta\alpha - \epsilon\beta) - (\beta\gamma - \alpha\delta)\} = 0; \end{aligned}$$

the equation of the axis $d_{23}i_{12}i_{13}$ is obtained herefrom by the interchange of ϵ and ζ.

10. The points d_{24} and d_{34} will lie upon the first-mentioned axis if only d_{24} lies upon this axis, viz. if we have

$$\begin{aligned} & \{\alpha\beta(\delta\epsilon - \gamma\zeta) + \gamma\delta(\zeta\alpha - \epsilon\beta) + \epsilon\zeta(\beta\gamma - \alpha\delta)\}(\beta + E - \alpha - Z) \\ & + \{(\beta-\alpha)(\gamma\delta - \epsilon\zeta) + (\delta-\gamma)(\epsilon\zeta - \alpha\beta) + (\zeta-\epsilon)(\alpha\beta - \gamma\delta)\}(\beta E - \alpha Z) \\ & + \{\ -(\delta\epsilon - \gamma\zeta) - (\zeta\alpha - \epsilon\beta) - (\beta\gamma - \alpha\delta)\}\{-\alpha Z(\beta + E) + \beta E(\alpha + Z)\} = 0. \end{aligned}$$

Reducing this equation, the factor $\alpha - \beta$ divides out, and we finally obtain

$$\begin{aligned} & (\gamma-\alpha)(\beta\delta + \zeta Z)(\epsilon - E) \\ & + (\beta-\delta)(\alpha\gamma + \epsilon E)(\zeta - Z) \\ & + (\alpha\delta - \beta\gamma)(\epsilon\zeta - EZ) \\ & + (\alpha\beta - \gamma\delta)(\epsilon Z - \zeta E) = 0; \end{aligned}$$

say this is

$$A + BE + CZ + DEZ = 0,$$

where

$$\begin{aligned} A &= \quad . \quad (\gamma-\alpha)\beta\delta\epsilon + (\beta-\delta)\alpha\gamma\zeta + (\alpha\delta - \beta\gamma)\epsilon\zeta, \\ B &= -(\gamma-\alpha)\beta\delta \quad . \quad -(\alpha\beta - \gamma\delta)\zeta + (\beta-\delta)\epsilon\zeta, \\ C &= -(\beta-\delta)\alpha\gamma + (\alpha\beta - \gamma\delta)\epsilon \quad . \quad + (\gamma-\alpha)\epsilon\zeta, \\ D &= -(\alpha\delta - \beta\gamma) - (\beta-\delta)\epsilon - (\gamma-\alpha)\zeta \quad . \quad , \end{aligned}$$

viz. this is the condition for the existence of the axis $d_{23}d_{13}d_{12}d_{43}d_{42}$.

We interchange herein ϵ, ζ and also E, Z, and we thus obtain

$$A' + C'E + B'Z + D'EZ = 0,$$

where

$$\begin{aligned}
A' &= \quad . \quad (\beta - \delta)\,\alpha\gamma\epsilon + (\gamma - \alpha)\,\beta\delta\zeta + (\alpha\delta - \beta\gamma)\,\epsilon\zeta,\\
B' &= -(\gamma - \alpha)\,\beta\delta - (\alpha\beta - \gamma\delta)\,\epsilon \quad . \quad + (\beta - \delta)\,\epsilon\zeta,\\
C' &= -(\beta - \delta)\,\alpha\gamma \quad . \quad + (\alpha\beta - \gamma\delta)\,\zeta + (\gamma - \alpha)\,\epsilon\zeta,\\
D' &= -(\alpha\delta - \beta\gamma) - (\gamma - \alpha)\,\epsilon - (\beta - \delta)\,\zeta \quad . \quad ,
\end{aligned}$$

viz. this is the condition for the existence of the axis $d_{23}i_{13}i_{12}i_{43}i_{42}$.

11. I remark that we have

$$A' = A + \mathrm{a}\,(\epsilon - \zeta), \quad B' = B + \mathrm{b}\,(\epsilon - \zeta),$$
$$C' = C + \mathrm{c}\,(\epsilon - \zeta), \quad D' = D + \mathrm{d}\,(\epsilon - \zeta),$$

where

$$\begin{aligned}
\mathrm{a} &= (\beta - \delta)\,\alpha\gamma - (\gamma - \alpha)\,\beta\delta = \alpha\beta\,(\gamma + \delta) - \gamma\delta\,(\alpha + \beta),\\
\mathrm{b} &= \mathrm{c} = \gamma\delta - \alpha\beta,\\
\mathrm{d} &= \alpha + \beta - \gamma - \delta,
\end{aligned}$$

and further that

$$B - C = \alpha\beta\,(\gamma + \delta) - \gamma\delta\,(\alpha + \beta) + (\gamma\delta - \alpha\beta)(\epsilon + \zeta) + (\alpha + \beta - \gamma - \delta)\,\epsilon\zeta,$$

say this is Π.

12. It thus appears that, for the determination of E, Z, we have

$$A + BE + CZ + DEZ = 0,$$
$$A' + C'E + B'Z + D'EZ = 0.$$

Eliminating Z, we find

$$\frac{A + BE}{A' + C'E} = \frac{C + DE}{B' + D'E},$$

that is,

$$(AC' - A'C) + (AD' - A'D + BB' - CC')\,E + (BD' - B'D)\,E^2 = 0\,;$$

upon reducing the coefficients of this equation it appears that they contain each of them the factor Π, and throwing out this factor, the equation is

$$\begin{aligned}
&\epsilon\,[\alpha\beta\,(\gamma - \delta) + \gamma\delta\,(\beta - \alpha) + (\alpha\delta - \beta\gamma)\,\zeta]\\
&+ [-\alpha\beta\,(\gamma - \delta) - \gamma\delta\,(\beta - \alpha) + (\alpha\delta - \beta\gamma)\,\epsilon + (\beta\gamma - \alpha\delta)\,\zeta + (\beta + \gamma - \alpha - \delta)\,\epsilon\zeta]\,E\\
&+ [\beta\gamma - \alpha\delta - (\beta + \gamma - \alpha - \delta)]\,E^2 = 0\,;
\end{aligned}$$

this contains obviously the factor $E - \epsilon$, or throwing out this factor, we have for E the simple equation

$$\{\alpha\beta\,(\gamma - \delta) + \gamma\delta\,(\beta - \alpha)\} + (\alpha\delta - \beta\gamma)(E + \zeta) + (\beta - \alpha + \gamma - \delta)\,\zeta E = 0.$$

In a similar manner it may be shown that the two equations give for Z the like simple equation

$$\{\alpha\beta\,(\gamma - \delta) + \gamma\delta\,(\beta - \alpha)\} + (\alpha\delta - \beta\gamma)(Z + \epsilon) + (\beta - \alpha + \gamma - \delta)\,\epsilon Z = 0,$$

viz. starting from the chords 1, 2, 3 which depend on the parameters (ϵ, ζ), (α, β), (γ, δ) respectively, these last two equations give the parameters (E, Z) of the chord 4.