946.

NOTE ON THE THEORY OF ORTHOMORPHOSIS.

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THE equation of any given curve whatever, $\Theta = 0$, may be expressed in the form

$$\phi(x+iy)+\phi(x-iy)=0.$$

Let χ be any odd function; then since

$$\phi(x-iy)=-\phi(x+iy),$$

we have

$$\chi\phi(x-iy)=\chi\{-\phi(x+iy)\}=-\chi\phi(x+iy),$$

that is,

$$\chi\phi(x+iy)+\chi\phi(x-iy)=0.$$

Assuming that Θ is a real function, that is, a function with real coefficients, then also $\phi(x+iy)$ will be a function with real coefficients, or say a real function of $x+iy$; the function χ may be real or imaginary, but if imaginary, then the i of the coefficients does not change its sign in the passage from $\chi\phi(x+iy)$ to $\chi\phi(x-iy)$.

In proof of the assumed theorem, imagine the equation $\Theta=0$ expressed as an equation between $x+iy$ and $x-iy$, or, supposing it solved in regard to $x-iy$, take the form of it to be $x-iy=f(x+iy)$: let u_n be a function of n satisfying the equation of differences $u_{n+1}=fu_n$; and let $\phi(x+iy)$ be determined as a function of $x+iy$ by the elimination of n from the equations

$$x+iy=u_n,\quad \phi(x+iy)=\cos n\pi;$$

we thence have

$$x-iy=fu_n,\quad =u_{n+1},$$

and consequently

$$\phi(x-iy)=\cos(n+1)\pi,$$

that is,

$$\phi(x+iy)+\phi(x-iy)=0,$$

viz. this equation is a transformation of the equation $\Theta=0$, and thus it appears that the equation $\Theta=0$ can always be thrown into the last-mentioned form.

As an example, take the equation $y=ax+b$: which, putting for a moment $\xi=x+iy$, $\eta=x-iy$, is

$$\frac{1}{2i}(\xi-\eta)=\tfrac{1}{2}a(\xi+\eta)+b,$$

that is,

$$\eta=\frac{i+a}{i-a}\xi+\frac{2b}{i-a};$$

we have therefore

$$u_{n+1}=\frac{i+a}{i-a}u_n+\frac{2b}{i-a},$$

a solution of which is

$$u_n=\left(\frac{i+a}{i-a}\right)^n-\frac{b}{a};$$

putting this $=\xi$, we have

$$n=\frac{1}{\log\frac{i+a}{i-a}}\log\left(\xi+\frac{b}{a}\right),$$

and thence

$$\phi\xi=\cos\frac{\pi}{\log\frac{i+a}{i-a}}\log\left(\xi+\frac{b}{a}\right),$$

where observe that, writing $a+i=Re^{i\alpha}$ and therefore $a-i=Re^{-i\alpha}$, we have

$$\cos\alpha=\frac{a}{\sqrt{(a^2+1)}},\quad \sin\alpha=\frac{1}{\sqrt{(a^2+1)}},$$

or say $\cot\alpha=a$, and then

$$\frac{i+a}{i-a}=e^{2\alpha i+i\pi},\ \text{ or }\ \log\frac{i+a}{i-a}=i(2\alpha+\pi),$$

whence

$$\phi\xi=\cos\frac{\pi}{i(2\alpha+\pi)}\log\left(\xi+\frac{b}{a}\right),\ =\cosh\frac{\pi}{2\alpha+\pi}\log\left(\xi+\frac{b}{a}\right),$$

a real function of ξ.

In verification of the equation $\phi\xi+\phi\eta=0$, we have

$$\phi\eta=\cos\frac{\pi}{\log\frac{i+a}{i-a}}\log\left(\eta+\frac{b}{a}\right),$$

where

$$\log\left(\eta+\frac{b}{a}\right)=\log\left(\frac{i+a}{i-a}\,\xi+\frac{2b}{i-a}+\frac{b}{a}\right)=\log\frac{i+a}{i-a}\left(\xi+\frac{b}{a}\right),$$

$$=\log\frac{i+a}{i-a}+\log\left(\xi+\frac{b}{a}\right),$$

and thence

$$\phi\eta=\cos\frac{\pi}{\log\dfrac{i+a}{i-a}}\left\{\log\frac{i+a}{i-a}+\log\left(\xi+\frac{b}{a}\right)\right\}$$

$$=\cos\left\{\pi+\frac{\pi}{\log\dfrac{i+a}{i-a}}\log\left(\xi+\frac{b}{a}\right)\right\},\quad =-\cos\frac{\pi}{\log\dfrac{i+a}{i-a}}\log\left(\xi+\frac{b}{a}\right),$$

that is, $\phi\eta=-\phi\xi$, or $\phi\xi+\phi\eta=0$, the equation in question.

I remark, in passing, that the same equation $y=ax+b$ might have been put in the form $\phi x+\phi y=0$, viz. assuming

$$\phi x=\cos\frac{\pi}{\log a}\log\left(x-\frac{b}{1-a}\right),$$

then

$$\phi y=\cos\frac{\pi}{\log a}\log\left(ax+b-\frac{b}{1-a}\right)=\cos\frac{\pi}{\log a}\log a\left(x-\frac{b}{1-a}\right)$$

$$=\cos\frac{\pi}{\log a}\left\{\log a+\log\left(x-\frac{b}{1-a}\right)\right\}$$

$$=\cos\left\{\pi+\frac{\pi}{\log a}\log\left(x-\frac{b}{1-a}\right)\right\}$$

$$=-\cos\frac{\pi}{\log a}\log\left(x-\frac{b}{1-a}\right)=-\phi x,$$

that is, $\phi x+\phi y=0$.

If $b=0$, then

$$y=ax \text{ and } \phi x=\cos\frac{\pi\log x}{\log a};$$

in fact, repeating the proof for this particular case,

$$\phi y=\cos\pi\,\frac{\log ax}{\log a}=\cos\pi\left(1+\frac{\log x}{\log a}\right)=-\cos\frac{\pi\log x}{\log a},\quad =-\phi x;$$

that is,

$$\phi x+\phi y=0.$$

Considering then $(x,\ y)$ as the coordinates of a point on the curve $\Theta=0$, we have, as above,

$$\chi\phi\,(x+iy)+\chi\phi\,(x-iy)=0,$$

where ϕ is a real function determined as above, and χ is any real or imaginary odd function. This being so, assume

$$x_1+iy_1=e^{\chi\phi\,(x+iy)},$$

then also

$$x_1 - iy_1 = e^{\chi\phi(x-iy)},$$

and consequently

$$x_1^2 + y_1^2 = e^{\chi\phi(x+iy)+\chi\phi(x-iy)} = 1,$$

that is, we have the circumference of the circle $x_1^2 + y_1^2 - 1 = 0$ corresponding to the given curve $\Theta = 0$.

Suppose that the curve $\Theta = 0$ is a closed curve: and then writing

$$\xi + i\eta = \chi\phi(x+iy),$$

and therefore

$$\xi - i\eta = \chi\phi(x-iy),$$

we thence have

$$2\xi = \chi\phi(x+iy) + \chi\phi(x-iy),$$

a real function of (x, y).

(1) Assume that it is possible to find χ, such that ξ as defined by this last equation shall be throughout the area of the curve $\Theta = 0$ finite and continuous, except only that in the neighbourhood of a given point, taken to be the point $x = 0$, $y = 0$, it is $= \log\sqrt{(x^2 + y^2)}$.

(2) At the boundary of the area $\Theta = 0$, ξ is $= 0$.

(3) Throughout the area, ξ satisfies the partial differential equation

$$\frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} = 0.$$

These conditions being satisfied, the equation

$$x_1 + iy_1 = e^{\xi+i\eta},$$

that is,

$$x_1 + iy_1 = e^{\chi\phi(x+iy)},$$

gives an orthomorphosis of the area $\Theta = 0$ into the circle $x_1^2 + y_1^2 - 1 = 0$, the point $x = 0$, $y = 0$ corresponding to the centre of the circle; (2) and (3) are satisfied as above: it remains only to satisfy (1), viz. the function χ is determined not by any equation—*but only by this condition as to finiteness and continuity;* and if it be thus determined, then the foregoing equation $x_1 + iy_1 = e^{\chi\phi(x+iy)}$ gives the required orthomorphosis.

For instance, let the curve $\Theta = 0$ be the parabola $y^2 = 4(1 - x)$, which may be regarded as a closed curve bounding the infinite parabolic area. We have $2x = \xi + \eta$, $2iy = \xi - \eta$, whence the equation is

$$-\tfrac{1}{4}(\xi - \eta)^2 = 4 - 2\xi - 2\eta,$$

that is,

$$\xi^2 - 2\xi\eta + \eta^2 - 8\xi - 8\eta + 16 = 0,$$

whence $\sqrt{\xi} + \sqrt{\eta} - 2 = 0$, or writing this in the form

$$(\sqrt{\xi} - 1) + (\sqrt{\eta} - 1) = 0,$$

we have $\phi\xi = \sqrt{\xi} - 1$, and assuming that χ can be found so that the condition as to finiteness and continuity is satisfied, then the orthomorphosis is given by

$$x_1 + iy_1 = \exp\chi\,(\sqrt{\xi} - 1), \quad = \exp\chi\,\{\sqrt{(x+iy)} - 1\}.$$

Assuming

$$\tfrac{1}{2}\chi\omega = -\tfrac{1}{2}i\pi\omega + \log\frac{1 - i\exp(\ \ \tfrac{1}{2}i\pi\omega)}{1 - i\exp(-\tfrac{1}{2}i\pi\omega)},$$

which is obviously an odd function, we have

$$\exp\tfrac{1}{2}\chi\omega = \frac{1}{\exp\tfrac{1}{2}i\pi\omega}\,\frac{1 - i\exp(\ \ \tfrac{1}{2}i\pi\omega)}{1 - i\exp(-\tfrac{1}{2}i\pi\omega)},$$

$$= \frac{1 - \exp\tfrac{1}{2}i\pi\omega}{\exp\tfrac{1}{2}i\pi\omega - 1}, \quad = \frac{i\,(1 - i\exp\tfrac{1}{2}i\pi\omega)}{1 + i\exp\tfrac{1}{2}i\pi\omega},$$

which is

$$= \tan\tfrac{1}{4}\pi\,(\omega + 1),$$

and hence, for ω writing $\sqrt{(x+iy)} - 1$, we have

$$x_1 + iy_1 = \exp\chi\,\{\sqrt{(x+iy)} - 1\}, \quad = \tan^2\tfrac{1}{4}\pi\sqrt{(x+iy)}.$$

This satisfies the required conditions as to finiteness and continuity; and in particular, we have

$$\xi + i\eta = \log\tan^2\tfrac{1}{4}\pi\sqrt{(x+iy)},$$

so that, x and y being small,

$$\xi + i\eta = \log\frac{\pi^2}{16}(x+iy), \quad \xi - i\eta = \log\frac{\pi^2}{16}(x-iy),$$

that is,

$$\xi = \log\frac{\pi^2}{16}\sqrt{(x^2+y^2)}.$$

Hence we have the known result: the orthomorphosis of the parabola $y^2 = 4(1-x)$ into the circle $x_1^2 + y_1^2 - 1 = 0$ is given by the equation $x_1 + iy_1 = \tan^2\frac{1}{4}\pi\sqrt{(x+iy)}$.

Consider the ellipse, where $a^2 - b^2 = 1$, or say

$$\frac{x^2}{\frac{1}{4}\left(M + \frac{1}{M}\right)^2} + \frac{y^2}{\frac{1}{4}\left(M - \frac{1}{M}\right)^2} = 1.$$

I show, by a less direct process, how to express this equation in the required form $\phi\xi + \phi\eta = 0$. In fact, writing

$$\xi = x + iy, \quad \eta = x - iy,$$

the equation of the ellipse is the rationalised form of

$$i\eta + \sqrt{(1-\eta^2)} = M^2\{i\xi + \sqrt{(1-\xi^2)}\}.$$

To show that this is so, call for a moment the right-hand side Ω, the equation is

$$\sqrt{(1-\eta^2)} = \Omega - i\eta,$$

hence

$$1 - \eta^2 = \Omega^2 - 2\Omega i\eta - \eta^2,$$

$$2\Omega i\eta = \Omega^2 - 1,$$

or

$$2i\eta = \Omega - \frac{1}{\Omega} = M^2\{i\xi + \sqrt{(1-\xi^2)}\} + \frac{1}{M^2}\{i\xi - \sqrt{(1-\xi^2)}\},$$

$$= \left(M^2 + \frac{1}{M^2}\right) i\xi + \left(M^2 - \frac{1}{M^2}\right)\sqrt{(1-\xi^2)},$$

therefore

$$2i\eta - \left(M^2 + \frac{1}{M^2}\right) i\xi = \left(M^2 - \frac{1}{M^2}\right)\sqrt{(1-\xi^2)}$$

$$-4\eta^2 + 4\left(M^2 + \frac{1}{M^2}\right)\xi\eta + \left(M^4 + 2 + \frac{1}{M^4}\right)(-\xi)^2 = \left(M^4 - 2 + \frac{1}{M^4}\right) - \left(M^4 - 2 + \frac{1}{M^4}\right)\xi^2,$$

that is,

$$-4\eta^2 - 4\xi^2 + 4\left(M^2 + \frac{1}{M^2}\right)\xi\eta = \left(M^2 - \frac{1}{M^2}\right)^2,$$

or say

$$-\xi^2 - \eta^2 + \left(M^2 + \frac{1}{M^2}\right)\xi\eta - \tfrac{1}{4}\left(M^2 - \frac{1}{M^2}\right)^2 = 0:$$

viz. substituting for ξ, η their values, this is

$$-2(x^2 - y^2) + \left(M^2 + \frac{1}{M^2}\right)(x^2 + y^2) - \tfrac{1}{4}\left(M^2 - \frac{1}{M^2}\right)^2 = 0,$$

that is,

$$\left(M - \frac{1}{M}\right)^2 x^2 + \left(M + \frac{1}{M}\right)^2 y^2 - \tfrac{1}{4}\left(M^2 - \frac{1}{M^2}\right)^2 = 0,$$

or finally, it is

$$\frac{x^2}{\tfrac{1}{4}\left(M + \frac{1}{M}\right)^2} + \frac{y^2}{\tfrac{1}{4}\left(M - \frac{1}{M}\right)^2} - 1 = 0,$$

as it should be.

Starting then from the relation

$$i\eta + \sqrt{(1-\eta^2)} = M^2\{i\xi + \sqrt{(1-\xi^2)}\},$$

and writing

$$\phi\xi = \cos\frac{\pi}{2\log M}\log\{i\xi + \sqrt{(1-\xi^2)}\},$$

we have

$$\phi\eta = \cos\frac{\pi}{2\log M}\log M^2\{i\xi + \sqrt{(1-\xi^2)}\},$$

$$= \cos\frac{\pi}{2\log M}[\log M^2 + \log\{i\xi + \sqrt{(1-\xi^2)}\}]$$

$$= \cos\left[\pi + \frac{\pi}{2\log M}\log\{i\xi - \sqrt{(1-\xi^2)}\}\right]$$

$$= -\cos\frac{\pi}{2\log M}\log\{i\xi + \sqrt{(1-\xi^2)}\}, \quad = -\phi\xi,$$

that is, we have

$$\phi\xi + \phi\eta = 0,$$

as the required transformation of the equation of the ellipse

$$\frac{x^2}{\frac{1}{4}\left(M+\frac{1}{M}\right)^2} + \frac{y^2}{\frac{1}{4}\left(M-\frac{1}{M}\right)^2} = 1.$$

We hence derive the known formula for the orthomorphosis of the ellipse into the circle $x_1^2 + y_1^2 - 1 = 0$.