945.

NOTE ON LACUNARY FUNCTIONS.

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THE present note is founded upon Poincaré's paper "Sur les fonctions à espaces lacunaires," *Amer. Math. Jour.*, t. XIV. (1892), pp. 201—221.

If the complex variable $z = x + iy$ is represented as usual by a point the coordinates of which are (x, y), and if U_0, U_1, U_2, ... denote an infinite series of given functions of z, then the equation

$$fz = U_0 + U_1 + U_2 + \dots$$

defines a function of z, but only for those values of z for which the series is convergent, or say for points within a certain region Θ; and within this region, it defines the successive derived functions $f'z, f''z, f'''z, \dots$.

Taking $l, = h + ik$, as an increment of z, we define the function of $z + l$, by the equation

$$f(z + l) = fz + \frac{l}{1} f'z + \frac{l^2}{1 \,.\, 2} f''z + \dots,$$

but only for values of l for which the series is convergent: it may very well be, and it is in general the case, that we thereby extend the definition of fz so as to make it applicable to points within a larger region Θ_1; and then considering fz as defined within this larger region Θ_1, we may pass from it to a still larger region Θ_2; and so on indefinitely, or until we cover the whole infinite plane.

For instance, the equation

$$fz = 1 + z + z^2 + z^3 + \dots$$

defines the function $1 \div (1-z)$ for values of z for which mod. $z<1$, that is, for points within the circle $x^2+y^2-1=0$; and this being so,

$$f(z+l) = \frac{1}{1-z} + \frac{l}{(1-z)^2} + \frac{l^2}{(1-z)^3} + \dots$$

extends the definition to the larger region for which this is a convergent series: the condition of convergency is

$$\text{mod.} \frac{l}{1-z} < 1, \text{ that is, mod.} \frac{h+ik}{1-x-iy} < 1, \text{ or } h^2+k^2 < (1-x)^2+y^2.$$

The condition is that the distance $\sqrt{(h^2+k^2)}$ must not exceed the distance $\sqrt{\{(1-x)^2+y^2\}}$ of the point $z=x+iy$, from the point $(x=1, y=0)$; the point z is strictly within the circle $x^2+y^2=1$, but taking it on the circumference, the condition is that the point $z+l$ must lie within a circle having its centre on the circle $x^2+y^2-1=0$ and passing through the point $(x=1, y=0)$. Taking $\cos\theta$ and $\sin\theta$ for the coordinates of the centre, the equation of this circle is

$$(x-\cos\theta)^2 + (y-\sin\theta)^2 = (1-\cos\theta)^2 + \sin^2\theta,$$

that is,

$$x^2+y^2-1-2\cos\theta\,(x-1) - 2y\sin\theta = 0;$$

and the envelope of these circles is

$$(x^2+y^2-1)^2 - 4(x-1)^2 - 4y^2 = 0;$$

or, as this may be written,

$$(x^2+y^2)^2 - 6(x^2+y^2) + 8x - 3 = 0,$$

or again in the form

$$(x^2+y^2-3)^2 + 4(2x-1) = 0,$$

or in the form

$$y^4 + 2y^2(x^2-3) + (x-1)^2(x+3) = 0.$$

The curve is a cuspidal Cartesian. To put this in evidence, observe that the equation may be written

$$-4y^2 + \{(x-1)^2+y^2\}\{(x-1)^2+y^2+4x-4\} = 0,$$

viz. writing

$$\begin{aligned} A &= x+iy-1, \\ B &= x-iy-1, \\ Z &= -1, \end{aligned}$$

then

$$\begin{aligned} A-B &= 2iy, \\ AB &= (x-1)^2+y^2, \\ A+B &= 2x-2, \end{aligned}$$

or the equation is

$$Z^2(A-B)^2 + AB\{AB - 2Z(A+B)\} = 0,$$

that is,

$$Z^2A^2 + Z^2B^2 + A^2B^2 - 2Z^2AB - 2ZA^2B - 2ZAB^2 = 0,$$

which is the equation of a bicuspidal quartic curve, having for cusps the vertices of the triangle $A = 0$, $B = 0$, $Z = 0$.

The region within which the function is $= \frac{1}{1-z}$ is thus extended to the area within the Cartesian curve, say this is the region Θ_1: starting from this curve instead of the circle (viz. by considering the envelope of the circle having its centre on the curve and passing through the point $x = 1$, $y = 0$), we obtain a second curve, a closed curve, which instead of having a cusp on the axis of x cuts this axis at right angles at a point the distance of which from the origin is greater than 1; and we thus extend the region to the area within this second curve, say this is the region Θ_2. And proceeding in this way, we ultimately extend the region to the whole of the infinite plane.

But the functions U_0, U_1, U_2, ... may be such that for every value whatever of l, for which the point $z + l$ is outside the region Θ, the series

$$fz + \frac{l}{1} f'z + \frac{l^2}{1 \cdot 2} f''z + \dots$$

is divergent, and we are in this case unable to define the function fz for points outside the region Θ_1: the function then exists only for points inside the region Θ, and for points outside this region it is non-existent; a function such as this, existing only for points within a certain region and not for the whole of the infinite plane, is said to be a *lacunary* function.