944.

ON PFAFF-INVARIANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXVI. (1893), pp. 195—205.]

1. THE functions which I propose to call Pfaff-invariants present themselves and play a leading part in the memoir, Clebsch, "Ueber das *Pfaff*sche Problem" (Zweite Abhandlung), *Crelle*, t. LXI. (1863), pp. 146—179: but it is interesting to consider them for their own sake as invariants, and in the notation which I have elsewhere used for the functions called Pfaffians. The great simplification effected by this notation is, I think, at once shown by the remark that Clebsch's expression R, which he defines by the periphrasis "Sei ferner R der rationale Ausdruck dessen Quadrat der Determinant der a_{ik} gleich ist" (*l.c.* p. 149), is nothing else than the Pfaffian $1234 \ldots 2n-1 \,.\, 2n$, and that its differential coefficients $R_{ik} = \frac{\partial R}{\partial a_{ik}}$ are the Pfaffians obtained from the foregoing by the mere omission of any two symbolic numbers i, k.

2. I call to mind that the symbols 12, 13, &c., made use of are throughout such that $12 = -21$, &c.; and that the definition of the successive Pfaffians 12, 1234, &c., is as follows:

$$12 \quad = 12,$$

$$1234 \quad = 12 \,.\, 34 \quad + 13 \,.\, 42 \quad + 14 \,.\, 23,$$

$$123456 = 12 \,.\, 3456 + 13 \,.\, 4562 + 14 \,.\, 5623 + 15 \,.\, 6234 + 16 \,.\, 2345,$$

in which last expression 3456 denotes the Pfaffian $34 \,.\, 56 + 35 \,.\, 64 + 36 \,.\, 45$, and similarly 4562, &c.; and so on for any even number of symbols. Of course, instead of the symbolic numbers 1, 2, 3, &c., we may have any other numbers (0 is frequently used in the sequel as a symbolic number), or we may have letters or other symbols.

3. I use also a function very analogous to a Pfaffian, which is expressed in the same notation, viz. this is

$$\begin{aligned}\phi\psi 12 &= \phi\psi 12,\\ \phi\psi 1234 &= \phi\psi 12\,.\,34 + \phi\psi 13\,.\,42 + \phi\psi 14\,.\,23\\ &\quad + \phi\psi 34\,.\,12 + \phi\psi 42\,.\,13 + \phi\psi 23\,.\,14,\\ \phi\psi 123456 &= \phi\psi 12\,.\,34\,.\,56 + \phi\psi 34\,.\,12\,.\,56 + \phi\psi 56\,.\,12\,.\,34 + \text{\&c.},\end{aligned}$$

viz. taking any term 12 . 34 . 56 of the Pfaffian 123456, $\phi\psi$ is connected successively with each of the binary symbols 12, 34, 56 of the term, so as to give rise to terms containing the quaternary symbols $\phi\psi 12$, &c. Such function may be called a co-Pfaffian.

4. To avoid suffixes I use different letters (x, y), (x, y, z), &c., as the case may be, associating these with the numbers (1, 2), (1, 2, 3), &c. In the case of a differential of an even number $2n$ of terms, for instance $Xdx + Ydy + Zdz + Wdw$, I consider the functions 1234, $\phi 01234$, and $\phi\psi 1234$, the first and second of which are Pfaffians, the last a co-Pfaffian, as explained above. To fix in connexion with the differential $Xdx + Ydy + Zdz + Wdw$ the meanings of these expressions, I assume

$$12 = \frac{dX}{dy} - \frac{dY}{dx}, \quad 13 = \frac{dX}{dz} - \frac{dZ}{dx}, \text{ \&c.},$$

(of course these imply $12 = -21$, &c.),

$$01 = -10 = X, \quad 02 = -20 = Y, \text{ \&c.};$$

ϕ is an arbitrary function of x, y, z, w, and I write

$$\phi 0 = -0\phi = 0, \quad \phi 1 = -1\phi = \frac{d\phi}{dx}, \quad \phi 2 = -2\phi = \frac{d\phi}{dy}, \text{ \&c.};$$

ψ is also an arbitrary function of x, y, z, w, and I write

$$\phi\psi 12 = \frac{d\phi}{dx}\frac{d\psi}{dy} - \frac{d\phi}{dy}\frac{d\psi}{dx}, \quad = \frac{\partial(\phi, \psi)}{\partial(x, y)}, \text{ \&c.}$$

(this implies $\phi\psi 21 = -\phi\psi 12$, &c.).

5. Thus, at full length, the functions are

$$\begin{aligned}1234 &= 12\,.\,34 + 13\,.\,42 + 14\,.\,23\\ &= \left(\frac{dX}{dy} - \frac{dY}{dx}\right)\left(\frac{dZ}{dw} - \frac{dW}{dz}\right) + \left(\frac{dX}{dz} - \frac{dZ}{dx}\right)\left(\frac{dW}{dy} - \frac{dY}{dw}\right) + \left(\frac{dX}{dw} - \frac{dW}{dx}\right)\left(\frac{dY}{dz} - \frac{dZ}{dy}\right),\end{aligned}$$

$$\begin{aligned}\phi 01234 &= \phi 0\,.\,1234 + \phi 1\,.\,2340 + \phi 2\,.\,3401 + \phi 3\,.\,4012 + \phi 4\,.\,0123\\ &= \frac{d\phi}{dx}(23\,.\,40 + 24\,.\,03 + 20\,.\,34)\\ &\quad + \frac{d\phi}{dy}(34\,.\,01 + 30\,.\,14 + 31\,.\,40)\\ &\quad + \frac{d\phi}{dz}(40\,.\,12 + 41\,.\,20 + 42\,.\,01)\\ &\quad + \frac{d\phi}{dw}(01\,.\,23 + 02\,.\,31 + 03\,.\,12)\end{aligned}$$

$$= \frac{d\phi}{dx}\left\{-W\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Z\left(\frac{dY}{dw}-\frac{dW}{dy}\right)-Y\left(\frac{dZ}{dw}-\frac{dW}{dz}\right)\right\}$$

$$+\frac{d\phi}{dy}\left\{\ X\left(\frac{dZ}{dw}-\frac{dW}{dz}\right)-Z\left(\frac{dX}{dw}-\frac{dW}{dx}\right)-W\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)\right\}$$

$$+\frac{d\phi}{dz}\left\{-W\left(\frac{dX}{dy}-\frac{dY}{dx}\right)-Y\left(\frac{dW}{dx}-\frac{dX}{dw}\right)+X\left(\frac{dW}{dy}-\frac{dY}{dw}\right)\right\}$$

$$+\frac{d\phi}{dw}\left\{\ X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Y\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right)\right\},$$

$$\phi\psi 1234 = \phi\psi 12\,.\,34+\phi\psi 13\,.\,24+\phi\psi 14\,.\,23$$

$$+\phi\psi 34\,.\,12+\phi\psi 24\,.\,13+\phi\psi 23\,.\,14$$

$$= \frac{\partial(\phi,\ \psi)}{\partial(x,\ y)}\left(\frac{dZ}{dw}-\frac{dW}{dz}\right)+\frac{\partial(\phi,\ \psi)}{\partial(x,\ z)}\left(\frac{dY}{dw}-\frac{dW}{dy}\right)+\frac{\partial(\phi,\ \psi)}{\partial(x,\ w)}\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)$$

$$+\frac{\partial(\phi,\ \psi)}{\partial(z,\ w)}\left(\frac{dX}{dy}-\frac{dY}{dx}\right)+\frac{\partial(\phi,\ \psi)}{\partial(y,\ w)}\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+\frac{\partial(\phi,\ \psi)}{\partial(y,\ z)}\left(\frac{dX}{dw}-\frac{dW}{dx}\right).$$

6. The invariantive property of the functions consists herein, viz. if we have

$$Xdx+Ydy+Zdz+Wdw=Pdp+Qdq+Rdr+Sds,$$

so that p, q, r, s, and thence also P, Q, R, S are functions each of them of x, y, z, w, then we have

$$1234\,\partial(x,\ y,\ z,\ w) = (1234)'\,\partial(p,\ q,\ r,\ s),$$

$$\phi 01234\,\partial(x,\ y,\ z,\ w) = (\phi 01234)'\,\partial(p,\ q,\ r,\ s),$$

$$\phi\psi 1234\,\partial(x,\ y,\ z,\ w) = (\phi\psi 1234)'\,\partial(p,\ q,\ r,\ s),$$

where the accented functions refer to (p, q, r, s, P, Q, R, S), and where for greater symmetry I have separated the symbolical numerator and denominator $\partial(p, q, r, s)$ and $\partial(x, y, z, w)$; each of these equations really contains

$$\frac{\partial(p,\ q,\ r,\ s)}{\partial(x,\ y,\ z,\ w)},$$

which is the functional determinant of (p, q, r, s) in regard to (x, y, z, w): or, if we please, it contains the reciprocal hereof

$$\frac{\partial(x,\ y,\ z,\ w)}{\partial(p,\ q,\ r,\ s)},$$

which is the functional determinant of (x, y, z, w) in regard to (p, q, r, s).

7. The equations give

$$\frac{\phi 01234}{1234}=\frac{(\phi 01234)'}{(1234)'},$$

$$\frac{\phi\psi 1234}{1234}=\frac{(\phi\psi 1234)'}{(1234)'},$$

and then the expressions on the left-hand are absolute invariants in respect to the transformation of

$$Xdx + Ydy + Zdz + Wdw \quad \text{into} \quad Pdp + Qdq + Rdr + Sds.$$

They are, in fact, (for $2n = 4$) Clebsch's derivatives (ϕ) and (ϕ, ψ).

8. For the Pfaffian reduction

$$Xdx + Ydy + Zdz + Wdw = Fdf + Gdg,$$

we may write

$$P, Q, R, S = F, G, 0, 0,$$

$$p, q, r, s = f, g, F, G,$$

viz. we take f, g, F, G as the new independent variables; we thus have

$$01' = F, \quad 02' = G, \quad 03' = 0, \quad 04' = 0,$$

$$12' = 0, \quad 13' = 1, \quad 14' = 0, \quad 23' = 0, \quad 24' = 1, \quad 34' = 0,$$

$$(1234)' = 12' \,.\, 34' + 13' \,.\, 42' + 14' \,.\, 23', \ = -1;$$

and similarly

$$(\phi 01234)' = -\left\{F\frac{d\phi}{dF} + G\frac{d\phi}{dG}\right\},$$

$$(\phi\psi 1234)' = -\left\{\frac{\partial(\phi, \psi)}{\partial(f, F)} + \frac{\partial(\phi, \psi)}{\partial(g, G)}\right\},$$

where, in the equations, the $-$ sign presents itself by reason that $2n$, $= 4$, is the double of an even number, or say that n is even; in the case of $2n$, the double of an odd number, that is, n odd, the sign would have been $+$.

9. We thus have

$$(\phi) = \frac{\phi 01234}{1234} = F\frac{d\phi}{dF} + G\frac{d\phi}{dG},$$

$$(\phi\psi) = \frac{\phi\psi 1234}{1234} = \frac{\partial(\phi, \psi)}{\partial(f, F)} + \frac{\partial(\phi, \psi)}{\partial(g, G)};$$

and in particular, by giving to ϕ and ψ the values f, g, F, G, we find

$$(f) = 0, \quad (g) = 0, \quad (F) = F, \quad (G) = G,$$

$$(f, g) = 0, \quad (f, F) = 1, \quad (f, G) = 0,$$

$$(F, G) = 0, \quad (g, F) = 0, \quad (g, G) = 1,$$

which are Clebsch's equations; in the case of $2n$ terms, the number of them is

$$n + n + \tfrac{1}{2}(n^2 - n) + \tfrac{1}{2}(n^2 - n) + n^2 = 2n + n^2 - n + n^2,$$

$= n(2n + 1)$, or $\tfrac{1}{2}2n(2n + 1)$, as it should be.

10. It may be remarked that we have

$$Fdf + Gdg = Fd\left(f + \frac{G}{F}g\right) - Fgd\frac{G}{F},$$

or writing this $=F'df'+G'dg'$, we have

$$F'=F,\quad f'=f+\frac{Gg}{F},\quad G'=+Fg,\quad g'=\frac{G}{F},$$

whence conversely

$$F=F',\quad f=f'+\frac{G'g'}{F'},\quad G=F'g',\quad g=-\frac{G'}{F'},\quad (Gg=-G'g').$$

The ten equations $(f)=0$, $(g)=0$, &c., ought then to lead to the corresponding ten equations $(f')=0$, $(g')=0$, &c., and it is easy to verify that they do so; for instance, we have

$$(f')=\left(f+\frac{Gg}{F}\right)=(f)+\frac{G}{F}(g)+g\left(\frac{G}{F}\right),$$

where

$$\left(\frac{G}{F}\right)=\frac{1}{F}(G)-\frac{G}{F^2}(F),\ =\frac{G}{F}-\frac{G}{F^2}F,\ =0,$$

and thus $(f')=0$. And again,

$$(f',\ g')=\left(f+\frac{Gg}{F},\ \frac{G}{F}\right)=\left(f,\ \frac{G}{F}\right)+\left(\frac{Gg}{F},\ \frac{G}{F}\right)=\left(f,\ \frac{G}{F}\right)+\frac{G}{F}\left(g,\ \frac{G}{F}\right)+g\left(\frac{G}{F},\ \frac{G}{F}\right),$$

where the last term vanishes; the remaining terms are

$$=\frac{1}{F}(f,\ G)-\frac{G}{F^2}(f,\ F)+\frac{G}{F^2}(g,\ G)-\frac{G}{F^2}(g,\ F),$$

$$=0-\frac{G}{F^2}+\frac{G}{F^2}-0,\ \text{ that is, }\ (f',\ g')=0.$$

There is, of course, the like transformation

$$Fdf+Gdg=G\left(dg+\frac{F}{G}f\right)-Gfd\frac{F}{G}.$$

11. I have, for better exhibiting the results, taken $2n=4$, but the most simple case for an even number of terms is $2n=2$. Here we have $Xdx+Ydy=Pdp+Qdq$, and the functions to be considered are

$$12,\ =12 \qquad = \frac{dX}{dy}-\frac{dY}{dx},$$

$$\phi 012,\ =\phi 0\,.\,12+\phi 1\,.\,20+\phi 2\,.\,01=-Y\frac{d\phi}{dx}+X\frac{d\phi}{dy},$$

$$\phi\psi 12,\ =\phi\psi 12 \qquad = \frac{\partial(\phi,\ \psi)}{\partial(x,\ y)}.$$

We have here

$$X=P\frac{dp}{dx}+Q\frac{dq}{dx},\quad Y=P\frac{dp}{dy}+Q\frac{dq}{dy},$$

and the invariantive properties are easily verified.

12. Thus

$$12=\frac{dX}{dy}-\frac{dY}{dx},\ =\left(\frac{dP}{dy}\frac{dp}{dx}-\frac{dP}{dx}\frac{dp}{dy}\right)+\left(\frac{dQ}{dy}\frac{dq}{dx}-\frac{dQ}{dx}\frac{dq}{dy}\right);$$

or, writing herein

$$\frac{dP}{dx}=\frac{dP}{dp}\frac{dp}{dx}+\frac{dP}{dq}\frac{dq}{dx},$$

and the like values for $\frac{dP}{dy}$ and for $\frac{dQ}{dx}$ and $\frac{dQ}{dy}$, we have

$$12=\left(\frac{dP}{dq}-\frac{dQ}{dp}\right)\left(\frac{dp}{dx}\frac{dq}{dy}-\frac{dp}{dy}\frac{dq}{dx}\right)=(12)'\frac{\partial(p,\ q)}{\partial(x,\ y)}.$$

Similarly, we find

$$\phi 012=-Y\frac{d\phi}{dx}+X\frac{d\phi}{dy}=\left(-Q\frac{d\phi}{dp}+P\frac{d\phi}{dq}\right)\left(\frac{dp}{dx}\frac{dq}{dy}-\frac{dp}{dy}\frac{dq}{dx}\right)=(\phi 012)'\frac{\partial(p,\ q)}{\partial(x,\ y)},$$

and

$$\phi\psi 12=\frac{d\phi}{dx}\frac{d\psi}{dy}-\frac{d\phi}{dy}\frac{d\psi}{dx}=\left(\frac{d\phi}{dp}\frac{d\psi}{dq}-\frac{d\phi}{dq}\frac{d\psi}{dp}\right)\left(\frac{dp}{dx}\frac{dq}{dy}-\frac{dp}{dy}\frac{dq}{dx}\right)=(\phi\psi 12)'\frac{\partial(p,\ q)}{\partial(x,\ y)}.$$

We thus have

$$\begin{aligned}12\,\partial(x,\ y)&=(12)'\,\partial(p,\ q),\\ \phi 012\,\partial(x,\ y)&=(\phi 012)'\,\partial(p,\ q),\\ \phi\psi 12\,\partial(x,\ y)&=(\phi\psi 12)'\,\partial(p,\ q);\end{aligned}$$

or say

$$\frac{\phi 012}{12}=\frac{(\phi 012)'}{(12)'},\ \text{and}\ \frac{\phi\psi 12}{12}=\frac{(\phi\psi 12)'}{(12)'}.$$

The proof is the same in principle for $2n=4$, or any other even value.

13. The theory is very similar in the case of an odd number $2n+1$ of terms; thus $2n+1=3$, the forms are

$$0123,\ \phi 123,\ \text{and}\ \phi\psi 0123,$$

the first and second of which are Pfaffians, the third of them co-Pfaffian: the developed expression of this last is

$$\begin{aligned}\phi\psi 0123=&\ \phi\psi 01\,.\,23+\phi\psi 02\,.\,31+\phi\psi 03\,.\,12\\ &+\phi\psi 23\,.\,01+\phi\psi 31\,.\,02+\phi\psi 12\,.\,03,\end{aligned}$$

and to fix the meaning hereof we write

$$\phi\psi 01=0,\quad \phi\psi 02=0,\quad \phi\psi 03=0.$$

Hence, the differential expression being $Xdx+Ydy+Zdz$, we have

$$\begin{aligned}0123&=01\,.\,23+02\,.\,31+03\,.\,12\\ &=X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Y\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)-Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right),\\ \phi 123&=\phi 1\,.\,23+\phi 2\,.\,31+\phi 3\,.\,12\\ &=\frac{d\phi}{dx}\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+\frac{d\phi}{dy}\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+\frac{d\phi}{dz}\left(\frac{dX}{dy}-\frac{dY}{dx}\right),\\ \phi\psi 0123&=\phi\psi 23\,.\,01+\phi\psi 31\,.\,02+\phi\psi 12\,.\,03\\ &=X\frac{\partial(\phi,\ \psi)}{\partial(y,\ z)}+Y\frac{\partial(\phi,\ \psi)}{\partial(z,\ x)}+Z\frac{\partial(\phi,\ \psi)}{\partial(x,\ y)}.\end{aligned}$$

14. For the transformation

$$Xdx + Ydy + Zdz = Pdp + Qdq + Rdr,$$

we have

$$0123\,\partial(x,\ y,\ z) = \quad (0123)'\,\partial(p,\ q,\ r),$$

$$\phi 123\,\partial(x,\ y,\ z) = \quad (\phi 123)'\,\partial(p,\ q,\ r),$$

$$\phi\psi 0123\,\partial(x,\ y,\ z) = (\phi\psi 0123)'\,\partial(p,\ q,\ r),$$

and consequently

$$\frac{\phi 123}{0123} = \frac{(\phi 123)'}{(0123)'},$$

$$\frac{\phi\psi 0123}{0123} = \frac{(\phi\psi 0123)'}{(0123)'};$$

so that the left-hand functions are absolute invariants.

15. If in particular, $Xdx + Ydy + Zdz = df + Gdg$, then we may write

$$P,\ Q,\ R = 1,\ G,\ 0,$$

$$p,\ q,\ r = f,\ g,\ G.$$

Hence

$$01' = 1,\quad 02' = G,\quad 03' = 0;\quad 23' = 1,\quad 31' = 0,\quad 12' = 0,$$

and therefore

$$(0123)' = 1,\quad (\phi 123)' = \frac{d\phi}{df},\quad (\phi\psi 0123)' = \frac{\partial(\phi,\ \psi)}{\partial(g,\ G)} + G\frac{\partial(\phi,\ \psi)}{\partial(f,\ G)},$$

or say

$$= \frac{\partial(\phi,\ \psi)}{\partial(g,\ G)} - G\frac{\partial(\phi,\ \psi)}{\partial(f,\ G)};$$

and we thus have

$$0123\,\partial(x,\ y,\ z) = \partial(f,\ g,\ G),$$

$$\phi 123\,\partial(x,\ y,\ z) = \frac{d\phi}{df}\,\partial(f,\ g,\ G),$$

$$\phi\psi 0123\,\partial(x,\ y,\ z) = \left\{\frac{\partial(\phi,\ \psi)}{\partial(g,\ G)} - G\frac{\partial(\phi,\ \psi)}{\partial(f,\ G)}\right\}\partial(f,\ g,\ G),$$

and then

$$(\phi) = \frac{\phi 123}{0123} = \frac{d\phi}{df},$$

$$(\phi,\ \psi) = \frac{\phi\psi 0123}{0123} = \frac{\partial(\phi,\ \psi)}{\partial(g,\ G)} - G\frac{\partial(\phi,\ \psi)}{\partial(f,\ G)};$$

viz. we thus have derivatives (ϕ) and $(\phi,\ \psi)$ analogous to (but quite different in form from) those of Clebsch in the case of an even number of terms.

In particular, writing $\phi, \psi = f,\ g,\ G$, we obtain

$$(f) = 1,\quad (g) = 0,\quad (G) = 0;\quad (f,\ g) = 0,\quad (f,\ G) = -G,\quad (g,\ G) = 1,$$

which are the analogues of Clebsch's formula.

16. It is interesting to compare the formula for the two cases

$$Xdx + Ydy + Zdz + Wdw = Fdf + Gdg,$$

and

$$Xdx + Ydy + Zdz = df + Gdg.$$

In the former case f and g are symmetrically related to each other, and we may say that ($f=$ const. and $g=$ const.) is a solution of $Xdx + Ydy + Zdz + Wdw = 0$; we have $(f)=0$ and $(g)=0$. In the second case ($f=$ const. and $g=$ const.) is still a solution of $Xdx + Ydy + Zdz = 0$, but f and g are not symmetrically related to each other, and we have $(f)=1$, $(g)=0$. Moreover, in the first case $(G)=G$, but in the second case $(G)=0$, an equation of the same form as $(g)=0$; the reason is that we have here

$$Xdx + Ydy + Zdz = df + Gdg, \quad = d(f+Gg) - gdG,$$

so that, besides the solution ($f=$ const. and $g=$ const.), we have the solution

$$(f + Gg = \text{const. and } G = \text{const.}).$$

17. The remark just made may be further developed: we have

$$Xdx + Ydy + Zdz = df + Gdg, \quad = d(f+Gg) - gdG, \quad = df' + G'dg',$$

suppose, where $f' = f + Gg$, $G' = -g$, $g' = G$, and therefore also $f = f' + G'g'$, $G = g'$, $g = -G'$; the equations

$$(f)=1, \quad (g)=0, \quad (G)=0, \quad (f,\, g)=0, \quad (f,\, G)=-G, \quad (g,\, G)=1,$$

should lead to

$$(f')=1, \quad (g')=0, \quad (G')=0, \quad (f',\, g')=0, \quad (f',\, G')=-G', \quad (g',\, G')=1.$$

There is no difficulty in verifying this; thus the equations $(g)=0$, $(G)=0$, give at once $(g')=0$, $(G')=0$; and then the equation $(f)=1$ gives $(f'+G'g')=1$, that is,

$$(f') + G'(g') + g'(G') = 1, \text{ or } (f') = 1.$$

So again $(g,\, G)=1$ gives $(g',\, G')=1$; and then $(f,\, g)=0$ gives $(f'+G'g',\, G')=0$, that is,

$$(f',\, G') + G'(g',\, G') + g'(G',\, G') = 0, \text{ or } (f',\, G') = -G'.$$

And finally, $(f,\, G)=-G$ gives $(f'+G'g',\, g')+g'=0$, that is,

$$(f',\, g') + G'(g',\, g') + g'(G',\, g') + g' = 0, \text{ or } (f',\, g') = 0.$$

I stop to give the direct verification of the equations $(f)=1$, $(g)=0$, $(G)=0$. We have

$$Xdx + Ydy + Zdz = df + Gdg,$$

that is,

$$X = \frac{df}{dx} + G\frac{dg}{dx}, \quad Y = \frac{df}{dy} + G\frac{dg}{dy}, \quad Z = \frac{df}{dz} + G\frac{dg}{dz},$$

and thence

$$23 = \frac{dY}{dz} - \frac{dZ}{dy} = \frac{dG}{dz}\frac{dg}{dy} - \frac{dG}{dy}\frac{dg}{dz},$$

$$31 = \frac{dZ}{dx} - \frac{dX}{dz} = \frac{dG}{dx}\frac{dg}{dz} - \frac{dG}{dz}\frac{dg}{dx},$$

$$12 = \frac{dX}{dy} - \frac{dY}{dx} = \frac{dG}{dy}\frac{dg}{dx} - \frac{dG}{dx}\frac{dg}{dy}.$$

Hence, multiplying first by $\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}$, that is,

$$X - G\frac{dg}{dx}, \quad Y - G\frac{dg}{dy}, \quad Z - G\frac{dg}{dz},$$

and adding, we have

$$23\frac{df}{dx} + 31\frac{df}{dy} + 12\frac{df}{dz} = X23 + Y31 + Z12,$$

that is, $f123 = 0123$, or $(f) = 1$.

And then multiplying secondly by $\frac{dg}{dx}, \frac{dg}{dy}, \frac{dg}{dz}$ and adding, and thirdly by $\frac{dG}{dx}, \frac{dG}{dy}, \frac{dG}{dz}$ and adding, we obtain

$$23\frac{dg}{dx} + 31\frac{dg}{dy} + 12\frac{dg}{dz} = 0, \text{ that is, } (g) = 0,$$

and

$$23\frac{dG}{dx} + 31\frac{dG}{dy} + 12\frac{dG}{dz} = 0, \text{ that is, } (G) = 0.$$

To exhibit more clearly the formulæ for any odd number of terms, I take $2n + 1 = 5$,

$$Xdx + Ydy + Zdz + Wdw + Tdt = df + Gdg + Hdh.$$

We have here

$$(\phi) = \frac{\phi 12345}{012345} = \frac{d\phi}{df},$$

$$(\phi\psi) = \frac{\phi\psi 012345}{012345} + \frac{\partial(\phi, \psi)}{\partial(g, G)} + \frac{\partial(\phi, \psi)}{\partial(h, H)} - G\frac{\partial(\phi, \psi)}{\partial(f, G)} - H\frac{\partial(\phi, \psi)}{\partial(f, H)};$$

and in particular,

$$(f) = 1; \quad (g) = 0, \quad (h) = 0; \quad (G) = 0, \quad (H) = 0;$$

$$(f, g) = 0; \quad (f, h) = 0; \quad (f, G) = -G, \quad (f, H) = -H;$$

$$(g, h) = 0; \quad (G, H) = 0; \quad (g, G) = 1, \quad (g, H) = 0;$$

$$(h, G) = 0, \quad (h, H) = 1;$$

in all

$$1 + 2n + 2n + \tfrac{1}{2}(n^2 - n) + \tfrac{1}{2}(n^2 - n) + n^2,$$

$$= 1 + 4n + n^2 - n + n^2, \quad = 2n^2 + 3n + 1, \quad = \tfrac{1}{2}(2n+1)(2n+2)$$

equations.

We can, by what precedes, at once express the conditions which must be satisfied in order that a differential expression $X_1dx_1 + X_2dx_2 + \ldots + X_\nu dx_\nu$, may be reducible to one of the special forms df, Fdf, $df + F_1df_1$, &c.; viz. if we have

$$\begin{aligned} X_1dx_1 + X_2dx_2 + \ldots + X_\nu dx_\nu &= df, && \text{then } 12 = 0, \text{ \&c.} \\ &= Fdf, && 0123 = 0, \text{ \&c.} \\ &= df + F_1df_1, && 1234 = 0, \text{ \&c.} \\ &= Fdf + F_1df_1, && 012345 = 0, \text{ \&c.,} \\ &\text{\&c.,} && \text{\&c.,} \end{aligned}$$

where the numbers 12, 1234, 12345, &c., represent any combinations out of the numbers 1, 2, 3, ..., ν. Of course, if ν is not sufficiently large to furnish such a combination, then there is no condition to be satisfied; thus if

$$X_1dx_1 + X_2dx_2 + X_3dx_3 = df + F_1df_1,$$

there is no condition to be satisfied.