764.

THE BINOMIAL EQUATION $x^p - 1 = 0$: QUINQUISECTION.

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THE theory should be precisely analogous to those for the trisection and quartisection (see my paper, "The Binomial Equation $x^p - 1 = 0$, Trisection and Quartisection," *Proceedings of the London Mathematical Society*, vol. XI. (1879), pp. 4—17, [731]), only I have not been able to carry it so far. We have in the present case five periods X, Y, Z, W, T, the actual expressions for which, $X = \eta^1 + \ldots$, $Y = \eta^3 + \ldots$, etc., with Reuschle's selected prime root g, can be (for the primes $5n+1$ under 100) at once written down by means of the table given, pp. 16, 17, of that paper; [see this volume, pp. 95, 96]. The relations between the periods are of the form

	X	Y	Z	W	T
$X^2 =$	a	b	c	d	e
$XY =$	f	g	h	i	j
$XZ =$	k	l	m	n	o;

that is, we have

$$X^2 = (a, b, c, d, e \between X, Y, Z, W, T),$$

and thence, by cyclical permutations,

$$Y^2 = (e, a, b, c, d \between \quad\quad ,, \quad\quad), \text{ etc.};$$

viz. from the value of X^2 we have those of Y^2, Z^2, W^2, T^2; from the value of XY those of YZ, ZW, WT, TX; and from the value of XZ those of YW, ZT, WX, TY.

From the equation $X + Y + Z + W + T = -1$, multiplying by X and then substituting for X^2, XY, &c., their values, we obtain

$$\begin{aligned} -a &= 1 + f + k + m + g, \\ -b &= g + l + n + h, \\ -c &= h + m + o + i, \\ -d &= i + n + k + j, \\ -e &= j + o + l + f, \end{aligned}$$

which determine (a, b, c, d, e) in terms of (f, g, h, i, j) and (k, l, m, n, o). It is, moreover, easy to prove that

$$\begin{aligned} f + g + h + i + j &= \tfrac{1}{5}(p-1), \\ k + l + m + n + o &= \tfrac{1}{5}(p-1), \end{aligned}$$

whence also

$$a + b + c + d + e = -1 - \tfrac{4}{5}(p-1).$$

We obtain other relations between the coefficients by considering the two triple products XYZ and XYW: these are all that need be considered, since the other triple products are deducible from them by cyclical permutations. From the first of these we have

$$X . YZ = Y . XZ = Z . XY,$$

and from the second

$$X . YW = Y . XW = W . XY;$$

and if we herein substitute for YZ, XZ, &c., their values, and then in the resulting equations for X^2, XY, &c., their values as linear functions of X, Y, Z, W, T, we obtain in all $5 . 2 . 2 = 20$ quadric relations between the 15 coefficients; or if we substitute for (a, b, c, d, e) their foregoing values, in all 20 relations between the 10 coefficients (f, g, h, i, j) and (k, l, m, n, o). These are at most equivalent to 8 independent equations, since we have, besides, the sums $f+g+h+i+j$ and $k+l+m+n+o$ each $= \frac{1}{5}(p-1)$; but I have not succeeded in finding the connexions between them, or even in ascertaining to how many independent equations they are equivalent.

For any given prime $p = 5n + 1$, the values of the coefficients, and also the coefficients of the quintic equation for the periods, could of course be calculated directly from the expressions of the periods; but for the primes under 100, that is, for the values 11, 31, 41, 61, 71, they are at once obtained from Reuschle. We have thus the two Tables, the former giving the coefficients $a, b, \ldots, n, o$, and the latter the coefficients of the quintic equations.

TABLE 1.

p	a / f / k	b / g / l	c / h / m	d / i / n	e / j / o
11	-2	-1	-2	-2	-2
	1	0	0	1	0
	0	0	0	1	1
31	-4	-6	-6	-4	-5
	0	1	2	1	2
	0	2	2	1	1
41	-8	-5	-6	-6	-8
	3	0	2	1	2
	2	2	2	1	1
61	-10	-9	-12	-8	-10
	3	2	2	3	2
	0	2	4	3	3
71	-14	-10	-12	-9	-12
	4	2	3	2	3
	2	3	5	2	2

TABLE 2 OF THE QUINTIC EQUATIONS.

COEFFICIENTS OF

p	η^5	η^4	η^3	η^2	η^1	1	$=0.$
11	1	1	-4	-3	$+3$	$+1$	
31	1	1	-12	-2	$+1$	$+5$	
41	1	1	-16	$+5$	$+21$	-9	
61	1	1	-24	-17	$+41$	-23	
71	1	1	-28	$+37$	$+25$	$+1$	