760.

REDUCTION OF $\int \frac{dx}{(1-x^3)^{\frac{2}{3}}}$ TO ELLIPTIC INTEGRALS.

[From the *Messenger of Mathematics*, vol. XI. (1882), pp. 142, 143.]

WRITING s, c, d for the sn, cn, and dn of u to a modulus k, which will be determined, and denoting by θ a constant which will also be determined, the formula of reduction is

$$x = \frac{-1 + \theta scd}{1 + \theta scd}.$$

To find from this the value of y, $= \sqrt[3]{(1-x^3)}$, putting for shortness $X = \theta scd$, the formula is $x = \frac{-1+X}{1+X}$, and we thence have

$$y^3, = 1 - x^3, = \frac{2(1+3X^2)}{(1+X)^3},$$

where

$$1 + 3X^2 = 1 + 3\theta^2 s^2 (1-s^2)(1-k^2 s^2),$$

$$= 1 + 3\theta^2 s^2 - 3\theta^2 (1+k^2) s^4 + 3\theta^2 k^2 s^6,$$

may be put equal to $(1+\theta^2 s^2)^3$, that is,

$$= 1 + 3\theta^2 s^2 + 3\theta^4 s^4 \qquad + \theta^6 s^6;$$

viz. this will be the case if

$$3\theta^4 = -3\theta^2(1+k^2), \quad \theta^6 = 3\theta^2 k^2;$$

that is,

$$\theta^2 = -1 - k^2, \qquad \theta^4 = 3k^2;$$

these give

$$k^4 - k^2 + 1 = 0;$$

that is, $k^2 = \omega$, if $\omega = -\frac{1}{2} + \frac{1}{2} i\sqrt{3}$, an imaginary cube root of unity; and then

$$\theta^2 = -1 + \omega, \quad = \omega^2(\omega^2 - \omega), \quad = -i\omega^2\sqrt{3};$$

that is,

$$\theta = \pm \frac{(1-\sqrt{3}) - i(1+\sqrt{3})}{2\sqrt{2}} \sqrt[4]{3},$$

as may be verified by squaring.

Hence finally, θ and k denoting the values just obtained,

$$x = \frac{-1+\theta scd}{1+\theta scd},$$

$$y = \sqrt[3]{(1-x^3)} = \frac{\sqrt[3]{2}\,(1+\theta^2 s^2)}{1+\theta scd};$$

or, writing as before, $X = \theta scd$, we have

$$dx = \frac{2dX}{(1+X)^2}, \quad y^2 = \frac{2^{\frac{2}{3}}(1+\theta^2 s^2)^2}{(1+X)^2};$$

whence

$$\frac{dx}{(1-x^3)^{\frac{2}{3}}}, \; = \frac{dx}{y^2}, \; = \frac{2^{\frac{1}{3}} dX}{(1+\theta^2 s^2)^2},$$

and then

$$dX = \theta\{1 - 2(1+k^2)s^2 + 3k^2 s^4\}\,du, \quad = \theta(1+\theta^2 s^2)^2\,du;$$

that is,

$$\frac{dx}{(1-x^3)^{\frac{2}{3}}} = 2^{\frac{1}{3}}\theta\,.\,du;$$

or say

$$\int \frac{dx}{(1-x^3)^{\frac{2}{3}}} = 2^{\frac{1}{3}}\theta u,$$

the required formula.