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A GEOMETRICAL CONSTRUCTION RELATING TO IMAGINARY QUANTITIES.

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LET A, B, C be given imaginary quantities, and let it be required to construct the roots of the quadric equation

$$\frac{1}{X-A}+\frac{1}{X-B}+\frac{1}{X-C}=0.$$

The equation is

$$(X-B)(X-C)+(X-C)(X-A)+(X-A)(X-B)=0,$$

that is,

$$3X^2-2(A+B+C)X+BC+CA+AB=0,$$

and we have therefore

$$3X-(A+B+C)=\pm\sqrt{\{(A+B+C)^2-3(BC+CA+AB)\}},$$
$$=\pm\sqrt{\{A^2+B^2+C^2-BC-CA-AB\}};$$

or as this may be written

$$X=\tfrac{1}{3}(A+B+C)\pm\sqrt{\{\tfrac{1}{3}(A+B\omega+C\omega^2)\,.\,\tfrac{1}{3}(A+B\omega^2+C\omega)\}},$$

where ω is an imaginary cube root of unity,

$$=\cos 120^\circ+i\sin 120^\circ \text{ suppose.}$$

Taking an arbitrary point O as the origin, let the imaginary quantity A, $=\alpha+\alpha' i$ suppose, be represented by the point A, coordinates α and α'; and in like manner the imaginary quantities B and C by the points B and C respectively.

Then $B\omega$, $B\omega^2$ are represented by points B_1, B_2, obtained by rotating the point B about the origin through angles of 120° and 240° respectively; $C\omega^2$, $C\omega$ are represented by points C_1, C_2 obtained by rotating the point C about the origin through angles of 240° and 480° (=120°) respectively: and

$$\tfrac{1}{3}(A+B+C),\quad \tfrac{1}{3}(A+B\omega+C\omega^2),\quad \tfrac{1}{3}(A+B\omega^2+C\omega)$$

are represented by the points G, G_1, G_2 which are the C.G.'s of the triangles ABC, AB_1C_1, AB_2C_2 respectively. The formula therefore is

$$X = OG \pm \sqrt{(OG_1 \,.\, OG_2)},$$

where, if a, a′ are the coordinates of G, then OG is written to denote the imaginary quantity $\mathrm{a}+\mathrm{a}'i$; and the like as regards OG_1, OG_2. Taking $\sqrt{(OG_1 \,.\, OG_2)} = OH$, we then have H a point such, that the distance OH from the origin is = geometric mean of the distances OG_1, OG_2, and that the radial direction* of the distance OH bisects the radial directions of the distances OG_1, OG_2 respectively. Finally, measuring off from G in the radial direction OH, and in the opposite radial direction, the distances GX', GX'' each $= OH$; we have the two points X', X'' representing the two roots X.

The construction is somewhat simplified if we take for the origin the point G; for then $OG=0$, and we have $X = \pm\sqrt{(GG_1 \,.\, GG_2)}$, so that the points X', X'' are in fact the point H, and the opposite point in regard to G.

The theory of the more general equation

$$\frac{p}{X-A}+\frac{q}{X-B}+\frac{r}{X-C}=0,$$

(p, q, r real) is somewhat similar, but the construction is less simple; we have

$$(p+q+r)X^2 - \{(q+r)A+(r+p)B+(p+q)C\}X + pBC + qCA + rAB = 0.$$

Writing herein $q+r$, $r+p$, $p+q = l$, m, n, the equation becomes

$$(l+m+n)X^2 - 2(lA+mB+nC)X + (-l+m+n)BC + (l-m+n)CA + (l+m-n)AB = 0,$$

that is,

$$\{(l+m+n)X - lA - mB - nC\}^2$$
$$= (lA+mB+nC)^2 + \{l^2-(m+n)^2\}BC + \{m^2-(n+l)^2\}CA + \{n^2-(l+m)^2\}AB.$$

Here the right-hand side is

$$= l^2A^2 + m^2B^2 + n^2C^2 + (l^2-m^2-n^2)BC + (-l^2+m^2-n^2)CA + (-l^2-m^2+n^2)AB,$$

which is

$$= -l^2(C-A)(A-B) - m^2(A-B)(B-C) - n^2(C-A)(A-B),$$

and consequently is a product of two linear factors; these, in fact, are

$$\frac{1}{l}\{l^2A + \tfrac{1}{2}(-l^2-m^2+n^2 \pm \sqrt{\Delta})B + \tfrac{1}{2}(-l^2+m^2-n^2 \mp \sqrt{\Delta})C\},$$

* Radial direction is, I think, a convenient expression for the direction of a line considered as drawn as a radius of a circle from the centre, and not as a diameter in two opposite radial directions.

where

$$\Delta = l^4 + m^4 + n^4 - 2m^2n^2 - 2n^2l^2 - 2l^2m^2.$$

It is to be observed that $\Delta, = (l^2 - m^2 - n^2)^2 - 4m^2n^2$, is negative; hence, calling the factors $fA + gB + hC$, $f'A + g'B + h'C$ respectively, the coefficients f, g, h, and f', g', h' are imaginary; moreover $f + g + h = 0$, $f' + g' + h' = 0$.

The values of X thus are

$$(l + m + n) X = lA + mB + nC \pm \surd\{(fA + gB + hC)(f'A + g'B + h'C)\},$$

and then passing to the geometrical representation, we have $\dfrac{lA + mB + nC}{l + m + n}$ represented by the point which is the C.G. of weights l, m, n at the points A, B, C respectively; on account of the imaginary values of the coefficients the construction is not immediately applicable to the factors

$$fA + gB + hC, \quad f'A + g'B + h'C;$$

but a construction, such as was used for the factors

$$A + \omega B + \omega^2 C, \quad A + \omega^2 B + \omega C,$$

might be found without difficulty.