748.

ON THE BITANGENTS OF A QUARTIC.

[From *Salmon's Higher Plane Curves,* (3rd *ed.*, 1879), pp. 387—389.]

THE equations of the 28 bitangents of a quartic curve were obtained in a very elegant form by Riemann in the paper "Zur Theorie der Abel'schen Functionen für den Fall $p=3$," *Ges. Werke,* Leipzig, 1876, pp. 456—472; and see also Weber's *Theorie der Abel'schen Functionen vom Geschlecht* 3," Berlin, 1876. Riemann connects the several bitangents with the characteristics of the 28 odd functions, thus obtaining for them an algorithm which it is worth while to explain, but they will be given also with the algorithm employed p. 231 *et seq.* of the present work*, which is in fact the more simple one. The characteristic of a triple θ-function is a symbol of the form

$$\begin{matrix} \alpha\,\beta\,\gamma, \\ \alpha'\beta'\gamma', \end{matrix}$$

where each of the letters is $=0$ or 1; there are thus in all 64 such symbols, but they are considered as odd or even according as the sum $\alpha\alpha'+\beta\beta'+\gamma\gamma'$ is odd or even; and the numbers of the odd and even characteristics are 28 and 36 respectively; and, as already mentioned, the 28 odd characteristics correspond to the 28 bitangents respectively.

We have x, y, z trilinear coordinates, α, β, γ, α', β', γ' constants chosen at pleasure, and then α'', β'', γ'' determinate constants, such that the equations

$$x+\quad y+\quad z+\xi+\eta+\zeta=0,$$

$$\alpha\, x+\beta\, y+\gamma\, z+\frac{\xi}{\alpha}+\frac{\eta}{\beta}+\frac{\zeta}{\gamma}=0,$$

$$\alpha' x+\beta' y+\gamma' z+\frac{\xi}{\alpha'}+\frac{\eta}{\beta'}+\frac{\zeta}{\gamma'}=0,$$

$$\alpha'' x+\beta'' y+\gamma'' z+\frac{\xi}{\alpha''}+\frac{\eta}{\beta''}+\frac{\zeta}{\gamma''}=0,$$

[* That is, Salmon's *Higher Plane Curves.*]

are equivalent to three independent equations; this being so, they determine ξ, η, ζ, each of them as a linear function of (x, y, z); and the equations of the bitangents of the curve $\sqrt{(x\xi)} + \sqrt{(y\eta)} + \sqrt{(z\zeta)} = 0$ (see Weber, p. 100) are

18	111 111	$x = 0,$
28	001 011	$y = 0,$
38	011 001	$z = 0,$
23	010 010	$\xi = 0,$
13	100 110	$\eta = 0,$
12	110 100	$\zeta = 0,$
48	101 100	$x + y + z = 0,$
14	010 011	$\xi + y + z = 0,$
58	100 101	$\alpha x + \beta y + \gamma z = 0,$
15	011 010	$\frac{\xi}{\alpha} + \beta y + \gamma z = 0,$
68	110 010	$\alpha' x + \beta' y + \gamma' z = 0,$
16	001 101	$\frac{\xi}{\alpha'} + \beta' y + \gamma' z = 0,$
78	010 110	$\alpha'' x + \beta'' y + \gamma'' z = 0,$
17	101 001	$\frac{\xi}{\alpha''} + \beta'' y + \gamma'' z = 0,$
24	100 111	$x + \eta + z = 0,$
34	110 101	$x + y + \zeta = 0,$
25	101 110	$\alpha x + \frac{\eta}{\beta} + \gamma z = 0,$
35	111 100	$\alpha x + \beta y + \frac{\zeta}{\gamma} = 0,$

26	111 001	$\alpha' x + \frac{\eta}{\beta'} + \gamma' z = 0,$
36	101 011	$\alpha' x + \beta' y + \frac{\zeta}{\gamma'} = 0,$
27	011 101	$\alpha'' x + \frac{\eta}{\beta''} + \gamma'' z = 0,$
37	001 111	$\alpha'' x + \beta'' y + \frac{\zeta}{\gamma''} = 0,$
67	100 100	$\frac{x}{1-\beta\gamma} + \frac{y}{1-\gamma\alpha} + \frac{z}{1-\alpha\beta} = 0,$
57	110 011	$\frac{x}{1-\beta'\gamma'} + \frac{y}{1-\gamma'\alpha'} + \frac{z}{1-\alpha'\beta'} = 0,$
56	010 111	$\frac{x}{1-\beta''\gamma''} + \frac{y}{1-\gamma''\alpha''} + \frac{z}{1-\alpha''\beta''} = 0,$
45	001 001	$\frac{\xi}{\alpha(1-\beta\gamma)} + \frac{\eta}{\beta(1-\gamma\alpha)} + \frac{\zeta}{\gamma(1-\alpha\beta)} = 0,$
46	011 110	$\frac{\xi}{\alpha'(1-\beta'\gamma')} + \frac{\eta}{\beta'(1-\gamma'\alpha')} + \frac{\zeta}{\gamma'(1-\alpha'\beta')} = 0,$
47	111 010	$\frac{\xi}{\alpha''(1-\beta''\gamma'')} + \frac{\eta}{\beta''(1-\gamma''\alpha'')} + \frac{\zeta}{\gamma''(1-\alpha''\beta'')} = 0.$

The whole number of ways in which the equation of the curve can be expressed in a form such as $\sqrt{(x\xi)} + \sqrt{(y\eta)} + \sqrt{(z\zeta)} = 0$ is 1260; viz. the three pairs of bitangents entering into the equation of the curve are of one of the types

12.34,	13.24,	14.23	⊠	No. is	70
12.34,	13.24,	56.78	□ | |	„	630
13.23,	14.24,	15.25	◈	„	560
					1260.

It may be remarked that, selecting at pleasure any two pairs out of a system of three pairs, the type is always □ or \|\|\|\|, viz. (see p. 233) the four bitangents are such that their points of contact are situate on a conic.