744.

TABLE OF $\Delta^m 0^n \div \Pi(m)$ UP TO $m = n = 20$.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XIII. Part I. (1881), pp. 1—4. Read October 27, 1879.]

THE differences of the powers of zero, $\Delta^m 0^n$, present themselves in the Calculus of Finite Differences, and especially in the applications of Herschel's theorem,

$$f(e^t) = f(1+\Delta)\, e^{t \cdot 0},$$

for the expansion of the function of an exponential. A small Table up to $\Delta^{10}0^{10}$ is given in Herschel's *Examples* (Camb. 1820), and is reproduced in the treatise on Finite Differences (1843) in the *Encyclopædia Metropolitana*. But, as is known, the successive differences $\Delta 0^n$, $\Delta^2 0^n$, $\Delta^3 0^n$, ... are divisible by 1, 1.2, 1.2.3, ... and generally $\Delta^m 0^n$ is divisible by $1.2.3\ldots m$, $= \Pi(m)$; these quotients are much smaller numbers, and it is therefore desirable to tabulate them rather than the undivided differences $\Delta^m 0^n$: moreover, it is easier to calculate them. A table of the quotients $\Delta^m 0^n \div \Pi(m)$, up to $m = n = 12$ is in fact given by Grunert, *Crelle*, t. XXV. (1843), p. 279, but without any explanation in the heading of the meaning of the tabulated numbers $C_n{}^k$, $= \Delta^n 0^k \div \Pi(n)$, and without using for their determination the convenient formula $C_n{}^{k+1} = nC_n{}^k + C_{n-1}{}^k$ given by Björling in a paper, *Crelle*, t. XXVIII. (1844), p. 284. The formula in question, say

$$\frac{\Delta^m 0^{n+1}}{\Pi(m)} = m\,\frac{\Delta^m 0^n}{\Pi(m)} + \frac{\Delta^{m-1} 0^n}{\Pi(m-1)},$$

is given in the second edition (by Moulton) of Boole's *Calculus of Finite Differences*, (London, 1872), p. 28, under the form

$$\Delta^m 0^n = m\,(\Delta^{m-1} 0^{n-1} + \Delta^m 0^{n-1}).$$

It occurred to me that it would be desirable to extend the table of the quotients $\Delta^m 0^n \div \Pi(m)$, up to $m = n = 20$. The calculation is effected very readily by means of the foregoing theorem, which is used in the following form; viz. any column of the table for instance the fifth, being

$$\begin{array}{lll} A, & \text{then the following column is} & A, \\ B, & \ldots & 2B + A, \\ C, & \ldots & 3C + B, \\ D, & \ldots & 4D + C, \\ E, & \ldots & 5E + D, \\ & & + E; \end{array}$$

and then we obtain a good verification by taking the sum of the terms in the new column, and comparing it with the value as calculated from the formula,

$$\text{Sum} = 2A + 3B + 4C + 5D + 6E.$$

Observe that, in the two calculations, we take successive multiples such as $4D$ and $5D$ of each term of the preceding column, and that the verification is thus a safeguard against any error of multiplication or addition.

TABLE, No. 1, OF $\Delta^m 0^n \div \Pi(m)$.

Ind. Δ	0^1	0^2	0^3	0^4	0^5	0^6	0^7	0^8	0^9	0^{10}	0^{11}	0^{12}	0^{13}	0^{14}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255	511	1 023	2 047	4 095	8 191
3			1	6	25	90	301	966	3 025	9 330	28 501	86 526	261 625	788 970
4				1	10	65	350	1 701	7 770	34 105	145 750	611 501	2 532 530	10 391 745
5					1	15	140	1 050	6 951	42 525	246 730	1 379 400	7 508 501	40 075 035
6						1	21	266	2 646	22 827	179 487	1 323 652	9 321 312	63 436 373
7							1	28	462	5 880	63 987	627 396	5 715 424	49 329 280
8								1	36	750	11 880	159 027	1 899 612	20 912 320
9									1	45	1 155	22 275	359 502	5 135 130
10										1	55	1 705	39 325	752 752
11											1	66	2 431	66 066
12												1	78	3 367
13													1	91
14														1
15														
16														
17														
18														
19														
20														

Ind. Δ	0^{15}	0^{16}	0^{17}	0^{18}	0^{19}	0^{20}	
1	1	1	1	1	1	1	1
2	16 383	32 767	65 535	131 071	262 143	524 287	2
3	2 375 101	7 141 686	21 457 825	64 439 010	193 448 101	580 606 446	3
4	42 355 950	171 798 901	694 337 290	2 798 806 985	11 259 666 950	45 232 115 901	4
5	210 766 920	1 096 190 550	5 652 751 651	28 958 095 545	147 589 284 710	749 206 090 500	5
6	420 693 273	2 734 926 558	17 505 749 898	110 687 251 039	693 081 601 779	4 306 078 895 384	6
7	408 741 333	3 281 882 604	25 708 104 786	197 462 483 400	1 492 924 634 839	11 143 554 045 652	7
8	216 627 840	2 141 764 053	20 415 995 028	189 036 065 010	1 709 751 003 480	15 170 932 662 679	8
9	67 128 490	820 784 250	9 528 822 303	106 175 395 755	1 144 614 626 805	12 011 282 644 725	9
10	12 662 650	193 754 990	2 758 334 150	37 112 163 803	477 297 033 785	5 917 584 964 655	10
11	1 479 478	28 936 908	512 060 978	8 391 004 908	129 413 217 791	1 900 842 429 486	11
12	106 470	2 757 118	62 022 324	1 256 328 866	23 466 951 300	411 016 633 391	12
13	4 550	165 620	4 910 178	125 854 638	2 892 439 160	61 068 660 380	13
14	105	6 020	249 900	8 408 778	243 577 530	6 302 524 580	14
15	1	120	7 820	367 200	13 916 778	452 329 200	15
16		1	136	9 996	527 136	22 350 954	16
17			1	153	12 597	741 285	17
18				1	171	15 675	18
19					1	190	19
20						1	20

Writing down the sloping lines as columns thus:

1 (0)	2 (2)	3 (4)	4 (6)	5 (8)	6 (10)	7 (12)	8 etc. (14) etc.
1							
1	1						
1	3	1					
1	6	7	1				
1	10	25	15	1			
1	15	65	90	31	1		
1	21	140	350	301	63	1	
1	28	266	1 050	1 701	966	127	
1	36	462	2 646	6 951	7 770	3 025	
1	45	750	5 880	22 827	42 525	34 105	
1	55	1 155	11 880	63 987	179 487	246 730	
1	66	1 705	22 275	159 027	627 396	1 323 652	
1	78	2 431	39 325	359 502	1 899 612	5 715 424	
1	91	3 367	66 066	752 752	5 135 130	20 912 320	
1	105	4 550	106 470	1 479 478	12 662 650	67 128 490	
1	120	6 020	165 620	2 757 118	28 936 908	193 754 990	
1	136	7 820	249 900	4 910 178	62 022 324	512 060 978	
1	153	9 996	367 200	8 408 778	125 854 638	1 256 328 866	
1	171	12 597	527 136	13 916 778	243 577 530	2 892 439 160	
1	190	15 675	741 285	22 350 954	452 329 200	6 302 524 580	
20	19	18	17	16	15	14	13 etc.

it appears by inspection that, in the second column the second differences, are constant, in the third column the fourth differences, in the fourth column the sixth differences, and so on, are constant; and we thence deduce the law of the numbers in the successive columns: viz. this can be done up to column 7, in which we have 14 numbers in order to find the 12th differences: but in column 8 we have only 13 numbers, and therefore cannot find the 14th differences. The differences are given in the following

TABLE, No. 2 (*explanation infrà*).

Ind. Δ	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1		2	6	14	30	62	126
2		1	12	61	240	841	2 772
3			10	124	890	5 060	25 410
4			3	131	1 830	16 990	127 953
5				70	2 226	35 216	401 436
6				15	1 600	47 062	836 976
7					630	40 796	1 196 532
8					105	21 225	1 182 195
9						10 930	795 718
10						945	349 020
11							90 090
12							10 395

We have, by means of this Table, the general expressions of $\Delta^r 0^r$, $\Delta^{r-1}0^r$, $\Delta^{r-2}0^r$, ... up to $\Delta^{r-6}0^r$, viz. the formulæ are

$$\Delta^r 0^r \div \Pi(r) \quad = 1,$$

$$\Delta^{r-1}0^r \div \Pi(r-1) = 1 + 2\binom{r-2}{1}^1 + 1\binom{r-2}{2}^2,$$

$$\Delta^{r-2}0^r \div \Pi(r-2) = 1 + 6\binom{r-3}{1}^1 + 12\binom{r-3}{2}^2 + 10\binom{r-3}{3}^3 + 3\binom{r-3}{4}^4,$$

&c., &c.,

where the numerical coefficients are the numbers in the successive columns of the table; and where for shortness $\binom{r-m}{k}^k$ is written to denote the binomial coefficient $\frac{[r-m]^k}{[k]^k}$. For instance, $r=10$, we have

$$\Delta^8 0^{10} \div \Pi(8) = 1 + 6\,.\,7 + 12\,.\,21 + 10\,.\,35 + 3\,.\,35, \; = 750,$$

agreeing with the principal Table. It will be observed that, in the successive columns of the Table, the last terms are 1, 1, 1.3, 1.3.5, 1.3.5.7, 1.3.5.7.9, and 1.3.5.7.9.11. This is itself a good verification: I further verified the last column by calculating from it the value of $\Delta^{14}0^{20} \div \Pi(14)$, $= 6\,302\,524\,580$ as above. The Table shows that we have $\Delta^{r-m}0^r \div \Pi(r-m)$ given as an algebraical rational and integral function of r, of the degree $2m$. But the terms from the top of a column, $\Delta 0^r = 1$, $\Delta^2 0^r \div 1\,.\,2 = 2^{r-1} - 1$, &c., are not algebraical functions of r.

22 *October*, 1879.