741.

ON A THEOREM OF ABEL'S RELATING TO A QUINTIC EQUATION.

THE theorem in question is given, *Œuvres Complètes*, [Christiania, 1881], t. II., p. 266, as an extract from a letter to Crelle dated 14th March, 1826, as follows:

"Si une équation du cinquième degré dont les coefficients sont *des nombres rationnels* est résoluble algébriquement, on peut donner aux racines la forme suivante :

$$x = c + A a^{\frac{1}{5}} a_1^{\frac{2}{5}} a_2^{\frac{4}{5}} a_3^{\frac{3}{5}} + A_1 a_1^{\frac{1}{5}} a_2^{\frac{2}{5}} a_3^{\frac{4}{5}} a^{\frac{3}{5}} + A_2 a_2^{\frac{1}{5}} a_3^{\frac{2}{5}} a^{\frac{4}{5}} a_1^{\frac{3}{5}} + A_3 a_3^{\frac{1}{5}} a^{\frac{2}{5}} a_1^{\frac{4}{5}} a_2^{\frac{3}{5}},$$

où

$$a \ = m + n \sqrt{(1 + e^2)} + \sqrt{[h \, (1 + e^2 + \sqrt{(1 + e^2)})]},$$
$$a_1 = m - n \sqrt{(1 + e^2)} + \sqrt{[h \, (1 + e^2 - \sqrt{(1 + e^2)})]},$$
$$a_2 = m + n \sqrt{(1 + e^2)} - \sqrt{[h \, (1 + e^2 + \sqrt{(1 + e^2)})]},$$
$$a_3 = m - n \sqrt{(1 + e^2)} - \sqrt{[h \, (1 + e^2 - \sqrt{(1 + e^2)})]},$$

$$A \ = K + K'a \ + K''a_2 + K'''aa_2, \quad A_1 = K + K'a_1 + K''a_3 + K'''a_1a_3,$$
$$A_2 = K + K'a_2 + K''a \ + K'''aa_2, \quad A_3 = K + K'a_3 + K''a_1 + K'''a_3.$$

Les quantités c, h, e, m, n, K, K', K'', K''' sont des nombres *rationnels*. Mais de cette manière l'équation $x^5 + ax + b = 0$ n'est pas résoluble tant que a et b sont des quantités quelconques. J'ai trouvé de pareils théorèmes pour les équations du 7ème, 11ème, 13ème, etc. degré."

It is easy to see that x is the root of a quintic equation, the coefficients of which are rational and integral functions of a, a_1, a_2, a_3: these coefficients are not symmetrical functions of a, a_1, a_2, a_3, but they are functions which remain unaltered

by the cyclical change a into a_1, a_1 into a_2, a_2 into a_3, a_3 into a. But the coefficients of the quintic equation must be rational functions of c, h, e, m, n, K, K', K'', K''': hence regarding a, a_1, a_2, a_3, as the roots of a quartic equation, the coefficients of this equation being rational functions of m, n, e, h, this equation must be such that every rational function of the roots, unchanged by the aforesaid cyclical change of the roots, shall be rationally expressible in terms of these quantities m, n, e, h: or, what is the same thing, the group of the quartic equation, using the term "group of the equation" in the sense assigned to it by Galois, must be $aa_1a_2a_3$, $a_1a_2a_3a$, $a_2a_3aa_1$, $a_3aa_1a_2$. And conversely, the quartic equation being of this form, x will be the root of a quintic equation, the coefficients whereof are rational and integral functions of c, h, e, m, n, K, K', K'', K'''.

To investigate the form of a quartic equation having the property just referred to, let it be proposed to find γ, γ' functions of e, h, such that $\gamma^2 + \gamma'^2$ is a rational function of e, h, but that $\gamma^2 - \gamma'^2$, $\gamma\gamma'$ are rational multiples of the same quadric radical $\sqrt{\theta}$. Assume that we have

$$\gamma^2 - \gamma'^2 = 2p\sqrt{\theta}, \quad \gamma\gamma' = q\sqrt{\theta};$$

then

$$(\gamma^2 + \gamma'^2)^2 = 4(p^2 + q^2)\,\theta;$$

that $\gamma^2 + \gamma'^2$ may be rational, we must have $p^2 + q^2 = \lambda^2\theta$, or say $p^2 + q^2 = h^2\theta$; hence, $\theta = \dfrac{p^2}{h^2} + \dfrac{q^2}{h^2}$ must be a sum of two squares, or, assuming one of these equal to unity and the other of them equal to e^2, say $\theta = 1 + e^2$, we satisfy the required equation by taking $p = h$, $q = he$: viz. we thus have

$$\gamma^2 - \gamma'^2 = 2h\sqrt{1+e^2}, \quad \gamma\gamma' = he\sqrt{1+e^2}, \quad \gamma^2 + \gamma'^2 = 2h(1+e^2);$$

and thence also

$$\gamma^2 = h(1 + e^2 + \sqrt{1+e^2}), \quad \gamma'^2 = h(1 + e^2 - \sqrt{1+e^2}),$$

the roots of these expressions, or values of γ, γ', being such that

$$\gamma\gamma' = he\sqrt{1+e^2}.$$

Taking now α rational, $= m$ suppose, and β a rational multiple of

$$\sqrt{1+e^2}, \ = h\sqrt{1+e^2},$$

suppose; it is easy to see that the quartic equation which has for its roots

$$a, a_1, a_2, a_3 = \alpha + \beta + \gamma, \ \alpha - \beta + \gamma', \ \alpha + \beta - \gamma, \ \alpha - \beta - \gamma',$$

has the property in question, viz. that every rational function of the roots unchangeable by the cyclical change a into a_1, a_1 into a_2, a_2 into a_3, a_3 into a, is rationally expressible in terms of e, h, m, n.

It will be sufficient to give the proof in the case of a rational and integral function; such a function, unchangeable as aforesaid, is of the form

$$\phi(a, a_1, a_2, a_3) + \phi(a_1, a_2, a_3, a) + \phi(a_2, a_3, a, a_1) + \phi(a_3, a, a_1, a_2);$$

and if $\phi(a, a_1, a_2, a_3)$ contains a term $\alpha^m \beta^n \gamma^p \gamma'^q$, then the other three functions will contain respectively the terms

$$\alpha^m (-\beta)^n \gamma'^p (-\gamma)^q, \quad \alpha^m \beta^n (-\gamma)^p (-\gamma')^q, \quad \alpha^m (-\beta)^n (-\gamma')^p (\gamma)^q;$$

viz. the sum of the four terms is

$$= \alpha^m \beta^n [\{1 + (-)^{p+q} 1\} \gamma^p \gamma'^q + \{(-)^{n+p} 1 + (-)^{n+q} 1\} \gamma^q \gamma'^p].$$

This obviously vanishes unless p and q are both even, or both odd; and the cases to be considered are 1°, n even, p and q even; 2°, n odd, p and q even; 3°, n even, p and q odd; 4°, n odd, p and q odd. Writing, for greater distinctness, $2n$ or $2n+1$ for n, according as n is even or odd, and similarly for p and q, the term is, in the four cases respectively,

$$= 2\alpha^m \beta^{2n} \quad (\gamma^{2p} \gamma'^{2q} + \gamma^{2q} \gamma'^{2p}),$$
$$= 2\alpha^m \beta^{2n+1} (\gamma^{2p} \gamma'^{2q} - \gamma^{2q} \gamma'^{2p}),$$
$$= 2\alpha^m \beta^{2n} \quad (\gamma^{2p+1} \gamma'^{2q+1} - \gamma^{2q+1} \gamma'^{2p+1}),$$
$$= 2\alpha^m \beta^{2n+1} (\gamma^{2p+1} \gamma'^{2q+1} + \gamma^{2q+1} \gamma'^{2p+1}).$$

The second, third, and fourth expressions contain the factors

$$\beta (\gamma^2 - \gamma'^2), \quad \gamma \gamma' (\gamma^2 - \gamma'^2), \quad \beta \gamma \gamma',$$

respectively; and the first expression as it stands, and the other three divested of these factors respectively are rational functions of α, β^2, γ^2, γ'^2, that is, they are rational functions of m, n, e, h. But the omitted factors $\beta (\gamma^2 - \gamma'^2)$, $\gamma \gamma' (\gamma^2 - \gamma'^2)$, $\beta \gamma \gamma'$, $= 2nh (1 + e^2)$, $2h^2 e (1 + e^2)$, $nhe (1 + e^2)$ are rational functions of n, h, e; hence each of the original four expressions is a rational function of m, n, h, e; and the entire function

$$\phi(a, a_1, a_2, a_3) + \phi(a_1, a_2, a_3, a) + \phi(a_2, a_3, a, a_1) + \phi(a_3, a, a_1, a_2)$$

is a rational function of m, n, h, e.

Replacing α, β, γ, γ' by their values, the roots of the quartic equation are

$$m + n \sqrt{(1 + e^2)} + \sqrt{[h (1 + e^2 + \sqrt{(1 + e^2)})]},$$
$$m - n \sqrt{(1 + e^2)} + \sqrt{[h (1 + e^2 - \sqrt{(1 + e^2)})]},$$
$$m + n \sqrt{(1 + e^2)} - \sqrt{[h (1 + e^2 + \sqrt{(1 + e^2)})]},$$
$$m - n \sqrt{(1 + e^2)} - \sqrt{[h (1 + e^2 - \sqrt{(1 + e^2)})]}.$$

And I stop to remark that taking m, n, e, $h = -\frac{1}{4}$, $+\frac{1}{4}$, 2, $-\frac{1}{8}$ respectively, the roots are

$$-\frac{1}{4} + \frac{1}{4} \sqrt{5} + \sqrt{[-\frac{1}{8} (5 + \sqrt{5})]},$$
$$-\frac{1}{4} - \frac{1}{4} \sqrt{5} + \sqrt{[-\frac{1}{8} (5 - \sqrt{5})]},$$
$$-\frac{1}{4} + \frac{1}{4} \sqrt{5} - \sqrt{[-\frac{1}{8} (5 + \sqrt{5})]},$$
$$-\frac{1}{4} - \frac{1}{4} \sqrt{5} - \sqrt{[-\frac{1}{8} (5 - \sqrt{5})]},$$

viz. these are the imaginary fifth roots of unity, or roots r, r^2, r^4, r^3 of the quartic equation $x^4 + x^3 + x^2 + x + 1 = 0$; which equation, as is well known, has the group $rr^2r^4r^3$, $r^2r^4r^3r$, $r^4r^3rr^2$, $r^3rr^2r^4$.

Reverting to Abel's expression for x, and writing this for a moment in the form

$$x = c + p + s + r + q,$$

the quintic equation in x is

$$
\begin{aligned}
0 = \ & (x - c)^5 \\
& + (x - c)^3 . - 5\,(pr + qs) \\
& + (x - c)^2 . - 5\,(p^2s + q^2p + r^2q + s^2r) \\
& + (x - c) . - 5\,(p^3q + q^3r + r^3s + s^3p) + 5\,(p^2r^2 + q^2s^2) - 5pqrs \\
& + (x - c)^0 . - \quad (p^5 + q^5 + r^5 + s^5) \\
& \qquad + 5\,(p^3rs + q^3sp + r^3pq + s^3qr) \\
& \qquad - 5\,(p^2q^2r + q^2r^2s + r^2s^2p + s^2p^2q).
\end{aligned}
$$

If we substitute herein for p, q, r, s their values, then, altering the order of the terms, the final result is found to be

$$
\begin{aligned}
0 = \ & (x - c)^5 \\
& + (x - c)^3 . - 5\,(AA_2 + A_1A_3)\,aa_1a_2a_3 \\
& + (x - c)^2 . - 5\,(A^2A_1a_2a_3 + A_1^2A_2a_3a + A_2^2A_3aa_1 + A_3^2Aa_1a_2)\,aa_1a_2a_3 \\
& + (x - c) . - 5\,(A^3A_3a_1a_2^2a_3 + A_1^3Aa_2a_3^2a + A_2^3A_1a_3a^2a_1 + A_3^3A_2aa_1^2a_2)\,aa_1a_2a_3 \\
& \qquad + 5\,(A^2A_2^2 + A_2^2A_3^2 - A_1A_2A_3A)\,(aa_1a_2a_3)^2 \\
& + (x - c)^0 . - \quad (A^5a_1a_2^3a_3^2 + A_1^5a_2a_3^3a_1^2 + A_2^5a_3a^3a_1^2 + A_3^5aa_1^3a_2^2)\,aa_1a_2a_3 \\
& \qquad + 5\,(A^3A_1A_2a_2a_3 + A_1^3A_2A_3a_3a + A_2^3A_3Aaa_1 + A_3^3AA_1a_1a_2)\,(aa_1a_2a_3)^2 \\
& \qquad - 5\,(A^2A_3^2A_2a_1a_2 + A_1^2A^2A_3a_2a_3 + A_2^2A_1^2Aa_3a + A_3^2A_2^2A_1aa_1)\,(aa_1a_2a_3)^2 \, ;
\end{aligned}
$$

viz. considering herein A, A_1, A_2, A_3 as standing for their values

$$K + K'a + K''a_2 + K'''aa_2, \ \&\text{c}.$$

respectively, each coefficient is a function of a, a_1, a_2, a_3, which is unaltered by the cyclical change of these values and therefore is a rational function of

$$m, \ n, \ e, \ h, \ K, \ K', \ K'', \ K'''.$$