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ON A PAIR OF DIFFERENTIAL EQUATIONS IN THE LUNAR THEORY.

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I CONSIDER the differential equations

$$\frac{d}{dt}\frac{d\rho}{dt} - \rho\left(\frac{dv}{dt}\right)^2 + \frac{1}{\rho^2} = km^2\rho\ \{\tfrac{1}{2} + \tfrac{3}{2}\cos(2v - 2mt)\},$$

$$\frac{d}{dt}\left(\rho^2\frac{dv}{dt}\right) = jm^2\rho^2\ \{\ -\tfrac{3}{2}\sin(2v - 2mt)\},$$

which when $j = k = 1$ give the following equations in the lunar theory $(D = t - mt)$:

$$\begin{aligned}\frac{1}{\rho} = 1 &+ \tfrac{1}{6}m^2 - \tfrac{179}{288}m^4 - \tfrac{97}{48}m^5 - \tfrac{757}{162}m^6 - \tfrac{4039}{432}m^7 - \tfrac{34751189}{1990656}m^8 - \tfrac{155067635}{4976640}m^9 \\ &+ \cos 2D\,[m^2 + \tfrac{19}{6}m^3 + \tfrac{131}{18}m^4 + \tfrac{383}{27}m^5 + \tfrac{510565}{20736}m^6 + \tfrac{23140781}{622080}m^7 \\ &\qquad + \tfrac{355021217}{9331200}m^8 + \tfrac{27888590059}{34992000}m^9] \\ &+ \cos 4D\,[\tfrac{7}{8}m^4 + \tfrac{2737}{480}m^5 + \tfrac{162869}{7200}m^6 + \tfrac{7554833}{108000}m^7 + \tfrac{2389416723}{12960000}m^8 + \tfrac{2335230125283}{5443200000}m^9] \\ &+ \cos 6D\,[\tfrac{219}{256}m^6 + \tfrac{151339}{17920}m^7 + \tfrac{29887443}{627200}m^8 + \tfrac{98978623957}{444528000}m^9] \\ &+ \cos 8D\,[\tfrac{2701}{3072}m^8 + \tfrac{70033633}{6021120}m^9],\end{aligned}$$

or as far as m^7,

$$\begin{aligned}\rho = 1 &- \tfrac{1}{6}m^2 + \tfrac{331}{288}m^4 + \tfrac{83}{16}m^5 + \tfrac{42775}{2592}m^6 + \tfrac{4787}{108}m^7 \\ &+ \cos 2D\,[-m^2 - \tfrac{19}{6}m^3 - \tfrac{125}{18}m^4 - \tfrac{709}{54}m^5 - \tfrac{485173}{20736}m^6 - \tfrac{24487949}{622080}m^7] \\ &+ \cos 4D\,[-\tfrac{3}{8}m^4 - \tfrac{1217}{480}m^5 - \tfrac{74069}{7200}m^6 - \tfrac{1749779}{54000}m^7] \\ &+ \cos 6D\,[-\tfrac{59}{256}m^6 - \tfrac{126193}{53760}m^7],\end{aligned}$$

$\Big(\frac{1}{\rho}$ is given by M. Delaunay only as far as m^5, the additional terms of $\frac{1}{\rho}$ and expression for ρ were kindly communicated to me by Prof. Adams$\Big)$; and

$$\begin{aligned}
v = t \\
&+ \sin 2D\left(\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3 + \tfrac{893}{72}m^4 + \tfrac{2855}{108}m^5 + \tfrac{8304449}{165888}m^6\right. \\
&\qquad\qquad \left. + \tfrac{102859909}{1244160}m^7 + \tfrac{7596606727}{74649600}m^8 - \tfrac{8051418161}{111974400}m^9\right) \\
&+ \sin 4D\left(\tfrac{201}{256}m^4 + \tfrac{649}{120}m^5 + \tfrac{647623}{28800}m^6 + \tfrac{31363361}{432000}m^7 + \tfrac{123030377303}{414720000}m^8\right) \\
&+ \sin 6D\left(\tfrac{3715}{6144}m^6 + \tfrac{664571}{107520}m^7\right)
\end{aligned}$$

(Delaunay, t. II. pp. 815, 836, 845).

To integrate the original equations write

$$\rho = 1 + \rho_1 + \rho_2 + \dots,$$
$$v = t + v_1 + v_2 + \dots,$$

where the suffixes indicate the degrees in the coefficients k, j conjointly: the equations for ρ_n, v_n take the form

$$\frac{d}{dt}\frac{d\rho_n}{dt} - 3\rho_n - 2\frac{dv_n}{dt} + V_n = Q_n,$$

$$\frac{d}{dt}\left(\frac{dv_n}{dt} + 2\rho_n + U_n\right) \qquad = P_n,$$

where V_n, U_n, P_n, Q_n do not contain ρ_n or v_n. From the second equation we have

$$\frac{dv_n}{dt} + 2\rho_n + U_n = \Omega_n + \int P_n\, dt,$$

where Ω_n is a constant of integration, the integral $\int P_n\, dt$ containing only periodic terms; and then adding twice this to the first equation we have

$$\frac{d}{dt}\frac{d\rho_n}{dt} + \rho_n + V_n + 2U_n = 2\Omega_n + Q_n + 2\int P_n\, dt$$

which determines ρ_n; and substituting its value in the other equation we have $\frac{dv_n}{dt}$, and thence v_n; the constant Ω_n is determined so that $\frac{dv_n}{dt}$ may contain no constant term. We have

$$V_1 = 0,$$

$$V_2 = -\left(\frac{dv_1}{dt}\right)^2 - 2\rho_1\frac{dv_1}{dt} + 3\rho_1^2,$$

$$V_3 = -2\frac{dv_1}{dt}\frac{dv_2}{dt} - 2\rho_1\frac{dv_2}{dt} - \rho_1\left(\frac{dv_1}{dt}\right)^2 - 2\rho_2\frac{dv_2}{dt} + 6\rho_1\rho_2 - 4\rho_1^3,$$

&c.

$$U_1 = 0,$$

$$U_2 = 2\rho_1\frac{dv_1}{dt} + \rho_1^2,$$

$$U_3 = 2\rho_1\frac{dv_2}{dt} + (2\rho_2 + \rho_1^2)\frac{dv_1}{dt} + 2\rho_1\rho_2,$$

&c.

$$Q_1 = km^2 (\tfrac{1}{2} + \tfrac{3}{2}\cos 2D),$$
$$Q_2 = km^2 \{3v_1 \sin 2D + \rho_1 (\tfrac{1}{2} + \tfrac{3}{2}\cos 2D)\},$$
$$Q_3 = km^2 \{-3v_2 \sin 2D - 3v_1^2 \cos 2D + \rho_1 v_1 . 3 \sin 2D + \rho_2 (\tfrac{1}{2} + \tfrac{3}{2}\cos 2D)\},$$
&c.

$$P_1 = jm^2 (-\tfrac{3}{2}\sin 2D),$$
$$P_2 = jm^2 (-3v_1 \cos D - 3\rho_1 \sin 2D),$$
$$P_3 = jm^2 \{-3v_2 \cos 2D + 3v_1^2 \sin 2D - 6\rho_1 v_1 \cos 2D + (2\rho_2 + \rho_1^2) . -\tfrac{3}{2}\sin 2D\},$$
&c.

In particular attending to the values of P_1, Q_1 the equations for ρ_1, v_1 are in their original form

$$\frac{d}{dt}\frac{d\rho_1}{dt} - 3\rho_1 + 2\frac{dv_1}{dt} = km^2 (\tfrac{1}{2} + \tfrac{3}{2}\cos 2D),$$
$$\frac{d}{dt}\left(\frac{dv_1}{dt} + 2\rho_1\right) \quad = jm^2 (\ -\tfrac{3}{2}\sin 2D),$$

whence in the transformed form they are

$$\frac{dv_1}{dt} + 2\rho_1 = \Omega_1 + \frac{3jm^2}{4(1-m)}\cos 2D,$$

and

$$\frac{d^2\rho_1}{dt^2} + \rho_1 = 2\Omega_1 + km^2(\tfrac{1}{2} + \tfrac{3}{2}\cos 2D) + \frac{\tfrac{3}{2}jm^2}{1-m}\cos 2D.$$

Thus the constant term of ρ_1 is $2\Omega_1 + \tfrac{1}{2}km^2$, giving in $\frac{dv_1}{dt}$ a constant term $-3\Omega_1 - km^2$ this must vanish, or we have $\Omega_1 = -\tfrac{1}{3}km^2$; and the equations thus become

$$\frac{dv_1}{dt} + 2\rho_1 = -\tfrac{1}{3}km^2 + \frac{3jm^2}{4(1-m)}\cos 2D,$$
$$\frac{d^2\rho_1}{dt^2} + \rho_1 = -\tfrac{1}{6}km^2 + \left(\tfrac{3}{2}km^2 + \frac{\tfrac{3}{2}jm^2}{1-m}\right)\cos 2D,$$

and then completing the integration

$$\rho_1 = -\tfrac{1}{6}km^2 + \left\{\frac{-\tfrac{3}{2}km^2}{3 - 8m + 4m^2} + \frac{-\tfrac{3}{2}jm^2}{(1-m)(3-8m+4m^2)}\right\}\cos 2D,$$
$$v_1 = \left\{\frac{\tfrac{3}{2}km^2}{(1-m)(3-8m+4m^2)} + \frac{\tfrac{3}{8}jm^2(7-8m+4m^2)}{(1-m)^2(3-8m+4m^2)}\right\}\sin 2D,$$

which are the accurate values of ρ_1 and v_1.

Expanding as far as m^6 we have

$$\rho_1 = k(-\tfrac{1}{6}m^2) + \cos 2D\{\ k(-\tfrac{1}{2}m^2 - \tfrac{4}{3}m^3 - \tfrac{26}{9}m^4 - \tfrac{160}{27}m^5 - \tfrac{968}{81}m^6)$$
$$+j(-\tfrac{1}{2}m^2 - \tfrac{11}{6}m^3 - \tfrac{85}{18}m^4 - \tfrac{575}{54}m^5 - \tfrac{3661}{162}m^6)\},$$

which for $j = k$ is $\qquad = k(-\ m^2 - \tfrac{19}{6}m^3 - \tfrac{137}{18}m^4 - \tfrac{895}{54}m^5 - \tfrac{5597}{162}m^6)$,

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and

$$v_1 = \sin 2D \left\{ k\left(\tfrac{1}{2} m^2 + \tfrac{11}{6} m^3 + \tfrac{85}{18} m^4 + \tfrac{575}{54} m^5 + \tfrac{3661}{162} m^6 \right) + j\left(\tfrac{7}{8} m^2 + \tfrac{37}{12} m^3 + \tfrac{589}{72} m^4 + \tfrac{1037}{54} m^5 + \tfrac{27331}{648} m^6 \right) \right\},$$

which for $j = k$ is

$$= k\left(\tfrac{11}{8} m^2 + \tfrac{59}{12} m^3 + \tfrac{929}{72} m^4 + \tfrac{896}{27} m^5 + \tfrac{41975}{648} m^6 \right).$$

I have, not in general, but for the value $j = k$, calculated ρ_2 and v_2 as far as m^6: I have not made the calculation for ρ_3 and v_3, but their values may be deduced from the foregoing values of ρ, v; the final expressions (when $j = k$) of ρ, $= 1 + \rho_1 + \rho_2 + \rho_3 + \ldots$ and v, $= t + v_1 + v_2 + v_3 \ldots$ are

$$\begin{aligned}
\rho = 1 & \\
& + k\left(-\tfrac{1}{6} m^2\right) \\
& + k^2\left(\tfrac{331}{288} m^4 + \tfrac{83}{16} m^5 + \tfrac{5113}{288} m^6\right) \\
& + k^3\left(- \tfrac{1621}{1296} m^6\right) \\
+ \cos 2D \Big\{ & k\left(- m^2 - \tfrac{19}{6} m^3 - \tfrac{137}{18} m^4 - \tfrac{895}{54} m^5 - \tfrac{5597}{162} m^6\right) \\
& + k^2\left(\tfrac{2}{3} m^4 + \tfrac{31}{9} m^5 + \tfrac{329}{27} m^6\right) \\
& + k^3\left(- \tfrac{2381}{2304} m^6\right)\Big\} \\
+ \cos 4D \Big\{ & k^2\left(- \tfrac{3}{8} m^4 - \tfrac{1217}{480} m^5 - \tfrac{76589}{7200} m^6\right) \\
& + k^3\left(+ \tfrac{7}{20} m^6\right)\Big\} \\
+ \cos 6D \Big\{ & k^3\left(- \tfrac{59}{256} m^6\right)\Big\},
\end{aligned}$$

and

$$\begin{aligned}
v = t & \\
+ \sin 2D \Big\{ & k\left(\tfrac{11}{8} m^2 + \tfrac{59}{12} m^3 + \tfrac{929}{72} m^4 + \tfrac{896}{27} m^5 + \tfrac{41975}{648} m^6\right) \\
& + k^2\left(- \tfrac{1}{2} m^4 - \tfrac{41}{12} m^5 - \tfrac{43}{3} m^6\right) \\
& + k^3\left(- \tfrac{783}{2048} m^6\right)\Big\} \\
+ \sin 4D \Big\{ & k^2\left(\tfrac{201}{256} m^4 + \tfrac{649}{120} m^5 + \tfrac{665263}{28800} m^6\right) \\
& + k^3\left(- \tfrac{49}{80} m^6\right)\Big\} \\
+ \sin 6D \Big\{ & k^3\left(+ \tfrac{3715}{6144} m^6\right)\Big\};
\end{aligned}$$

which for $k = 1$ agree with the foregoing formulæ (verifying them as far as m^5); the present formulæ exhibit the manner in which the expressions depend on the several powers of the disturbing force.