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ON AN EXPRESSION FOR THE ANGULAR DISTANCE OF TWO PLANETS.

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If for the planet m, referred to any fixed plane and origin of longitudes, we have

v, the longitude in orbit,

θ, the longitude of node,

ϕ, the inclination,

and similarly for the planet m' referred to the same fixed plane and origin of longitudes, if the corresponding quantities are v', θ', ϕ'; then the angular distance of the two planets will of course be expressible in terms of $v, \theta, \phi, v', \theta', \phi'$, but I am not aware that the actual expression has been given. To obtain it in the most simple manner, I write further for the planet m:

$\theta + x$, the reduced longitude,

y, the latitude,

z, the distance from node,

so that z $(= v - \theta)$, x, y, are the hypothenuse, base, and perpendicular of a right-angled spherical triangle, the base angle of which is $= \phi$. And similarly $\theta' + x'$, y', z', have the like significations for the planet m'. I write also r, r', for the distances of the two planets respectively.

This being so, the rectangular coordinates of the planet m are

$$r \cos y \cos (\theta + x),$$

$$r \cos y \sin (\theta + x),$$

$$r \sin y.$$

But observing that from the right-angled triangle we have

$$\cos z = \cos x \cos y,$$

$$\cos\phi = \tan x \cot z,$$

$$\sin x = \cot\phi \tan y,$$

$$\sin y = \sin\phi \sin z,$$

and therefore also

$$\sin x \cos y = \cot\phi \sin y, = \cos\phi \sin z,$$

the expressions for the coordinates become

$$r(\cos z \cos\theta - \sin z \sin\theta \cos\phi),$$

$$r(\cos z \sin\theta + \sin z \cos\theta \cos\phi),$$

$$r(\qquad \sin z \sin\phi).$$

Forming the analogous expressions for the coordinates of m', then if H be the angular distance of the two planets, we deduce at once the expression for $\cos H$, viz. this is

$$\begin{aligned}\operatorname{Cos} H = \; & (\cos z\cos\theta - \sin z\sin\theta\cos\phi)(\cos z'\cos\theta' - \sin z'\sin\theta'\cos\phi') \\ & + (\cos z\sin\theta + \sin z\cos\theta\cos\phi)(\cos z'\sin\theta' + \sin z'\cos\theta'\cos\phi') \\ & + (\qquad \sin z\sin\phi \qquad)(\qquad \sin z'\sin\phi' \qquad),\end{aligned}$$

or, multiplying out, this is

$$\begin{aligned}\operatorname{Cos} H = \; & \cos z\cos z' \cos(\theta-\theta') \\ & + \cos z\sin z' \sin(\theta-\theta')\cos\phi' \\ & - \sin z\cos z' \sin(\theta-\theta')\cos\phi \\ & + \sin z\sin z' (\cos(\theta-\theta')\cos\phi\cos\phi' + \sin\phi\sin\phi'),\end{aligned}$$

say this is

$$\begin{aligned} = \; & A\cos z\cos z' \\ & + B\cos z\sin z' \\ & + C\sin z\cos z' \\ & + D\sin z\sin z',\end{aligned}$$

viz. it is

$$\begin{aligned} = \; & \cos(z-z').\ \tfrac{1}{2}A + \tfrac{1}{2}D \\ & + \sin(z-z').-\tfrac{1}{2}B + \tfrac{1}{2}C \\ & + \cos(z+z').\ \tfrac{1}{2}A - \tfrac{1}{2}D \\ & + \sin(z+z').\ \tfrac{1}{2}B + \tfrac{1}{2}C.\end{aligned}$$

But we have

$$z - z' = v - v' - \theta + \theta', \quad z + z' = v + v' - \theta - \theta',$$

whence the expression becomes

$$\begin{aligned}\operatorname{Cos} H = \quad & \cos(v-v') \,.\, (\tfrac{1}{2}A+\tfrac{1}{2}D)\cos(\theta-\theta') - (-\tfrac{1}{2}B+\tfrac{1}{2}C)\sin(\theta-\theta') \\ & + \sin(v-v') \,.\, (\tfrac{1}{2}A+\tfrac{1}{2}D)\sin(\theta-\theta') + (-\tfrac{1}{2}B+\tfrac{1}{2}C)\cos(\theta-\theta') \\ & + \cos(v+v') \,.\, (\tfrac{1}{2}A-\tfrac{1}{2}D)\cos(\theta+\theta') - (\ \tfrac{1}{2}B+\tfrac{1}{2}C)\sin(\theta+\theta') \\ & + \sin(v+v') \,.\, (\tfrac{1}{2}A-\tfrac{1}{2}D)\sin(\theta+\theta') + (\ \tfrac{1}{2}B+\tfrac{1}{2}C)\cos(\theta+\theta'),\end{aligned}$$

or substituting for A, B, C, D, their values, and after a few easy reductions, we find

$$\begin{aligned}\operatorname{Cos} H = \quad & \cos(v-v')\begin{Bmatrix}\tfrac{1}{2}+\tfrac{1}{2}\cos\phi\cos\phi' - \tfrac{1}{2}(1-\cos\phi)(1-\cos\phi')\sin^2(\theta-\theta') \\ +\tfrac{1}{2}\sin\phi\sin\phi' \qquad \cos(\theta-\theta')\end{Bmatrix} \\ & + \sin(v-v')\begin{Bmatrix}\tfrac{1}{2}(1-\cos\phi)(1-\cos\phi')\sin(\theta-\theta')\cos(\theta-\theta') \\ +\tfrac{1}{2}\sin\phi\sin\phi' \qquad \sin(\theta-\theta')\end{Bmatrix} \\ & + \cos(v+v')\begin{Bmatrix}\tfrac{1}{2}(1-\cos\phi\cos\phi')\cos(\theta-\theta')\cos(\theta+\theta') \\ +\tfrac{1}{2}(\cos\phi-\cos\phi') \quad \sin(\theta-\theta')\sin(\theta+\theta') \\ -\tfrac{1}{2}\sin\phi\sin\phi' \qquad \cos(\theta+\theta')\end{Bmatrix} \\ & + \sin(v+v')\begin{Bmatrix}\tfrac{1}{2}(1-\cos\phi\cos\phi')\cos(\theta-\theta')\sin(\theta+\theta') \\ -\tfrac{1}{2}(\cos\phi-\cos\phi') \quad \sin(\theta-\theta')\cos(\theta+\theta') \\ -\tfrac{1}{2}\sin\phi\sin\phi' \qquad \sin(\theta+\theta')\end{Bmatrix}\end{aligned}$$

For $\phi=\phi'=0$, the formula becomes, as of course it should do,

$$\operatorname{Cos} H = \cos(v-v').$$

It may be added, that if f, f' are the true anomalies, ω, ω' the longitudes of pericentre in orbit, then $v=\omega+f$, $v'=\omega'+f'$; and we thence have for $\cos H$, formulæ of the like form, containing $\cos f\cos f'$, $\cos f\sin f'$, $\sin f\cos f'$, $\sin f\sin f'$, or containing $\cos(f-f')$, $\sin(f-f')$, $\cos(f+f')$, $\sin(f+f')$, respectively, in place of the like functions of z, z', but with of course altered values of the coefficients.