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EXPRESSIONS FOR PLANA'S e, γ IN TERMS OF THE ELLIPTIC e, γ.

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THE coefficient of $\sin cnt$ in Plana's expression for the true longitude v (see Plana, t. I. p. 574), putting therein $E' = \epsilon' = e'$, that is, neglecting the terms which depend on the variation of the solar excentricity, is

$$
\begin{aligned}
= \; & e \quad \left(2 + \tfrac{3}{2} m^2 - \tfrac{75}{64} m^3 - \tfrac{6659}{256} m^4 - \tfrac{4884375}{36864} m^5 - \tfrac{65756819}{147456} m^6 \right) \\
& + e^3 \quad \left(- \tfrac{1}{4} - 17 m^2 - \tfrac{3195}{32} m^3 - \tfrac{4635997}{7680} m^4 \right) \\
& + e^5 \quad \left(\tfrac{5}{96} + \tfrac{66863}{6144} m^2 \right) \\
& + e^7 \quad \left(\tfrac{5921}{161280} \right) \\
& + e\gamma^2 \quad \left(- \tfrac{1}{2} - \tfrac{63}{32} m^2 + \tfrac{1467}{256} m^3 + \tfrac{22857}{512} m^4 \right) \\
& + e^3\gamma^2 \quad \left(\tfrac{23}{16} - \tfrac{405}{128} m + \tfrac{16029}{512} m^2 \right) \\
& + e^5\gamma^2 \quad \left(- \tfrac{195}{128} \right) \\
& + e\gamma^4 \quad \left(- \tfrac{3}{8} + \tfrac{135}{256} m + \tfrac{3749}{2048} m^2 \right) \\
& + e^3\gamma^4 \quad \left(\tfrac{1761}{1280} \right) \\
& + e\gamma^6 \quad \left(- \tfrac{5}{16} \right) \\
& + ee'^2 \quad \left(\left(- \tfrac{45}{4} + \tfrac{9}{4} = \right) - 9m^2 + \left(- \tfrac{6455}{64} + \tfrac{165}{42} = \right) - \tfrac{6125}{64} m^3 \right. \\
& \qquad\qquad \left. + \left(- \tfrac{281095}{512} - \tfrac{147}{256} = \right) - \tfrac{281389}{512} m^4 \right) \\
& + e^3e'^2 \quad \left(\left(- \tfrac{12831}{480} - \tfrac{33}{2} = \right) - \tfrac{20751}{480} m^2 \right) \\
& + ee^2\gamma^2 \quad \left(\left(\tfrac{5171}{128} - \tfrac{453}{128} = \right) + \tfrac{2359}{64} m^2 \right) \\
& + ee'^4 \quad \left(\left(\tfrac{45}{16} - \tfrac{2025}{64} = \right) - \tfrac{1845}{64} m^2 \right) \\
& + eb^4 \quad \left(- \tfrac{135}{32} m^2 \right) \\
& + ee'^2b^4 \left(- \tfrac{75}{16} \right).
\end{aligned}
$$

Taking this to the fifth order only, and comparing it with the coefficient in the elliptic theory, we have

Plana.		Elliptic.
$= e\ (\ 2 - \frac{3}{2}m^2 - \frac{75}{64}m^3 - \frac{6659}{256}m^4)$	$=$	$e\ (\ 2)$
$+e^3\ (-\frac{1}{4} - 17m^2)$		$+e^3(-\frac{1}{4})$
$+e^5\ (\ \frac{5}{96})$		$+e^5(\ \frac{5}{96})$
$+e\gamma^2\ (-\frac{1}{2} - \frac{63}{32}m^2)$		
$+e^3\gamma^2\ (\ \frac{23}{16})$		
$+e\gamma^4\ (-\frac{3}{8})$		
$+ee'^2\ (-9m^2).$		

The coefficient of $\sin gnt$ in Plana's expression for the latitude (see t. I. p. 704) is

$$
\begin{aligned}
= \gamma\ &(\ 1 + \tfrac{33}{128}m^3 + \tfrac{241}{512}m^4 - \tfrac{82495}{24576}m^5)\\
+\gamma e^2\ &(-1 - \tfrac{31}{512}m^2 - \tfrac{7977}{256}m^3)\\
+\gamma e^4\ &(\ \tfrac{5}{64} + \tfrac{945}{512}m)\\
+\gamma^3\ &(-\tfrac{3}{8} + \tfrac{5}{128}m^2 + \tfrac{69}{256}m^3)\\
+\gamma^3 e^2\ &(\ \tfrac{7}{32} - \tfrac{405}{256}m)\\
+\gamma^5\ &(\tfrac{1}{4})\\
+\gamma e'^2\ &(\ \tfrac{27}{8}m^2 - \tfrac{113}{128}m^3).
\end{aligned}
$$

But according to the calculation of Prof. Adams (quoted by M. Delaunay, *Comptes Rendus*, t. LIV. (1862), this should be

$$
\begin{aligned}
= \gamma\ &(\ 1 + \tfrac{33}{128}m^3 - \tfrac{1}{512}m^4 - \tfrac{82497}{24576}m^5 - \tfrac{4801697}{294012}m^6)\\
+\gamma e^2\ &(-1 - \tfrac{1111}{256}m^2 - \tfrac{7977}{256}m^3)\\
+\gamma e^4\ &(\ \tfrac{3}{16} - \tfrac{135}{512}m)\\
+\gamma^3\ &(-\tfrac{3}{8} + \tfrac{5}{128}m^2 - \tfrac{15}{128}m^3)\\
+\gamma^3 e^2\ &(\ \tfrac{23}{32} + \tfrac{135}{256}m)\\
+\gamma^5\ &(\ \tfrac{15}{64})\\
+\gamma e'^2\ &(\ \tfrac{9}{8}m^2 - \tfrac{113}{128}m^3 + \tfrac{3521}{1024}m^4).
\end{aligned}
$$

Adopting this as the true expression according to Plana's theory, taking it to the fifth order only, and comparing with the elliptic value of the same coefficient, we have

Plana.		Elliptic.
$\gamma\ (\ 1 + \frac{33}{128}m^3 - \frac{1}{512}m^4)$	$=$	$\gamma\ (\ 1)$
$+\gamma e^2\ (-1 - \frac{1111}{256}m^2)$		$+\gamma e^2\ (-1)$
$+\gamma e^4\ (\ \frac{3}{16})$		$+\gamma e^4\ (\ \frac{7}{64})$
$+\gamma^3\ (-\frac{3}{8} + \frac{5}{128}m^2)$		$+\gamma^3\ (-\frac{3}{8})$
$+\gamma^3 e^2\ (\ \frac{23}{32})$		$+\gamma^3 e^2\ (\ \frac{3}{8})$
$+\gamma^5\ (\ \frac{15}{64})$		$+\gamma^5\ (\ \frac{55}{64}).$
$+\gamma e'^2\ (\ \frac{9}{8}m^2)$		

We have thus two equations for the determination of Plana's e, γ in terms of the elliptic e, γ. And the solution of these equations give

$$
\begin{aligned}
& \qquad\qquad\qquad \text{Elliptic.} \\
e\ (\text{Plana}) = {} & e\ \left(1 - \tfrac{3}{4}\, m^2 + \tfrac{75}{128}\, m^3 + \tfrac{6947}{512}\, m^4\right) \\
& + e^5\ \left(\tfrac{263}{32}\, m^2\right) \\
& + \gamma^2 e\ \left(\tfrac{1}{4} + \tfrac{39}{64}\, m^2\right) \\
& + \gamma^2 e^3 \left(-\tfrac{5}{8}\right) \\
& + \gamma^4 e \left(\tfrac{1}{4}\right) \\
& + ee'^2 \left(\tfrac{9}{2}\, m^2\right),
\end{aligned}
$$

$$
\begin{aligned}
\gamma\ (\text{Plana}) = {} & \gamma\ \left(1 - \tfrac{33}{128}\, m^2 + \tfrac{1}{512}\, m^4\right) \\
& + \gamma e^2 \left(\tfrac{727}{256}\, m^2\right) \\
& + \gamma e^4 \left(-\tfrac{5}{64}\right) \\
& + \gamma^3 \left(-\tfrac{5}{128}\, m^2\right) \\
& + \gamma^3 e^2 \left(\tfrac{5}{32}\right) \\
& + \gamma^5 \left(\tfrac{5}{8}\right) \\
& + \gamma e'^2 \left(-\tfrac{9}{8}\, m^2\right).
\end{aligned}
$$

I annex the verification of these expressions; we have

$$
\begin{aligned}
& \text{Plana.} && \text{Elliptic.} \\
& e\ \left(2 + \tfrac{3}{2}\, m^2 - \tfrac{75}{64}\, m^3 - \tfrac{6659}{256}\, m^4\right) = {} && e\ \left(2 - \tfrac{3}{2}\, m^2 + \tfrac{75}{64}\, m^3 + \tfrac{6947}{256}\, m^4\right. \\
& && \qquad + \tfrac{3}{2}\, m^2 \qquad\qquad - \tfrac{9}{8}\, m^4 \\
& && \qquad\qquad\qquad \left. - \tfrac{75}{64}\, m^3 - \tfrac{6659}{256}\, m^4\right) \\
& && + e^3\ \left(\tfrac{263}{32}\, m^2\right) \\
& && + e\gamma^2 \left(\tfrac{1}{2} + \tfrac{39}{32}\, m^2\right. \\
& && \qquad\quad \left. + \tfrac{3}{8}\, m^2\right) \\
& && + e^3\gamma^2 \left(-\tfrac{5}{4}\right) \\
& && + e\gamma^4 \left(\tfrac{1}{2}\right) \\
& && + ee'^2 \left(9m^2\right), \\
& e^3\ \left(-\tfrac{1}{4} - 17m^2\right) = {} && e^3\ \left(-\tfrac{1}{4} + \tfrac{9}{16}\, m^2\right. \\
& && \qquad\quad \left. - 17\, m^2\right) \\
& && + e^3\gamma^2 \left(-\tfrac{3}{16}\right), \\
& e^5\ \left(\tfrac{5}{96}\right) = {} && e^5\ \left(\tfrac{5}{96}\right), \\
& e\gamma^2\ \left(-\tfrac{1}{2} - \tfrac{63}{32}\, m^2\right) = {} && e\gamma^2 \left(-\tfrac{1}{2} - \tfrac{63}{32}\, m^2\right. \\
& && \qquad\quad \left. + \tfrac{3}{8}\, m^2\right) \\
& && + e\gamma^4 \left(-\tfrac{1}{8}\right) \\
& e^3\gamma^2 \left(\tfrac{23}{16}\right) = {} && e^3\gamma^2 \left(\tfrac{23}{16}\right) \\
& e\gamma^4 \left(-\tfrac{3}{8}\right) = {} && e\gamma^4 \left(-\tfrac{3}{8}\right) \\
& ee'^2 \left(-9m^2\right) = {} && ee'^2 \left(-9m^3\right),
\end{aligned}
$$

whence, adding, we have the first equation.

And, moreover,

$$\begin{aligned}
\gamma\ (1 + \tfrac{33}{128} m^3 - \tfrac{1}{512} m^4) &= \gamma\ \ (\ 1 - \tfrac{33}{128}\ m^3 + \tfrac{1}{512} m^4 \\
&\qquad\qquad + \tfrac{33}{128}\ m^3 - \tfrac{1}{512} m^4) \\
&\quad + \gamma e^2\ (\qquad \tfrac{727}{256}\ m^2) \\
&\quad + \gamma^4 e\ (-\tfrac{5}{64}) \\
&\quad + \gamma^3\ (\qquad - \tfrac{5}{128}\ m^2) \\
&\quad + \gamma^3 e^2\ (\tfrac{5}{32}) \\
&\quad + \gamma^5\ \ (\tfrac{5}{8}) \\
&\quad + \gamma e'^2\ (\qquad - \tfrac{9}{8}\ m^2), \\
\gamma e^2\ (-1 - \tfrac{1111}{256} m^2) &= \gamma e^2\ (-1 + \tfrac{3}{2}\ m^2 \\
&\qquad\qquad - \tfrac{1111}{256} m^2) \\
&\quad + \gamma^3 e^2\ (\qquad - \tfrac{1}{2}), \\
\gamma e^4\ (\tfrac{3}{16}) &= \gamma e^4\ (\tfrac{3}{16}), \\
\gamma^3\ \ (-\tfrac{3}{8} + \tfrac{5}{128}\ m^2) &= \gamma^3\ \ (-\tfrac{3}{8} + \tfrac{5}{128}\ m^2) \\
\gamma^3 e^2\ (\ \tfrac{23}{32}) &= \gamma^3 e^2\ (\ \tfrac{23}{32}) \\
\gamma^5\ \ (\ \tfrac{15}{64}) &= \gamma^5\ \ (\ \tfrac{15}{64}) \\
\gamma e'^2\ (\ \ \tfrac{9}{8}\ m^2) &= \gamma e'^2\ (\ \ \tfrac{9}{8}\ m^2),
\end{aligned}$$

whence, adding, we have the second equation.

It may be noticed that, taking the foregoing expressions only as far as the third order, we have

Plana.		Elliptic.
e	$=$	$e\ (1 + \tfrac{1}{4}\gamma^2 - \tfrac{3}{4} m^2)$,
γ	$=$	γ.

And moreover that, attending only to the terms which are independent of m, we have

$$e = e\ (1 + \tfrac{1}{4}\gamma^2 - \tfrac{5}{8}\ \gamma^2 + \tfrac{1}{4}\gamma^4),$$

$$\gamma = \gamma\ (1 - \tfrac{5}{64} e^4 + \tfrac{5}{32} e^2\gamma^2 - \tfrac{5}{8}\gamma^4).$$

which are formulæ that may be found useful.