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NOTE ON THE LUNAR THEORY.

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I ATTEND, in the expressions for the lunar coordinates, only to the coefficients independent of m. Plana's values, taken to the fourth order only, are as follows; for greater simplicity I write $a = 1$; and, instead of $nt +$ constant, $cnt +$ constant, $gnt +$ constant, I write l, c, g respectively; viz., l is the mean longitude, c the mean anomaly, g the mean distance from node: this being so, then r, v, y, denoting the radius vector longitude and latitude respectively, we have

$$\frac{1}{r}\,(\text{Plana}) =$$

$$\begin{array}{lllc}
 & e - \frac{1}{8}e^3 - \frac{1}{4}\gamma^2 e & \cos & c \\
+ & e^2 - \frac{1}{3}e^4 - \frac{1}{2}\gamma^2 e^2 & \text{,,} & 2c \\
+ & \frac{9}{8}e^3 & \text{,,} & 3c \\
+ & \frac{4}{3}e^4 & \text{,,} & 4c \\
- & \frac{5}{4}\gamma^2 e^2 & \text{,,} & 2g \\
- & \frac{5}{8}\gamma^2 e & \text{,,} & c - 2g
\end{array}$$

(but I omit Plana's term $+\ \frac{1}{8}\gamma^2 e^2 \quad \cos \quad 2c + 2g$ which should be $= 0$).

$$v\,(\text{Plana}) = l +$$

$$\begin{array}{lllc}
+ & 2\ e - \frac{1}{4}e^3 - \frac{1}{2}\gamma^2 e & \sin & c \\
+ & \frac{5}{4}e^2 - \frac{11}{24}e^4 - \frac{15}{16}\gamma^2 e^2 & \text{,,} & 2c \\
+ & \frac{13}{12}e^3 & \text{,,} & 3c \\
+ & \frac{103}{96}e^4 & \text{,,} & 4c \\
- & \frac{1}{4}\gamma^2 - \frac{9}{16}\gamma^2 e^2 + \frac{1}{8}\gamma^4 & \text{,,} & 2g \\
+ & \frac{3}{4}\gamma^2 e & \text{,,} & c - 2g
\end{array}$$

$$\begin{array}{llr}
-\tfrac{1}{2}\,\gamma^2 e & \sin & c+2g \\
-\tfrac{1}{8}\,\gamma^2 e^2 & ,, & 2c-2g \\
-\tfrac{13}{16}\,\gamma^2 e^2 & ,, & 2c+2g \\
+\tfrac{1}{32}\,\gamma^4 & ,, & 4g.
\end{array}$$

y (Plana) =

$$\begin{array}{llr}
\gamma \;-\; \gamma e^2 - \tfrac{3}{8}\gamma^3 & \sin & g \\
+\;\gamma e \;-\; \tfrac{5}{8}\,\gamma e^3 & ,, & c-g \\
+\;\gamma e \;-\; \tfrac{5}{4}\,\gamma e^3 - \tfrac{5}{8}\gamma^3 e & ,, & c+g \\
+\tfrac{3}{4}\,\gamma e^2 & ,, & 2c-g \\
+\tfrac{9}{8}\,\gamma e^2 & ,, & 2c+g \\
+\tfrac{17}{24}\,\gamma e^3 & ,, & 3c-g \\
+\tfrac{4}{3}\,\gamma e^3 & ,, & 3c+g \\
-\tfrac{1}{24}\,\gamma^3 & ,, & 3g \\
+\tfrac{1}{2}\,\gamma^3 e & ,, & c-3g \\
-\tfrac{1}{8}\,\gamma^3 e & ,, & c+3g.
\end{array}$$

To compare these with the elliptic values, it is necessary to write $e(1+\frac{1}{4}\gamma^2)$ in place of e. Making this change, or say reducing Plana's (e, γ) to the elliptic (e, γ), I write down in a first column the transformed coefficients, and in a second column the elliptic coefficients, as follows:

Plana, with Elliptic e, γ	Elliptic		
$\frac{1}{r} =$	$\frac{1}{r} =$		
1	1		
$+\;e - \frac{1}{8}e^3$	$+\;e - \frac{1}{8}e^3$	cos	c
$+\;e^2 - \frac{1}{3}e^4$	$+\;e^2 - \frac{1}{3}e^4$	,,	$2c$
$+\frac{9}{8}e^3$	$+\frac{9}{8}e^3$	,,	$3c$
$+\frac{4}{3}e^4$	$+\frac{4}{3}e^4$	,,	$4c$
$-\frac{5}{4}\gamma^2 e^2$	0	,,	$2g$
$-\frac{5}{8}\gamma^2 e$	0	,,	$c-2g.$

Plana, with Elliptic e, γ	Elliptic		
$v =$	$v =$		
l	l		
$+\;2\;\;e - \frac{1}{4}e^3$	$+\;2\;\;e - \frac{1}{4}e^3$	sin	c
$+\frac{5}{4}\;\;e^2 - \frac{11}{24}e^4 - \frac{5}{16}\gamma^2 e^2$	$+\;\frac{5}{4}\;\;e^2 - \frac{11}{24}e^4$	,,	$2c$
$+\frac{13}{12}\;e^3$	$+\frac{13}{12}\;e^3$	,,	$3c$
$+\frac{103}{96}\;e^4$	$+\frac{103}{96}\;e^4$	,,	$4c$
$-\;\frac{1}{4}\;\;\gamma^2 - \frac{9}{16}\gamma^2 e^2 + \frac{1}{8}\gamma^4$	$-\;\frac{1}{4}\;\;\gamma^2 + \gamma^2 e^2 + \frac{1}{8}\gamma^4$	,,	$2g$
$+\;\frac{3}{4}\;\;\gamma^2 e$	$-\;\frac{1}{2}\;\;\gamma^2 e$	,,	$c-2g$
$-\;\frac{1}{2}\;\;\gamma^2 e$	$-\;\frac{1}{2}\;\;\gamma^2 e$	,,	$c+2g$
$-\;\frac{1}{8}\;\;\gamma^2 e^2$	$+\;\frac{3}{16}\;\;\gamma^2 e^2$	,,	$2c-2g$
$-\;\frac{13}{16}\;\;\gamma^2 e^2$	$-\;\frac{13}{16}\;\;\gamma^2 e^2$	,,	$2c+2g$
$+\;\frac{1}{32}\;\;\gamma^4$	$+\;\frac{1}{32}\;\;\gamma^4$	,,	$4g.$

Plana, with Elliptic e, γ	Elliptic	
$y = \gamma - \gamma e^2 - \frac{3}{8}\gamma^3$	$y = +\gamma e - \gamma e^2 - \frac{3}{8}\gamma^3$	$\sin\ g$
$+\ \gamma e - \frac{5}{4}\gamma e^3 - \frac{3}{8}\gamma^3 e$	$+\ \gamma e - \frac{5}{4}\gamma e^3 - \frac{3}{8}\gamma^3 e$	„ $c+g$
$+\ \gamma e - \frac{5}{8}\gamma e^3 + \frac{1}{4}\gamma^3 e$	$+\ \gamma e \qquad - \frac{3}{8}\gamma^3 e$	„ $c-g$
$+\frac{3}{4}\gamma e^2$	$+\frac{1}{8}\gamma e^2$	„ $2c-g$
$+\frac{9}{8}\gamma e^2$	$+\frac{9}{8}\gamma e^2$	„ $2c+g$
$+\frac{17}{24}\gamma e^3$	$+\frac{1}{12}\gamma e^3$	„ $3c-g$
$+\frac{4}{3}\gamma e^3$	$+\frac{4}{3}\gamma e^3$	„ $3c+g$
$-\frac{1}{24}\gamma^3$	$-\frac{1}{24}\gamma^3$	„ $3g$
$+\frac{1}{2}\gamma^3 e$	$-\frac{1}{8}\gamma^3 e$	„ $c-3g$
$-\frac{1}{8}\gamma^3 e$	$-\frac{1}{8}\gamma^3 e$	„ $c+3g$,

where, for greater clearness, I remark that the values called "elliptic" of e, γ, c, g, refer to an ellipse, such that the longitude of the node, and the longitude (in orbit) of the pericentre, vary uniformly with the time,—viz., we have mean distance $=1$, excentricity $=e$, tangent of inclination $=\gamma$, mean longitude $=l$, mean anomaly $=c$, distance from node $=g$.

We have therefore

$$\begin{array}{rlll}
\delta\frac{1}{r} = & -\frac{5}{4}\gamma^2 e^2 & \cos & 2g \\
 & -\frac{5}{8}\gamma^2 e & \text{„} & c-2g \\
\delta v = & -\frac{5}{16}\gamma^2 e^2 & \sin & 2c \\
 & -\frac{25}{16}\gamma^2 e^2 & \text{„} & 2g \\
 & +\frac{5}{4}\gamma^2 e & \text{„} & c-2g \\
 & -\frac{5}{16}\gamma^2 e^2 & \text{„} & 2c-2g \\
\delta y = & -\frac{5}{8}\gamma e^3 + \frac{5}{8}\gamma^3 e & \text{„} & c-g \\
 & +\frac{5}{8}\gamma e^2 & \text{„} & 2c-g \\
 & +\frac{5}{8}\gamma e^3 & \text{„} & 3c-g \\
 & +\frac{5}{8}\gamma^3 e & \text{„} & c-3g,
\end{array}$$

viz., these are the increments to be added to the elliptic values of $\frac{1}{r}$, v, y, respectively, in order to obtain the disturbed values of $\frac{1}{r}$, v, y, attending only to the coefficients independent of m; *they represent, in fact, the lunar inequalities which rise two orders by integration.*

The elliptic values of $\frac{1}{r}$ and y are functions, and that of v, is equal $l+$, a function, of e, γ, c, g, and the foregoing disturbed values may be obtained by affecting each of the quantities e, γ, c, g, and l, with an inequality depending on the argument $2c-2g$, viz., these inequalities are

$$\delta e = -\tfrac{5}{8}\gamma^2 e \cos 2c-2g$$
$$\delta c = \tfrac{5}{8}\gamma^2 \sin 2c-2g$$
$$\delta\gamma = \tfrac{5}{8}\gamma e^2 \cos 2c-2g$$
$$\delta g = \tfrac{5}{8} e^2 \sin 2c-2g$$
$$\delta l = -\tfrac{5}{16}\gamma^2 e^2 \sin 2c-2g.$$

The verification may be effected without difficulty; thus, for instance, starting from the elliptic value of $\frac{1}{r}$, we have to the fourth order

$$\delta\frac{1}{r} = \delta\begin{pmatrix} e^2\cos c \\ +e\cos 2c\end{pmatrix} = \begin{pmatrix} -\ e\sin c \\ -2e^2\sin 2c\end{pmatrix}\delta c + \begin{pmatrix} \cos c \\ +2e\cos 2c\end{pmatrix}\delta e$$
$$= \tfrac{5}{8}\gamma^2 e\,(-\sin c\sin 2c-2g - \cos c\cos 2c-2g)$$
$$+\tfrac{5}{4}\gamma^2 e^2(-\sin 2c\sin 2c-2g - \cos 2c\cos 2c-2g)$$
$$= -\tfrac{5}{8}\gamma^2 e\cos c-g$$
$$-\tfrac{5}{4}\gamma^2 e^2\cos 2g,$$

which is right; and the verification of the values of δv, δy, may be effected in a similar manner.

I have, in order to fix the ideas, preferred to give in the first instance the foregoing *à posteriori* proof; but I now inquire generally as to the form of the values of $\frac{1}{r}$, v, y, or say of r, v, y, taking account only of coefficients independent of m; and I proceed to show that these may be obtained from the elliptic values expressed as above in terms of l, e, γ, c, g, by affecting l, e, γ, c, g, each with an inequality depending on the multiple sines or cosines of $c-g$.

Writing for greater simplicity $n=1$, we have $l=t+L$, $c=\mathrm{c}t+C$, $g=\mathrm{g}t+G$, where $\mathrm{c}=1-\tfrac{3}{4}m^2+\&c.$, $\mathrm{g}=1+\tfrac{3}{4}m^2+\&c.$; viz., c, g, are constants which differ from unity by terms involving m^2.

The required values of r, v, y, satisfy the *undisturbed* equations of motion, if after the differentiations we write in the coefficients (which coefficients are functions of m through c, g) $m=0$; that is, if we write in the coefficients $\mathrm{c}=1$, $\mathrm{g}=1$. In fact, the required values of r, v, y, are what the complete values become, upon writing in the coefficients of the complete values $m=0$; that is, the required values of r, v, y, differ from the complete values by terms the coefficients whereof contain m as a factor; and the disturbed equations differ from the undisturbed equations in that they contain the differential coefficients of the disturbing function; that is, terms the coefficients whereof have the factor m^2. Imagine the complete values of r, v, y, substituted in the disturbed equations of motion; the resulting equations are satisfied identically; and, therefore, whatever be the value of m; that is, they are satisfied if in these equations respectively we write $m=0$: it requires a little consideration to see that this is so, if *in the coefficients only* we write $m=0$; but recollecting that c, g, stand for functions $\mathrm{c}t+C$, $\mathrm{g}t+G$, so that, for example, $c-g$, $=(\mathrm{c}-\mathrm{g})t+C-G$, upon writing therein $m=0$, becomes equal, not to zero, but to the constant value $C-G$, the identity subsists in regard to the coefficient of the sine or cosine of each separate argument $\alpha c+\beta g$, and, consequently, it subsists notwithstanding that in the arguments c and g, instead of being each put $=1$, are left indeterminate. And granting this (viz. that the equations are satisfied if *in the coefficients only* we write $m=0$), then it is clear that, as above stated, the required values of r, v, y, satisfy the undisturbed equations of motion, if after the differentiations we write in the coefficients $\mathrm{c}=1$, $\mathrm{g}=1$.

The required values of r, v, y, are of the form $r=\phi(c, g)$, $y=\psi(c, g)$, $v=l+\chi(c, g)$, but writing $w=v+c-l$, $=c+\chi(c, g)$, the last mentioned property will equally subsist in regard to the functions r, w, y: in fact, v enters into the differential equations only through its differential coefficient $\frac{dv}{dt}$, and the differential coefficients of v and w, that is, of $l+\chi(c, g)$ and $c+\chi(c, g)$, differ only by the quantity $\mathrm{c}-1$, which becomes $=0$, in virtue of the assumed relations $\mathrm{c}=1$, $\mathrm{g}=1$.

Hence the undisturbed equations are satisfied by the values $r=\phi(c, g)$, $y=\psi(c, g)$, $w=c+\chi(c, g)$, when after the differentiations we write in the coefficients $\mathrm{c}=1$, $\mathrm{g}=1$; the foregoing values contain t through the quantities c, g, only; and we have, therefore, $\frac{d}{dt}=\mathrm{c}\frac{d}{dc}+\mathrm{g}\frac{d}{dg}$.

Hence, writing in the coefficients $\mathrm{c}=1$, $\mathrm{g}=1$, we have $\frac{d}{dt}=\frac{d}{dc}+\frac{d}{dg}$; that is, the values $r=\phi(c, g)$, $y=\psi(c, g)$, $w=\chi(c, g)$, regarding r, v, y, as functions of c, g, satisfy the partial differential equations obtained from the undisturbed equations of motion by writing therein $\frac{d}{dc}+\frac{d}{dg}$ in place of $\frac{d}{dt}$. Hence also, considering r, w, y, as functions of c and $c-g$, then observing that $\left(\frac{d}{dc}+\frac{d}{dg}\right)(c-g)$ is $=0$, the values of r, v, y, satisfy the partial differential equations obtained by writing $\frac{d}{dc}$ in place of $\frac{d}{dt}$; and inasmuch as these partial differential equations do not contain $\frac{d}{dg}$, they are to be integrated as ordinary differential equations in regard to c as the independent variable, the constants of integration being replaced by arbitrary functions of $c-g$.

Consider the pure elliptic values of r, v, y, in an elliptic orbit with the following elements: A, the mean distance; N, the mean motion ($N^2A^3=1$ and therefore $A=N^{-\frac{2}{3}}$); E, the excentricity; $Nt+D$, the mean anomaly; $Nt+H$, the mean distance from node; $Nt+K$, the mean longitude; then writing c in place of t, we have

$$\begin{aligned} r \quad &= N^{-\frac{2}{3}}\,\mathrm{elqr}\,(E,\ Nc+D), \\ v\,(=l-c+w) &= l-c+Nc+K+P\,(E,\ \Gamma,\ Nc+D,\ Nc+H), \\ y \quad &= \qquad\qquad\qquad\qquad Q\,(E,\ \Gamma,\ Nc+D,\ Nc+H), \end{aligned}$$

where N, E, Γ, D, H, K, are arbitrary functions of $c-g$: P and Q denote given functional expressions. But, in order that r, v, y, considered as functions of c and g may be of the proper form, it is necessary as regards N to write simply $N=1$; we have then

$$r = \text{elqr}\,(E,\ c+D),$$
$$v = l + K + P\,(E,\ \Gamma,\ c+D,\ c+H),$$
$$y = \quad Q\,(E,\ \Gamma,\ c+D,\ c+H),$$

where E, Γ, D, H, K, are arbitrary functions of $c-g$; or, what is the same thing, writing for these quantities respectively $e+\delta e$, $\gamma+\delta\gamma$, δc, $g-c+\delta g$, δl, where δe, $\delta\gamma$, δc, δg, δl are arbitrary functions of $c-g$, we have

$$r = \text{elqr}\,(e+\delta e,\ c+\delta c),$$
$$v = l + \delta l + P\,(e+\delta e,\ \gamma+\delta\gamma,\ c+\delta c,\ g+\delta g),$$
$$y = \quad Q\,(e+\delta e,\ \gamma+\delta\gamma,\ c+\delta c,\ g+\delta g),$$

that is, the values of r, v, y, are obtained from the elliptic values

$$r = \text{elqr}\,(e,\ c),$$
$$v = l + P\,(e,\ \gamma,\ c,\ g),$$
$$y = \quad Q\,(e,\ \gamma,\ c,\ g),$$

by affecting each of the quantities e, γ, c, g, l, with an inequality which is a function of $c-g$.