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NOTE ON PLANA'S LUNAR THEORY.

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I HAVE been much surprised to find that there is an error of the order $m^2\gamma^4$, arising from the omission of a factor $(1+\gamma^2)^{-1}$, in the expression for $\frac{d^2\delta u}{dv^2}+\delta u$, as given by the equation (II.)′ (*Théorie de la Lune*, t. I., p. 267), being the equation made use of in the theory for the determination of δu, the perturbation of the reciprocal of the radius vector. This error may probably be the cause of some of the discrepancies in the terms of the fourth and higher orders, between Plana's results and those of Pontécoulant and Delaunay.

Plana's equation (6), t. I., p. 260, is

$$\begin{aligned}\frac{d^2\delta u}{dv^2}+\delta u = \ & \mathrm{a}R'' \\ & +f(e,\ \gamma)\,Q'e\cos\left(cv-\int\varpi dv\right) \\ & -\left\{f(e,\ \gamma)\,(1+\gamma^2)\,(1+s_,^2)^{-\frac{3}{2}}-\frac{\mathrm{a}\,(1+s^2)^{-\frac{3}{2}}}{\mathrm{a}_,\psi\,(e,\ \gamma)}\right\} \\ & +f(e,\ \gamma)\,P\gamma^2\,(1+s_,^2)^{-\frac{3}{2}}\,\Theta,\end{aligned}$$

if for shortness

$$\Theta=\tfrac{3}{8}\gamma^2-(1+\tfrac{1}{2}\gamma^2)\cos\left(2gv-2\int\theta dv\right)+\tfrac{1}{8}\gamma^2\cos\left(4gv-4\int\theta dv\right).$$

R'' (p. 256) should be

$$R''=\frac{1}{\sigma\mathrm{a}_,\psi\,(e,\ \gamma)}\,\frac{\Omega_2-\dfrac{du}{dv}\dfrac{u^2dv}{d\Omega}-\sigma\,(1+s^2)^{-\frac{3}{2}}\,2\displaystyle\int Udv}{1+2\displaystyle\int Udv},$$

but, by an error which is implicitly corrected, the σ which multiplies $(1+s^2)^{-\frac{3}{2}}\, 2\int Udv$ is omitted. Hence the equation (6) becomes

$$\left(1+2\int Udv\right)\left(\frac{d^2\delta u}{dv^2}+\delta u\right)=\frac{\mathrm{a}}{\sigma \mathrm{a}_{,}\, \psi(e,\gamma)}\left\{\Omega_2-\frac{du}{dv}\,\frac{d\Omega}{u^2 dv}-\sigma(1+s^2)^{-\frac{3}{2}}\, 2\int Udv\right\}$$

$$+\left(1+2\int Udv\right) f(e,\gamma)\, Q'e\cos\left(cv-\int \varpi dv\right)$$

$$-\left(1+2\int Udv\right)\left\{f(e,\gamma)(1+\gamma^2)(1+s_{,}^2)^{-\frac{3}{2}}-\frac{\mathrm{a}(1+s^2)^{-\frac{3}{2}}}{\mathrm{a}_{,}\psi(e,\gamma)}\right\}$$

$$+\left(1+2\int Udv\right) f(e,\gamma)\, P\gamma^2(1+s_{,}^2)^{-\frac{3}{2}}\,\Theta,$$

in which equation

$$\Omega_2=\frac{d\Omega}{du}+\frac{s}{u}\frac{d\Omega}{ds},\ \text{pp. 26, 245},\quad \frac{d\Omega}{u^2 dv}=\frac{\sigma \mathrm{a}_{,}}{\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}}U,\ \text{p. 265},$$

$$\psi(e,\gamma)=\lambda^{-\frac{2}{3}}(1+\gamma^2)^{-\frac{1}{3}},\quad f(e,\gamma)=\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{1}{3}},\ \text{p. 261.}$$

But retaining for greater convenience the function $f(e,\gamma)$ in two of the terms, we have

$$\left(1+2\int Udv\right)\left(\frac{d^2\delta u}{dv^2}+\delta u\right)=$$

$$\frac{\mathrm{a}}{\sigma \mathrm{a}_{,}}\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{1}{3}}\left\{\frac{d\Omega}{du}+\frac{s}{u}\frac{d\Omega}{ds}-\frac{\sigma \mathrm{a}_{,}}{\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}}U\frac{du}{dv}-\sigma(1+s^2)^{-\frac{3}{2}}\, 2\int Udv\right\}$$

$$+\left(1+2\int Udv\right) f(e,\gamma)\, Q'e\cos\left(cv-\int \varpi dv\right)$$

$$-\left(1+2\int Udv\right)\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}\left\{(1+s_{,}^2)^{-\frac{3}{2}}-\frac{\mathrm{a}}{\mathrm{a}_{,}}(1+s^2)^{-\frac{3}{2}}\right\}$$

$$+\left(1+2\int Udv\right) f(e,\gamma)\, P\gamma^2(1+s_{,}^2)^{-\frac{3}{2}}\,\Theta$$

$$=\frac{\mathrm{a}}{\sigma \mathrm{a}_{,}}\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{1}{3}}\left(\frac{d\Omega}{du}+\frac{s}{u}\frac{d\Omega}{ds}\right)$$

$$-\mathrm{a}U\frac{du}{dv}$$

$$-\frac{\mathrm{a}}{\mathrm{a}_{,}}\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}(1+s^2)^{-\frac{3}{2}}\, 2\int Udv$$

$$+\left(1+2\int Udv\right) f(e,\gamma)\, Q'e\cos\left(cv-\int \varpi dv\right)$$

$$-\left(1+2\int Udv\right)\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}\left\{(1+s_{,}^2)^{-\frac{3}{2}}-\frac{\mathrm{a}}{\mathrm{a}_{,}}(1+s^2)^{-\frac{3}{2}}\right\}$$

$$+\left(1+2\int Udv\right) f(e,\gamma)\, P\gamma^2(1+s_{,}^2)^{-\frac{3}{2}}\,\Theta,$$

which is

$$= \frac{\mathrm{a}}{\sigma \mathrm{a}_{,}} \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} \left(\frac{d\Omega}{du} + \frac{s}{u}\frac{d\Omega}{dv}\right)$$

$$- \mathrm{a} U \frac{du}{dv}$$

$$- \frac{\mathrm{a}}{\mathrm{a}_{,}} \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} (1+s^2)^{-\frac{3}{2}}\, 2 \int U dv \text{ (destroyed by term } \textit{infrà}\text{)}$$

$$+ \left(1 + 2\int U dv\right) f(e,\gamma)\, Q' e \cos\left(cv - \int \varpi dv\right)$$

$$- \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} \left\{(1+s_{,}^2)^{-\frac{3}{2}} - \frac{\mathrm{a}}{\mathrm{a}_{,}} (1+s^2)^{-\frac{3}{2}}\right\}$$

$$- \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} (1+s_{,}^2)^{-\frac{3}{2}}\, 2 \int U dv$$

$$+ \frac{\mathrm{a}}{\mathrm{a}_{,}} \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} (1+s^2)^{-\frac{3}{2}}\, 2 \int U dv \text{ (destroyed by term } \textit{suprà}\text{)}$$

$$+ \left(1 + 2\int U dv\right) f(e,\gamma)\, P\gamma^2 (1+s_{,}^2)^{-\frac{3}{2}}\, \Theta;$$

or, putting $u = \frac{1}{\mathrm{a}}(u_{,} + \delta u)$, this becomes

$$\left(1 + 2\int U dv\right)\left(\frac{d^2\delta u}{dv^2} + \delta u\right) = \frac{\mathrm{a}}{\sigma \mathrm{a}_{,}} \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} \left(\frac{d\Omega}{du} + \frac{s}{u}\frac{d\Omega}{dv}\right)$$

$$- U\left(\frac{du_{,}}{dv} + \frac{d\delta u}{dv}\right)$$

$$- \lambda^{\frac{2}{3}} (1+\gamma)^{\frac{4}{3}} (1+s_{,}^2)^{-\frac{3}{2}}\, 2 \int U dv$$

$$+ \left(1 + 2\int U dv\right) f(e,\gamma)\, Q' e \cos\left(cv - \int \varpi dv\right)$$

$$- \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} \left\{(1+s_{,}^2)^{-\frac{3}{2}} - \frac{\mathrm{a}}{\mathrm{a}_{,}} (1+s^2)^{-\frac{3}{2}}\right\}$$

$$+ \left(1 + 2\int U dv\right) f(e,\gamma)\, P\gamma^2 (1+s_{,}^2)^{-\frac{3}{2}}\, \Theta,$$

agreeing with the Formula II. p. 265, except that in Plana's last term, instead of the factor $f(e, \gamma) \left(= \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{1}{3}}\right)$, we have the factor $\lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}}$. That is, the last term, as given by Plana, should be divided by $1+\gamma^2$. And this error is introduced from the formula (II.) into the formula (II.)′, p. 267, viz. the incorrect factor $\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{1}{3}}$ is there replaced by its value q; whereas, the true value being $\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}$, the factor in (II.)′ should be $=\dfrac{q}{1+\gamma^2}$.

The corrected formula (II.)′ is

$$\begin{aligned}
-\frac{d^2\delta u}{dv^2}-\delta u = &-Q'\frac{qe}{1+\gamma^2}\cos\left(cv-\int\varpi dv\right)\\
&+q\left\{(1+s_{,}^2)^{-\frac{3}{2}}-\frac{\mathrm{a}}{\mathrm{a}_{,}}(1+s^2)^{-\frac{3}{2}}\right\}\\
&+\mu^2(R_4+R_5)-\mu^2\left(\frac{du_{,}}{dv}+\frac{d\delta u}{dv}\right)R_{,}\\
&-2\mu^2\left\{\frac{d^2\delta u}{dv^2}+\delta u+q(1+s_{,}^2)^{-\frac{3}{2}}-\frac{Q'qe}{1+\gamma^2}\cos\left(cv-\int\varpi dv\right)\right\}\int R_{,}dv\\
&-Pq\gamma^3(1+\gamma^2)^{-1}(1+s_{,}^2)^{-\frac{3}{2}}\left(1-2\mu^2\int R_{,}dv\right)\times\\
&\left\{\tfrac{3}{8}\gamma^2-(1+\tfrac{1}{2}\gamma^2)\cos\left(2gv-2\int\theta dv\right)+\tfrac{1}{8}\gamma^2\cos\left(4gv-4\int\theta dv\right)\right\}.
\end{aligned}$$

Observing that P is of the order m^2, and that q is approximately equal to unity, the error in $\dfrac{d^2\delta u}{dv^2}+\delta u$ is of the order $m^2\gamma^4$, as noticed above. It may be right to mention that I obtained the correction in the first instance by starting from the fundamental equations, and not as here from the intermediate equation (6), so that there is not in that equation any error afterwards implicitly corrected in the transformation to (II.)′.