457.

ON THE QUARTIC SURFACES $(*\between U, V, W)^2 = 0$.

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I PROPOSE to myself for investigation the quartic surfaces represented by an equation

$$(*\between U, V, W)^2 = 0,$$

where U, V, W are quadric functions of the coordinates.

Such a surface has 8 nodes (conical points), viz., these are the points of intersection of the quadric surfaces $U=0$, $V=0$, $W=0$. It is to be observed, that not every quartic surface with 8 nodes is included under the above form; in fact the equation of a quartic surface contains (homogeneously) 35 coefficients, or say 34 arbitrary parameters; in order that a given point may be a node, 4 conditions must be satisfied, and it is consequently possible to find a quartic surface having 8 given points as nodes (and having in its equation $(34 - 8 \, . \, 4 =)$ 2 arbitrary parameters): but 8 given points are not in general the intersections of three quadric surfaces, and such a quartic surface is therefore not in general included under the above form. I think, however, that it may be assumed that the above form includes all the quartic surfaces having 8 nodes, points of intersection of three quadric surfaces. It will presently appear that, included in the form, we have surfaces where (instead of the 8 nodes) there is a nodal or cuspidal conic; and that these are the most general forms of such quartic surfaces. A quartic surface has at most 16 nodes, and the general form with 8 nodes must admit of being particularised so that the surface shall acquire any number not exceeding 8 of additional nodes. This does not show, but it is probable, that the above special form with 8 nodes can be particularised so that the surface shall in like manner acquire any number not exceeding 8 of additional nodes. Similarly, a quartic surface with a nodal conic may have besides 1, 2, 3, or 4 nodes; and it will be shown in the sequel how the form, particularised so as to give a nodal conic, may be further particularised so as to give the 1, 2, 3, or 4 nodes. So a quartic surface with a cuspidal conic may besides have 1 node, and it will be shown how the form, particularised so as to give a cuspidal conic, may be further particularised so as to give 1 node.

Starting from the equation $(*\between U, V, W)^2 = 0$, we may, by substituting for U, V, W linear functions of these expressions, transform the equation precisely in the manner of a conic, and therefore into any of the forms under which the equation of a conic can be exhibited; for instance, in the forms $aU^2 + bV^2 + cW^2 = 0$, $fVW + gWU + hUV = 0$, $UW - V^2 = 0$, &c. I attend at present only to the last-mentioned form $UW - V^2 = 0$, which, it thus appears, is equally general with the original form $(*\between U, V, W)^2 = 0$.

The quartic surface

$$UW - V^2 = 0$$

where U, V, W are any quadric functions of the coordinates, may be considered as the envelope of the quadric surface

$$(U, V, W\between \theta, 1)^2 = 0,$$

where θ is an arbitrary parameter. And it thus appears that it is very easy to reciprocate (in regard to any given quadric surface) the quartic surface. For the reciprocal of the quartic surface is clearly the envelope of the reciprocal of the variable quadric surface; this reciprocal is itself a quadric surface, and the reciprocal of the quartic surface is thus given in the same form as the original surface, viz., as the envelope of a quadric surface the equation whereof contains rationally the variable parameter θ; the equation of the reciprocal surface is consequently obtained by equating to zero the discriminant in regard to θ, of the equation of the reciprocal quadric surface.

It is to be observed that, inasmuch as the equation of the reciprocal quadric surface is of the third degree in the coefficients of the original quadric, it is in general of the degree 6 in the parameter θ; we have thus a sextic function of θ, the coefficients whereof are quadric functions of the coordinates; and the discriminant is a function of the order 10 in these coefficients, that is, of the order 20 in the coordinates. The reciprocal of the quartic surface is thus a surface of the order 20; this is right, for in a general quartic surface the order of the reciprocal surface is $= 36$, and the 8 nodes reduce the order by 16; $36 - 16 = 20$.

In the equation $UW - V^2 = 0$, or say $V^2 - UW = 0$; if U reduce itself to the square of a linear function, $U = P^2$, the equation becomes $V^2 - P^2W = 0$, which is the general form of the quartic surface having the nodal conic $V = 0$, $P = 0$. And if, moreover, W be the product of this same linear function P by another linear function Q, $W = PQ$, then the form is $V^2 - P^3Q = 0$, which is the general form of the quartic surface having the cuspidal conic $V = 0$, $P = 0$.

Writing for greater convenience x, y in the place of P, Q respectively, we have the quartic

$$(AA) \qquad V^2 - x^3y = 0,$$

having the cuspidal conic $V=0$, $x=0$; and which has besides the conic of plane contact $V=0$, $y=0$. In virtue of the cuspidal conic the reciprocal surface should be of the order 6; and by the foregoing method of obtaining the equation of the reciprocal surface, I will verify that this is so. To effect this as simply as possible, I fix the remaining coordinates z, w as follows. The line $x=0$, $y=0$ is not in general a tangent to the surface $V=0$; it therefore meets this surface in two points, and we may take $z=0$, $w=0$ to be the equations of the tangent planes at these two points respectively; we have thus $V=ax^2+2hxy+by^2+2nzw$. Introducing for convenience the numerical factor 2, and taking the equation of the surface to be

$$(ax^2+2hxy+by^2+2nzw)^2-2x^3y=0,$$

this is the envelope of the quadric surface

$$\theta^2x^2+2\theta(ax^2+2hxy+by^2+2nzw)+2xy=0,$$

which is a surface $(a, b, c, d, f, g, h, l, m, n\between x, y, z, w)^2=0$, where $a=\theta^2+2a\theta$, $b=2\theta b$, $h=2\theta h+1$, $n=2\theta n$, and where all the other coefficients vanish. Assuming, as usual, that the reciprocation is effected in regard to the surface $x^2+y^2+z^2+w^2=0$, the general equation is

$$\begin{aligned}
& x^2 . d(bc-f^2)-cm^2-bn^2+2fmn\\
+\ & y^2 . d(ca-g^2)-an^2-cl^2+2gnl\\
+\ & z^2 . d(ab-h^2)-bl^2-am^2+2hlm\\
+\ & w^2 . abc-af^2-bg^2-ch^2+2fgh\\
& +2yz . d(gh-af)+l^2f+amn-hnl-glm\\
& +2zx . d(hf-bg)+m^2g+bnl-flm-hmn\\
& +2xy . d(fg-ch)+n^2h+clm-gmn-fnl\\
& +2xw . -l(bc-f^2)-n(hf-bg)-m(fg-ch)\\
& +2yw . -m(ca-g^2)-l(fg-ch)-n(gh-af)\\
& +2zw . -n(ab-h^2)-m(gh-af)-l(hf-bg),
\end{aligned}$$

(I write down this general result as it will be useful for reference in other cases); in the present case this becomes simply

$$x^2 . -bn^2+y^2 . -an^2+2xy . nh^2+2zw(-nab+nh^2)=0,$$

where a, b, n, h have the foregoing values; the equation is thus only of the order 4 in regard to θ; but it in fact divides by $n\,(=2\theta n)$ and thus reduces itself to the third order, viz. it becomes

$$n(bx^2+ay^2)-2h^2xy+2(ab-h^2)zw=0,$$

or, substituting for a, b, n, h their values, this is

$$x^2 . 4bn\theta^2+y^2(2n\theta^3+4an\theta^2)+2zw(2b\theta^3+4ab\theta^2)-2(xy+zw)(2\theta h+1)^2=0,$$

say this is

$$(A, B, C, D\between\theta, 1)^3=0,$$

where A, B, C, D are each of them a quadric function of the coordinates; it being observed that C and D are respectively numerical multiples of the same function $xy+zw$. Hence equating the discriminant to zero, we have

$$A^2D^2+4AC^3+4B^3D-3B^2C^2-6ABCD=0,$$

which equation, inasmuch as every term contains either C or D as a factor, divides by $xy+zw$, and thus becomes an equation of the order 6 in the coordinates: that is the order of the reciprocal surface is $=6$. Multiplying by $\frac{3}{2}$ to avoid fractions, the actual values of A, B, C, D are

$$\begin{cases} A = \ \ 3\ \ (ny^2\ \ +2bzw), \\ B = \ \ 2\ \ \{bnx^2+any^2+2abzw-2h^2(xy+zw)\}, \\ C=-4h\,(xy\ \ +zw), \\ D=-3\ \ (xy\ \ +zw), \end{cases}$$

or say $A=3\alpha$, $B=2\beta$, $C=-4h\gamma$, $D=-3\gamma$; where $\gamma=xy+zw$; substituting these values and omitting the factor 3γ, the equation is

$$27\alpha^2\gamma-256h^3\alpha\gamma^2-32\beta^3-64h^2\beta^2\gamma-144h\alpha\beta\gamma=0,$$

which is an equation of the form $(*\between\alpha, \beta, \gamma)^3=0$. The sextic surface has thus singular points $\alpha=0, \beta=0, \gamma=0$, viz. these are the two points $(x=0, y=0, z=0)$, $(x=0, y=0, w=0)$ each four times. The further discussion of the sextic surface is reserved for another occasion.

I do not at present attempt to enumerate the particular cases of the surface $V^2-x^3y=0$, but content myself with the discussion of a particular case in which the order of the reciprocal surface is $=3$. Suppose that $V=0$ is a cone, $y=0$ a tangent plane to the cone (so that the conic $y=0$, $V=0$ breaks up into a line twice repeated), $x=0$ an arbitrary plane (so that we have still the proper cuspidal conic $x=0$, $V=0$). Any other tangent plane of the cone may be taken for the plane $z=0$; the plane containing the lines of contact of the two tangent planes for the plane $w=0$; the equation of the conic then is $V=dw^2+2fyz=0$; and the equation of the surface is

$$(AB) \qquad (dw^2+2fyz)^2-x^3y=0.$$

For convenience of comparison, I change x, y, w, z into y, w, z, x, and assign numerical values to the coefficients, writing the equation under the form

$$(AB) \qquad 27\,(4xw+z^2)^2-64y^3w=0.$$

The quartic is here the envelope of the quadric surface

$$\theta^2 . 4yw+\theta . 9\,(4xw+z^2)+12y^2=0,$$

viz., comparing with the general form

$$(a, b, c, d, f, g, h, l, m, n\between x, y, z, w)^2=0,$$

we have $b=12$, $c=9\theta$, $l=18\theta$, $m=2\theta^2$, all the other coefficients vanishing. The reciprocal equation is

$$x^2(-cm^2)+y^2(-cl^2)+z^2(-bl^2)+2xy\,.\,clm+2xw(-lbc)=0,$$

or substituting for b, c, l, m their values, this is found to be

$$\theta(\theta x-9y)^2+108(z^2+xw)=0.$$

Representing this by $(A, B, C, D\between\theta, 1)^3=0$, the discriminant in regard to θ would, in virtue of the values of A, B, C, contain D as a factor; the reason of this appears from the original form; in fact, forming the derived equation in regard to θ, this is found to be $(\theta x-9y)(\theta x-3y)=0$; the value $\theta x-9y=0$ gives as a factor of the discriminant z^2+xw; the value $\theta x-3y=0$ gives $3y(-6y)^2+108x(z^2+xw)$, that is the factor $y^3+x(z^2+xw)$; the complete value of the discriminant as obtained by substitution of the values of A, B, C, D being $x^3(z^2+xy)\{y^3+x(z^2+xw)\}$; the equation of the reciprocal surface is

$$y^3+x(z^2+xw)=0,$$

viz. this is a cubic surface, Prof. Schläfli's Case XX., having a uniplanar point $x=0$, $y=0$, $z=0$ reducing the class by 8, and so giving a reciprocal surface of the order $(12-8=)\,4$, viz. the surface $27(4xw+z^2)^2-64y^3w=0$. See the Memoir, Schläfli, "On the distribution of surfaces of the third order into species in reference to the absence or presence of singular points and the reality of their lines," *Phil. Trans.*, vol. CLIII. (1853), pp. 193—241.

I pass to the case of a surface

$$V^2-P^2U=0$$

having a nodal conic $V=0$, $P=0$, but not having in general any nodes. And I propose to show how the constants may be determined so that the surface shall have 1, 2, 3, or 4 nodes. It is to be remarked that in the above equation the plane $P=0$ is a determinate plane, but the quadric surface $V=0$ is not a determinate quadric, we may in fact substitute for it the quadric $V+\lambda P^2=0$, writing the equation under the form

$$(V+\lambda P^2)^2-P^2(U+2\lambda V+\lambda^2P^2),$$

so that we may without loss of generality, by means of the disposable constant λ, subject the surface $V=0$ to any single condition; for instance, take it to be a cone, or to pass through a given point, &c.

Taking the planes $x=0$, $y=0$, $z=0$, $w=0$ to be arbitrary planes, the implicit constant factors in these equations may be determined in such wise that the equation of the given plane $P=0$ shall be $x+y+z+w=0$. The equation of the surface will then be

$$\{(a, b, c, d, f, g, h, l, m, n\between x, y, z, w)^2\}^2$$
$$=(x+y+z+w)^2.(a', b', c', d', f', g', h', l', m', n'\between x, y, z, w)^2,$$

and we may assume that the node or nodes (if any) lie at a vertex or vertices of the tetrahedron $x=0$, $y=0$, $z=0$, $w=0$, say at the points A, B, C, D. The conditions for a node at each of these points are at once found to be

Node at A,	Node at B,	Node at C,	Node at D,
$2a^2 = a' + a'$	$2bh = h' + b'$	$2cg = g' + c'$	$2dl = l' + d'$
$2ah = h' + a'$	$2b^2 = b' + b'$	$2cf = f' + c'$	$2dm = m' + d'$
$2ag = g' + a'$	$2bf = f' + b'$	$2c^2 = c' + c'$	$2dn = n' + d'$
$2al = l' + a'$	$2bm = m' + b'$	$2cn = n' + c'$	$2d^2 = d' + d'$

The first set of equations gives

$$a' = a^2,\quad h' = 2ah - a^2,\quad g' = 2ag - g^2,\quad l' = 2al - a^2.$$

If the first and second sets are satisfied simultaneously, we have $2(a-b)h = a^2 - b^2$, that is $a=b$, or else $h = \frac{1}{2}(a+b)$; that is, the two sets may be satisfied in two different ways according as a and b are equal or unequal. Similarly the first, second, and third sets may be satisfied in three different ways and the four sets in five different ways according as there are or are not any equalities between a, b, c, and between a, b, c and d respectively. The several solutions are shown in the annexed table, viz., in the line I no set is satisfied; in the line II only the first set; in the lines III and IV the first and second sets; in the lines V to VII the first, second, and third sets; and in the lines VIII to XII the four sets.

[See next page for this Table, which should come in here.]

I is the general case, $V^2 = P^2U$, of a quartic and a nodal conic but without nodes. (AC).

II is the case of a single node; writing, as without loss of generality we may do, $a=0$, the equation is

$$[(0, b, c, d, f, g, h, l, m, n\between x, y, z, w)^2]^2 = (x+y+z+w)^2.(b, c, d, n, m, f\between y, z, w)^2,$$

viz. the quadric $U=0$ is here a cone having its vertex on the quadric $V=0$. (AD).

III and IV are two cases each of them with two nodes, viz. III, the equation is

$$\begin{aligned}\{(ax+by)(x+y)+cz^2+2nzw+dw^2+2x(gz+lw)+2y(fz+mw)\}^2&\\ = (x+y+z+w)^2\,[(ax+by)^2+c'z^2+2n'zw+d'w^2&\qquad (AE)\\ +2x\{(2ag-a^2)z+(2al-a^2)w\}&\\ +2y\{(2bf-b^2)z+(2bm-b^2)w\}],&\end{aligned}$$

where it is to be observed that the line $z=0$, $w=0$ joining the two nodes $(y=0, z=0, w=0)$ and $(w=0, z=0, x=0)$ is a line on the surface. Writing, as we may do, $a=0$, the equation assumes the more simple form

$$\begin{aligned}\{by(x+y)+cz^2+2nzw+dw^2+2x(gz+lw)+2y(fz+mw)\}^2&\qquad (AE)\\ = (x+y+z+w)^2\,[b^2y^2+c'z^2+2n'zw+d'w^2+2y\{(2bf-b^2)z+(2bm-b^2)w\}].&\end{aligned}$$

I	a	b	c	d	f	g	h	l	m	n	a'	b'	c'	d'	f'	g'	h'	l'	m'	n'
II	a	b	c	d	f	g	h	l	m	n	a^2	b'	c'	d'	f'	$2ag-a^2$	$2ah-a^2$	$2al-a^2$	m'	n'
III	a	b	c	d	f	g	$\frac{1}{2}(a+b)$	l	m	n	a^2	b^2	c'	d'	$2bf-b^2$	$2ag-a^2$	ab	$2al-a^2$	$2bm-b^2$	n'
IV	a	a	c	d	f	g	h	l	m	n	a^2	a^2	c'	d'	$2af-a^2$	$2ag-a^2$	$2ah-a^2$	$2al-a^2$	$2am-a^2$	n'
V	a	b	c	d	$\frac{1}{2}(b+c)$	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+b)$	l	m	n	a^2	b^2	c^2	d'	bc	ac	ab	$2al-a^2$	$2bm-b^2$	$2cn-c^2$
VI	a	a	c	d	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+c)$	h	l	m	n	a^2	a^2	c^2	d'	ac	ac	$2ah-a^2$	$2al-a^2$	$2am-a^2$	$2cn-c^2$
VII	a	a	a	d	f	g	h	l	m	n	a^2	a^2	a^2	d'	$2af-a^2$	$2ag-a^2$	$2ah-a^2$	$2al-a^2$	$2am-a^2$	$2an-a^2$
VIII	a	b	c	d	$\frac{1}{2}(b+c)$	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+b)$	$\frac{1}{2}(a+d)$	$\frac{1}{2}(b+d)$	$\frac{1}{2}(c+d)$	a^2	b^2	c^2	d^2	bc	ac	ab	ad	bd	cd
IX	a	a	c	d	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+c)$	h	$\frac{1}{2}(a+d)$	$\frac{1}{2}(a+d)$	$\frac{1}{2}(c+d)$	a^2	a^2	c^2	d^2	ac	ac	$2ah-a^2$	ad	ad	cd
X	a	a	a	d	f	g	h	$\frac{1}{2}(a+d)$	$\frac{1}{2}(a+d)$	$\frac{1}{2}(a+d)$	a^2	a^2	a^2	d^2	$2af-a^2$	$2ag-a^2$	$2ah-a^2$	ad	ad	ad
XI	a	a	c	c	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+c)$	h	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+c)$	n	a^2	a^2	c^2	c^2	ac	ac	$2ah-a^2$	ac	ac	$2cn-c^2$
XII	a	a	a	a	f	g	h	l	m	n	a^2	a^2	a^2	a^2	$2af-a^2$	$2ag-a^2$	$2ah-a^2$	$2al-a^2$	$2am-a^2$	$2an-a^2$

In IV the equation is

$$\{a(x^2+y^2)+2hxy+cz^2+dw^2+2nzw+2x(gz+lw)+2y(fz+mw)\}^2$$
$$=(x+y+z+w)^2[a^2(x^2+y^2)+2(2af-a^2)xy+c'z^2+2n'zw+d'w^2 \qquad (AF)$$
$$+2x\{(2ag-a^2)z+(2al-a^2)w\}$$
$$+2y\{(2af-a^2)z+(2am-a^2)w\}],$$

where it will be observed that the line $z=0$, $w=0$ joining the two nodes is *not* a line on the surface.

Writing, as we may do, $a=0$, the equation becomes

$$\{2hxy+cz^2+2nzw+dw^2+2x(gz+lw)+2y(fz+mw)\}^2$$
$$=(x+y+z+w)^2(c'z^2+2n'zw+d'w^2), \qquad (AF)$$

viz. the form is $V^2=P^2QR$, the quadric surface $U=0$ breaking up into the two planes $Q=0$, $R=0$; and the nodes being situate at the intersections of the line $Q=0$, $R=0$ with the surface $V=0$.

V, VI, VII are apparently cases with three nodes, but in fact VI is the only case of a proper quartic surface with three nodes. For in V the equation is

$$\{(ax+by+cz)(x+y+z)+dw^2+2w(lx+my+nz)\}^2$$
$$=(x+y+z+w)^2[(ax+by+cz)^2+d'w^2+2w\{(2al-a^2)x+(2bm-b^2)y+(2cn-c^2)z\}],$$

which is satisfied by $w=0$, and the surface thus breaks up into the plane $w=0$ and a cubic surface.

And in VII the equation is

$$\{a(x^2+y^2+z^2)+dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw\}^2$$
$$=(x+y+z+w)^2[a^2(x^2+y^2+z^2)+d'w^2$$
$$+2\{(2af-a^2)yz+(2ag-a^2)zx+(2ah-a^2)xy$$
$$+(2al-a^2)xw+(2am-a^2)yw+(2an-a^2)zw\}],$$

which putting $a=0$ is

$$(dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw)^2=(x+y+z+w)^2d'w^2,$$

viz. this is a pair of quadric surfaces.

In the remaining case VI the equation is

$$\{a(x^2+y^2)+2hxy+cz^2+dw^2+(a+c)(yz+zx)+2w(lx+my+nz)\}^2 \qquad (AG)$$
$$=(x+y+z+w)^2[a^2(x^2+y^2)+c^2z^2+d'w^2+2acz(x+y)+2(2ah-a^2)xy$$
$$+2w\{(2al-a^2)x+(2am-a^2)y+(2an-a^2)z\}],$$

which putting therein $a=0$ is

$$\{2hxy+dw^2+cz(x+y+z)+2w(lx+my+nz)\}^2$$
$$=(x+y+z+w)^2\{c^2z^2+d'w^2+2(2cn-c^2)zw\}, \qquad (AG)$$

which is a surface having the nodes A, B, C; and it is to be observed that the lines CA, CB, but *not* the line AB, are lines on the surface.

IX, X, XI, XII are apparently cases with four nodes, but it is only XI which is a proper quartic with four nodes. In fact IX is

$$\{(ax+ay+cz+dw)(x+y+z+w)+2(h-a)xy\}^2$$
$$=(x+y+z+w)^2\{(ax+ay+cz+dw)^2+4a(h-a)xy\},$$

which is satisfied if $x=0$ or if $y=0$; that is, the surface breaks up into the two planes $x=0$, $y=0$, and a quadric surface.

X is

$$\{a(x^2+y^2+z^2)+dw^2+2fyz+2gzx+2hxy+(a+d)(xw+yw+zw)\}^2$$
$$=(x+y+z+w)^2\{a^2(x^2+y^2+z^2)+d^2w^2$$
$$+2(2af-a^2)yz+2(2ag-a^2)zx+2(2af-a^2)xy+2ad(xw+yw+zw)\},$$

which putting therein $a=0$ is

$$\{dw(x+y+z+w)+2fyz+2gzx+2hxy\}^2=(x+y+z+w)^2d^2w^2,$$

and thus breaks up into two quadrics.

And XII is

$$\{a^2(x^2+y^2+z^2+w^2)+2fyz+2gzx+2hxy+2lxw+2myw+2nw\}^2$$
$$=(x+y+z+w)^2\{a^2(x^2+y^2+z^2+w^2)$$
$$+2(2af-a^2)yz+2(2ag-a^2)zx+2(2ah-a^2)xy$$
$$+2(2al-a^2)xw+2(2am-a^2)yw+2(2ah-a^2)zw\},$$

which putting $a=0$ is

$$(2fyz+2gzx+2hxy+2lxw+2myw+2nzw)^2=0,$$

and is thus a quadric surface twice repeated.

There remains XI, and here the equation is

$$\{(ax+ay+cz+cw)(x+y+z+w)+2(h-a)xy+2(n-c)zw\}^2$$
$$=(x+y+z+w)^2\{(ax+ay+cz+cw)^2+4a(h-a)xy+4c(n-c)zw\},$$

or writing $h+a$, $n+c$ in place of a, c respectively, this is

$$\{(ax+ay+cz+cw)(x+y+z+w)+2hxy+2nw\}^2$$
$$=(x+y+z+w)^2\}(ax+ay+cz+cw)^2+4ahxy+4cnzw\}^2, \qquad (AH)$$

or putting herein $a=0$ it is

$$\{c(z+w)(x+y+z+w)+2hxy+2nzw\}^2=c(x+y+z+w)^2\{c(z+w)^2+4nzw\}, \qquad (AH)$$

which may also be written

$$c(x+y+z+w)\{hxy(z+w) - nzw(x+y)\} + (hxy+nzw)^2 = 0, \qquad (AH)$$

the equation of a quartic surface with the four nodes A, B, C, D; it is to be observed that the lines AC, AD, BC, BD are, the lines AB and CD are *not*, lines on the surface.

A more simple form may be given to the equation as follows; using the second of the above forms, multiplying the equation by 4, and writing therein

$$p = \sqrt{(c)}\,(x+y+z+w),$$
$$qr = c(z+w)^2 + 4nzw,$$
$$st = c(x+y)^2 - 4hxw,$$

q, r, and s, t being the linear factors of the two quadric functions respectively, we have

$$qr - st = c(x+y+z+w)(-x-y+z+w) + 4hxy + 4nzw,$$

and thence

$$p^2 + qr - st = 2c(z+w)(x+y+z+w) + 4hxy + 4nzw,$$

wherefore the equation is

$$(p^2 + qr - st)^2 = 4p^2qr, \qquad (AH)$$

or, what is the same thing,

$$p + \sqrt{(qr)} + \sqrt{(st)} = 0, \qquad (AH)$$

where p, q, r, s, t are any linear functions of the coordinates; this is the equation of a quartic surface having the nodal conic $p=0$, $qr-st=0$; and the four nodes $(q=0, r=0, p^2-st=0)$ and $(s=0, t=0, p^2-qr=0)$. It includes the Cyclide, the equation of which may be written

$$b^2 = \sqrt{\{(ax-ek)^2 + b^2y^2\}} + \sqrt{\{(ex-ak)^2 - b^2z^2\}}.$$

I remark that Prof. Kummer in his most valuable Memoir, "Ueber die Flächen vierten Grades auf welchen Schaaren von Kegelschnitten liegen," *Crelle*, t. LXVI. (1864), pp. 66—76, has considered several of the cases of a quartic surface with a nodal conic, viz. no node, (AC); a single node, (AD); two nodes (the case AF); and four nodes, (AH); but he has not considered two nodes, the case (AE); nor three nodes, (AG).

In reference to the general case of a quartic surface with a nodal conic, some most interesting properties have recently been obtained by Prof. Clebsch, see *Berl. Monatsb.*, April 30, 1868, where it is shown that there are on the surface 16 right lines forming 20 systems of double-fours, analogous in some respect to the 27 lines and 36 systems of double-sixes of a cubic surface.