# 452.

## ON AN ANALYTICAL THEOREM FROM A NEW POINT OF VIEW.

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THE theorem is a well-known one, derived from the equation

$$(az^2 + 2bz + c)\,w^2 + 2\,(a'z^2 + 2b'z + c')\,w + a''z^2 + 2b''z + c'' = 0\,;$$

viz., considering this equation as establishing a relation between the variables $z$ and $w$, and writing it in the forms

$$2u = Aw^2 + 2Bw + C = A'z^2 + 2B'z + C' = 0,$$

(where, of course, $A$, $B$, $C$ are quadric functions of $z$, and $A'$, $B'$, $C'$ quadric functions of $w$,) we have

$$0 = \frac{du}{dw}\,dw + \frac{du}{dz}\,dz, \ = (Aw + B)\,dw + (A'z + B')\,dz;$$

but in virtue of the equation $u = 0$, we have $Aw + B = \sqrt{B^2 - AC}$, and $A'z + B' = \sqrt{B'^2 - A'C'}$, and the differential equation thus becomes

$$\frac{dw}{\sqrt{B'^2 - A'C'}} + \frac{dz}{\sqrt{B^2 - AC}} = 0,$$

where $B'^2 - A'C'$ and $B^2 - AC$ are quartic functions of $w$ and $z$ respectively. This is, of course, integrable (viz., the integral is the original equation $u = 0$); and it follows, from the theory of elliptic functions, that the two quartic functions must be linearly transformable into each other; viz., they must have the same absolute invariant $I^3 \div J^2$. It is, in fact, easy to verify, not only that this is so, but that the two functions have the same quadrinvariant $I$, and the same cubinvariant $J$.

The new point of view is, that we take the coefficients $a$, $b$, &c., to be homogeneous functions of $(x, y)$, their degrees being such that the equation $u=0$ is a quartic equation $(*\between x, y, z, w)^4 = 0$: viz., this equation now represents a quartic surface having a node (conical point) at the point $(x=0, y=0, z=0)$, and also a node at the point $(x=0, y=0, w=0)$, say, these points are $O$, $O'$ respectively. The equation $B'^2 - A'C' = 0$ gives the circumscribed sextic cone having $O$ for its vertex, and the equation $B^2 - AC = 0$ the circumscribed sextic cone having $O'$ for its vertex; each of these cones has the line $OO'$ $(x=0, y=0)$ for a nodal line, as appears geometrically, and also by the equations containing $z$, $w$ respectively in the degree 4. Considering $B'^2 - A'C'$ as a quartic function of $z$, its quadrinvariant is a function $(x, y)^8$, and its cubinvariant a function $(x, y)^{12}$; and similarly, considering $B^2 - AC$ as a quartic function of $w$, its invariants are functions $(x, y)^8$ and $(x, y)^{12}$. We have thus, between the two cones, a geometrical relation answering to the analytical one of the identity of the invariants; but the nature of this geometrical relation is not obvious; and it presents itself as an interesting subject of investigation.