740.

ON CERTAIN ALGEBRAICAL IDENTITIES.

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If P_0, P_1, P_2 are points on a circle, say the circle $x^2+y^2=1$, then it is possible to find functions of (P_0, P_1) and of (P_1, P_2) respectively, which are really independent of P_1, and consequently functions of only P_0 and P_2: the expression "function of a point or points" being here used to mean algebraical function of the coordinates of the point or points. Thus the functions of (P_0, P_1) and of (P_1, P_2) being $x_0x_1+y_0y_1$, $x_0y_1-x_1y_0$, and $x_1x_2+y_1y_2$, $x_1y_2-x_2y_1$, we have

$$(x_1x_2+y_1y_2)(x_0x_1+y_0y_1)+(x_1y_2-x_2y_1)(x_0y_1-x_1y_0)=x_0x_2+y_0y_2,$$

and another like equation. This depends obviously on the circumstance that the coordinates of a point of the circle are expressible by means of the functions sin, cos, $x=\cos u$, $y=\sin u$; and the identity written down is obtained by expressing the cosine of u_2-u_0, $=(u_2-u_1)+(u_1-u_0)$, in terms of the cosines and sines of u_2-u_1 and u_1-u_0.

Evidently the like property holds good for a curve, such that the coordinates of any point of it can be expressed by means of "additive" functions of a parameter u; where, by an additive function $f(u)$, is meant a function such that $f(u+v)$ is an algebraical function of $f(u)$, $f(v)$; the sine and cosine are each of them an additive function, because

$$\sin(u+v)=\sin u\sqrt{(1-\sin^2 v)}+\sin v\sqrt{(1-\sin^2 u)},$$

and, similarly, for the cosine. But it is convenient to consider pairs or groups $f(u)$, $\phi(u), \ldots$, where $f(u+v)$, $\phi(u+v), \ldots$ are each of them an algebraical (rational) function of $f(u), \phi(u), \ldots, f(v), \phi(v), \ldots$; the sine and cosine are such a group, and so also are the elliptic functions sn, cn, dn; but the H and Θ, or say the ϑ-functions generally, are not additive.

In the case of the elliptic functions, we may consider the quadriquadric curve

$$y^2 = 1 - x^2, \quad z^2 = 1 - k^2x^2,$$

so that the coordinates of a point on the curve are $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$. Taking then P_0, P_1, P_2, points on the curve, and (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), the coordinates of these points respectively, we have in the same way, from $u_2 - u_0 = (u_2 - u_1) + (u_1 - u_0)$, three equations, of which the first is

$$\frac{x_2y_0z_0 - x_0y_2z_2}{1 - k^2x_0^2x_2^2} = \frac{\begin{array}{l}(1 - k^2x_1^2x_2^2)(x_2y_1z_1 - x_1y_2z_2)(y_0y_1 + x_0z_0x_1z_1)(z_0z_1 + k^2x_0y_0x_1y_1)\\ + (1 - k^2x_0^2x_1^2)(x_1y_0z_0 - x_0y_1z_1)(y_1y_2 + x_1z_1x_2z_2)(z_1z_2 + k^2x_1y_1x_2y_2)\end{array}}{(1 - k^2x_0^2x_1^2)^2(1 - k^2x_1^2x_2^2)^2 - k^2(x_1y_0z_0 - x_0y_1z_1)^2(x_2y_1z_1 - x_1y_2z_2)^2}.$$

The form of the right-hand side is

$$\frac{A + Bx_1y_1z_1}{C + Dx_1y_1z_1},$$

where A, B, C, D are each of them rational as regards x_1^2; and it is easy to see that the equation can only subsist under the condition that we have separately

$$\frac{x_2y_0z_0 - x_0y_2z_2}{1 - k^2x_0^2x_2^2} = \frac{A}{C} = \frac{B}{D},$$

implying of course the identity $AD - BC = 0$. The values of B and D are found without difficulty; we, in fact, have

$$B = 2k^2(x_2^2 \quad - x_0^2 \quad)(x_1^2y_0z_0y_2z_2 + x_0x_2y_1^2z_1^2),$$

$$D = 2k^2(x_2y_0z_0 + x_0y_2z_2)(x_1^2y_0z_0y_2z_2 + x_0x_2y_1^2z_1^2),$$

so that, comparing the left-hand side with $B \div D$, we have the identity

$$x_2^2y_0^2z_0^2 - x_0^2y_2^2z_2^2 = (x_2^2 - x_0^2)(1 - k^2x_2^2x_0^2),$$

which is right. The comparison with $A \div C$ would be somewhat more difficult to effect.