737.

ON A COVARIANT FORMULA.

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STARTING from the equation

$$x_1 = x - \frac{fx}{f'x},$$

which presents itself in the Newton-Fourier problem, it is easy to see that, if a be a root of the equation $fx = 0$, then

$$x_1 - a, = \frac{(x-a)f'x - fx}{f'x},$$

contains the factor $(x-a)^2$, that is, the equation $(x - x_1)f'x - fx = 0$, considered as an equation in x containing the parameter x_1, will have a twofold root, if x_1 is equal to any root a of the equation $fx = 0$; and, consequently, the discriminant in regard to x of the function $(x - x_1)f'x - fx$ will contain the factor fx_1. But if fx be of the order n, then the discriminant is of the order $2n-2$ in x_1, and there is consequently a remaining factor ϕx_1 of the order $n-2$.

The like theorem applies to the homogeneous form

$$(xy_1 - x_1y)\left(\alpha \frac{d}{dx} + \beta \frac{d}{dy}\right) f(x, y) - (\alpha y_1 - \beta x_1) f(x, y),$$

which reduces itself to the foregoing on writing $\alpha = 1$, $\beta = 0$, $y = y_1 = 1$; or, changing the notation, say to the form

$$(\xi y - \eta x)\left(\alpha \frac{d}{d\xi} + \beta \frac{d}{d\eta}\right) f(\xi, \eta) - (\alpha y - \beta x) f(\xi, \eta),$$

viz. the discriminant hereof in regard to ξ, η, being a function, homogeneous of the order $2n-2$ in regard to x, y, to α, β, and to the coefficients of $f(\xi, \eta)$, will contain the factor $f(x, y)$, and there will be consequently a remaining factor of the order $n-2$ in (x, y), $2n-2$ in (α, β) and $2n-3$ in the coefficients of $f(\xi, \eta)$.

The most simple case is when $f(\xi, \eta)$ is the quadric function $(a, b, c)(\xi, \eta)^2$. The form here is

$$(\xi y-\eta x)\,2\,\{(a\alpha+b\beta)\,\xi+(b\alpha+c\beta)\,\eta\}-(\alpha y-\beta x)\,(a,\ b,\ c)(\xi,\ \eta)^2=(\mathrm{a},\ \mathrm{b},\ \mathrm{c})(\xi,\ \eta)^2,$$

where the coefficients are

$$\begin{aligned}
\mathrm{a} &= \quad 2y\,(a\alpha+b\beta)-a\,(\alpha y-\beta x), = \qquad a\beta x \qquad +(a\alpha+2b\beta)\,y,\\
\mathrm{b} &= \qquad y\,(b\alpha+c\beta)-x\,(a\alpha+b\beta)-b\,(\alpha y-\beta x),\\
&\qquad\qquad\qquad\qquad\qquad\qquad = -\quad a\alpha x \quad + \quad c\beta y \quad ,\\
\mathrm{c} &= -\ 2x\,(b\alpha+c\beta)-c\,(\alpha y-\beta x), = -(2b\alpha+c\beta)\,x- \quad c\alpha y \quad ;
\end{aligned}$$

and we then have

$$\begin{aligned}
\mathrm{ac}-\mathrm{b}^2 = &-(2b\alpha\beta+c\beta^2)\,ax^2\\
&-\{2ab\alpha^2+(2ac+4b^2)\,\alpha\beta+2bc\beta^2\}\,xy-(a\alpha^2+2b\alpha\beta)\,cy^2\\
&-ax^2\,.\,ax^2-\{-2ac\alpha\beta\}\,xy-c\beta^2\,.\,cy^2,
\end{aligned}$$

which is

$$=-(a\alpha^2+2b\alpha\beta+c\beta^2)\,(ax^2+2bxy+cy^2).$$

The discriminant is in this case

$$=-(a,\ b,\ c)(\alpha,\ \beta)^2\,.\,(a,\ b,\ c)(x,\ y)^2.$$

In the case of the cubic function $(a, b, c, d)(\xi, \eta)^3$, the form is

$$\begin{aligned}
(\xi y-x\eta)\,\{3\,(a\alpha+b\beta,\ b\alpha+c\beta,\ c\alpha+d\beta)(\xi,\ \eta)^2\}&\\
-(\alpha y-\beta x)\,(a,\ b,\ c,\ d)(\xi,\ \eta)^3&=(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d})(\xi,\ \eta)^3,
\end{aligned}$$

the values of the coefficients being

$$\begin{aligned}
\mathrm{a} &= \quad a\beta x \qquad\qquad +(2a\alpha+3b\beta)\,y,\\
\mathrm{b} &= -\,a\alpha x \qquad\qquad +(\ b\alpha+2c\beta)\,y,\\
\mathrm{c} &= -(2b\alpha+\ c\beta)\,x+ \qquad d\beta\ \ y,\\
\mathrm{d} &= -(3c\alpha+2d\beta)\,x- \qquad d\alpha\ \ y.
\end{aligned}$$

Attending only to the terms in x^2, we have

$$\begin{aligned}
\mathrm{ac}-\mathrm{b}^2 &= -\quad(a\alpha^2+2b\alpha\beta+c\beta^2)\,ax^2,\\
\mathrm{ad}-\mathrm{bc} &= -\,2\,(b\alpha^2+2c\alpha\beta+d\beta^2)\,ax^2,\\
\mathrm{bd}-\mathrm{c}^2 &= \quad\{(3ac-4b^2)\,\alpha^2+(2ad-4bc)\,\alpha\beta-c^2\beta^2\}\,x^2.
\end{aligned}$$

And hence, in

$$a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd, = (ad - bc)^2 - 4(ac - b^2)(bd - c^2),$$

we have the term

$$4ax^3 . x [a(b\alpha^2 + 2c\alpha\beta + d\beta^2)^2 + (a\alpha^2 + 2b\alpha\beta + c\beta^2)\{(3ac - 4b^2)\alpha^2 + (2ad - 4bc)\alpha\beta - c^2\beta^2\}];$$

then, forming the analogous term in y^4, and assuming that the whole divides by $(a, b, c, d \between x, y)^3$, and also expanding the $\alpha\beta$-functions within the square brackets, we find

$$\text{Discriminant} = 4(a, b, c, d \between x, y)^3 \text{ multiplied by}$$

$$x \left| \begin{array}{l} 3a^2c - 3ab^2 \\ 2a^2d + 6abc - 8b^3 \\ 6abd + 6ac^2 - 12b^2c \\ 6acd - 6bc^2 \\ ad^2 - c^3 \end{array} \right| \between \alpha, \beta)^4 + y \left| \begin{array}{l} a^2d - b^3 \\ 6abd - 6b^2c \\ 6acd + 6b^2d - 12bc^2 \\ 2ad^2 + 6bcd - 8c^3 \\ 3bd^2 - 3c^2d \end{array} \right| \between \alpha, \beta)^4.$$

Writing down the Hessian of $(a, b, c, d \between \alpha, \beta)^3$,

$$H = (ac - b^2, ad - bc, bd - c^2 \between \alpha, \beta)^2,$$

and the cubicovariant

$$\Phi = \left\{ \begin{array}{l} a^2d - 3abc + 2b^3 \\ abd - 2ac^2 + b^2c \\ -acd + 2b^2d - bc^2 \\ -ad^2 + 3bcd - 2c^3 \end{array} \right\} (x, y)^3,$$

it is easy to see that the coefficient of x is

$$= 3(a, b, c \between \alpha, \beta)^2 . (H - \beta\Phi);$$

hence also that of y is

$$= 3(b, c, d \between \alpha, \beta)^2 . (H + \alpha\Phi),$$

and the final result is that the discriminant $= 4(a, b, c, d \between x, y)^3$ multiplied by

$$\{3(a, b, c, d \between \alpha, \beta)^3 (x, y) H + (\alpha y - \beta x)\Phi\}.$$

It would be interesting to calculate the result for the quartic $(a, b, c, d, e \between \xi, \eta)^4$.

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