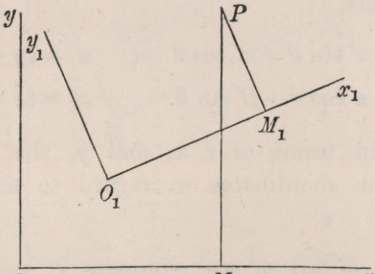
734.

ON THE KINEMATICS OF A PLANE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVI. (1879), pp. 1—8.]

It seems desirable to bring together under this title various questions which have been, or may be, proposed or discussed. We consider two planes in relative motion one upon the other, but, for convenience, they may be distinguished as a moving plane and a fixed plane, the first moving upon the second. Any point of the moving plane traces out on the fixed plane a curve, and any line of the moving plane envelopes on the fixed plane a curve; similarly, any point of the fixed plane traces out on the moving plane a curve, and any line of the fixed plane envelopes on the moving plane a curve. More generally, any curve of the moving plane envelopes on the fixed plane a curve, and any curve of the fixed plane envelopes on the moving plane a curve. There is, moreover, in the moving plane a curve which rolls upon a curve in the fixed plane, and these two curves (a single relative position being given) determine the motion.

Fig. 1.



The analytical theory presents no difficulty. Taking in the fixed plane the fixed axes Ox, Oy (fig. 1), and, fixed in the moveable plane so as to move with it, the axes O_1x_1, O_1y_1; then the position of the axes $O_1x_1y_1$ may be determined, say by

α, β, the coordinates of O_1 in regard to Oxy; and by θ, the inclination of O_1x_1 to Ox. And denoting by x, y, x_1, y_1 the coordinates of a point P in regard to the two sets of axes respectively, then

$$x = \alpha + x_1 \cos\theta - y_1 \sin\theta,$$
$$y = \beta + x_1 \sin\theta + y_1 \cos\theta;$$

or, what is the same thing,

$$x_1 = \quad (x-\alpha)\cos\theta + (y-\beta)\sin\theta,$$
$$y_1 = -(x-\alpha)\sin\theta + (y-\beta)\cos\theta;$$

or, as these last equations may be written,

$$x_1 = \alpha_1 + x\cos(-\theta) - y\sin(-\theta),$$
$$y_1 = \beta_1 + x\sin(-\theta) + y\cos(-\theta),$$

where α_1, β_1, $= -\alpha\cos\theta - \beta\sin\theta$, $\alpha\sin\theta - \beta\cos\theta$, are the coordinates of O referred to the axes $O_1x_1y_1$, and $-\theta$ is the inclination of Ox to O_1x_1.

When the motion is given, α, β, θ are given functions of a single variable parameter, say of t*; or, if we please, α, β are given functions of θ.

The velocities of a given point (x, y) are determined by the equations

$$x' = \alpha' - (x_1\sin\theta + y_1\cos\theta)\,\theta',$$
$$y' = \beta' + (x_1\cos\theta - y_1\sin\theta)\,\theta';$$

that is,

$$x' - \alpha' = -(y-\beta)\,\theta',$$
$$y' - \beta' = \quad (x-\alpha)\,\theta';$$

or, as these equations may also be written,

$$-(x'-\alpha')\sin\theta + (y'-\beta')\cos\theta = x_1\theta',$$
$$-(x'-\alpha')\cos\theta - (y'-\beta')\sin\theta = y_1\theta'.$$

Hence if $x'=0$, $y'=0$, we have

$$x_1\theta' = \alpha'\sin\theta - \beta'\cos\theta, \text{ or } \quad \alpha' = (y-\beta)\,\theta',$$
$$y_1\theta' = \alpha'\cos\theta + \beta'\sin\theta, \qquad -\beta' = (x-\alpha)\,\theta',$$

which equations determine in terms of t, x_1 and y_1 the coordinates in regard to the axes $O_1x_1y_1$, and x and y the coordinates in regard to the axes Oxy, of I, the centre of instantaneous rotation.

If from the expressions of x_1, y_1 we eliminate t, we obtain an equation between (x_1, y_1), which is that of the rolling curve in the moveable plane; and, similarly, if

* t may be regarded as denoting the time, and then the derived functions of x, y in regard to t will denote velocities; and, to simplify the expression of the theorems, it is convenient to do this.

from the expressions of x, y we eliminate t, we obtain a relation between (x, y), which is that of the rolled-on curve in the fixed plane.

The system may be written

$$x_1 = \frac{\alpha'}{\theta'}\sin\theta - \frac{\beta'}{\theta'}\cos\theta, \quad x = \alpha - \frac{\beta'}{\theta'},$$

$$y_1 = \frac{\alpha'}{\theta'}\cos\theta + \frac{\beta'}{\theta'}\sin\theta, \quad y = \beta + \frac{\alpha'}{\theta'};$$

or, if we take θ as the independent variable,

$$x_1 = \alpha'\sin\theta - \beta'\cos\theta, \quad x = \alpha - \beta',$$

$$y_1 = \alpha'\cos\theta + \beta'\sin\theta, \quad y = \beta + \alpha'.$$

To find the variations of I, we have

$$\begin{aligned} x_1' &= \alpha''\sin\theta - \beta''\cos\theta + \alpha'\cos\theta + \beta'\sin\theta, &&= \alpha''\sin\theta - \beta''\cos\theta + y_1, \\ y_1' &= \alpha''\cos\theta + \beta''\sin\theta - \alpha'\sin\theta + \beta'\cos\theta, &&= \alpha''\cos\theta + \beta''\sin\theta - x_1, \\ y' &= \beta' + \alpha'', && \\ x' &= \alpha' - \beta''. && \end{aligned}$$

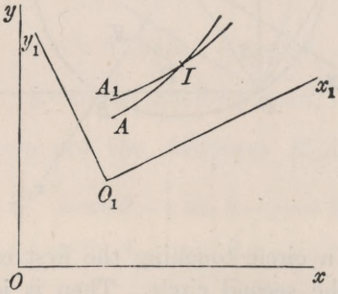
Hence

$$\begin{aligned} x_1' &= x'\cos\theta + y'\sin\theta, \quad \text{or } x' = x_1'\cos\theta - y_1'\sin\theta, \\ y_1' &= -x'\sin\theta + y'\cos\theta, \qquad\quad y' = x_1'\sin\theta + y_1'\cos\theta, \end{aligned}$$

values which give $x'^2 + y'^2 = x_1'^2 + y_1'^2$, which equation expresses that the motion is in fact a rolling one.

Imagine the two curves, and the initial relative position given; say the two points A, A_1 (fig. 2) were originally in contact, then the arcs AI, A_1I are equal, and, calling each of these s, and X, Y, X_1, Y_1 the coordinates of I in regard to the two

Fig. 2.



sets of axes respectively, we have X, Y, X_1, Y_1 given functions of s, such that $X'^2 + Y'^2 = 1$, $X_1'^2 + Y_1'^2 = 1$, the accents now denoting differentiation in regard to s. We have, from the figure,

$$\theta = \tan^{-1}\frac{Y'}{X'} - \tan^{-1}\frac{Y_1'}{X_1'};$$

or, what is the same thing,

$$\tan\theta = (Y'X_1' - Y_1'X) \div (X'X_1' + Y'Y_1'),$$

say

$$\sin\theta,\ \cos\theta = Y'X_1' - Y_1'X,\ X'X_1' + Y'Y_1';$$

and then, as before,

$$x = \alpha + x_1\cos\theta - y_1\sin\theta,$$
$$y = \beta + x_1\sin\theta + y_1\cos\theta;$$

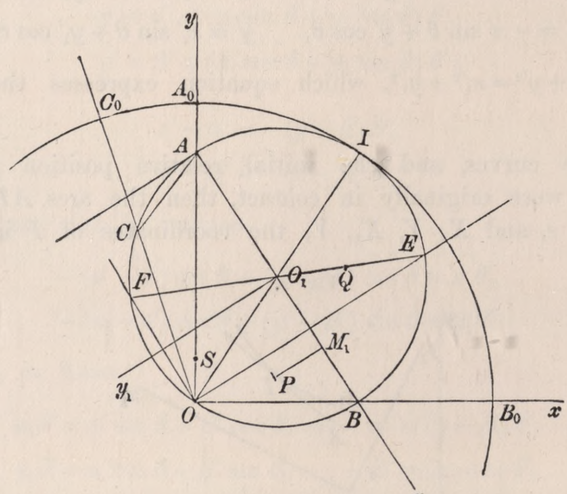
or, what is the same thing,

$$x - X = \cos\theta\,(x_1 - X_1) - \sin\theta\,(y_1 - Y_1),$$
$$y - Y = \sin\theta\,(x_1 - X_1) + \cos\theta\,(y_1 - Y_1),$$

where X, Y, X_1, Y_1, and therefore also θ, denote given functions of s. The formulæ will be of a like form if X, Y, X_1, Y_1 are given functions of a parameter t.

A well known but very interesting case is when two points of the moving plane describe right lines on the fixed plane. This may be discussed geometrically as follows: Suppose that we have the points A, C (fig. 3) describing the lines OA_0, OC_0, which meet in O; through A, C, O describe a circle, centre O_1, and with centre

Fig. 3.



O and radius $= 2OO_1$, describe a circle touching the first circle in a point I; and suppose that A_0, C_0 denote points on the second circle. Then it is at once seen that, considering the first or small circle as belonging to the moving plane, and the second or large circle as belonging to the fixed plane, the motion is in fact the rolling motion of the small upon the large circle; and, moreover, that each point of the small circle describes a right line, which is a diameter of the large circle. In fact, the angle IO_1C at the centre is the double of the angle IOC at the circumference; that is, it is the double of the angle IOC_0; and therefore (the radius of the small circle being half that of the large circle) the arcs IC, IC_0 are equal, so that the rolling motion will carry the point C along the radius OC_0, and will, in like manner, carry the point A along the radius OA_0, or the motion will be as originally assumed. And, in like manner, for any other point B of the small circle the motion will be along the radius OB_0; in particular, taking AB a diameter, the angle A_0OB_0 will be a right angle; and the motion is determined by means of the two points A, B describing respectively the two lines OA_0, OB_0 *at right angles to each other*, viz. there is no loss of generality in assuming that the two fixed lines are at right angles to each other. It thence at once follows, as will presently appear, that each point of the moving plane describes an ellipse (but we have the special case already referred to, each point on the small circle describes a right line, and also the special case, the centre O_1 of the small circle describes a circle). Considering any point Q of the moving plane, let the line QO_1 meet the small circle in the points E, F (or, what is the same thing, let E, F be the extremities of the diameter which passes through Q); then the points E, F describe the lines OE, OF at right angles to each other, and Q is a point on EF or on this line produced; clearly the locus is an ellipse having the lines OE, OF for the directions of its axes, and having the lengths of the semi-axes $=QF$, QE respectively.

Taking the points to be A, B moving along the two lines OB_0, OA_0 at right angles to each other, these lines may be taken for the axes Ox, Oy; the point O_1 for the origin of the coordinates x_1, y_1, the axes O_1x_1 being in the direction O_1B and O_1y_1 at right angles to it; calling the length $AB=2c$, we have $O_1A=O_1B=c$, and the angle ABO may be called θ (but this angle was previously taken with a contrary sign). We have then for the point P, having in regard to O_1x_1 and O_1y_1 the coordinates (x_1, y_1),

$$\left.\begin{aligned} x &= \alpha + x_1\cos\theta - y_1\sin\theta \\ y &= \beta - x_1\sin\theta - y_1\cos\theta \end{aligned}\right\},$$

where the sign of y_1 has been changed, and $\alpha = c\cos\theta$, $\beta = c\sin\theta$: the equations thus become

$$x = (c+x_1)\cos\theta - y_1\sin\theta,$$

$$y = (c-x_1)\sin\theta - y_1\cos\theta,$$

where observe that $c+x_1$, $c-x_1$ are the distances M_1A, M_1B respectively. And we have, conversely,

$$x_1 = \quad x\cos\theta - y\sin\theta - c\cos 2\theta,$$

$$y_1 = -x\sin\theta - y\cos\theta + c\sin 2\theta.$$

If, in particular, $y_1 = 0$, then

$$x,\ y = (c+x_1)\cos\theta,\ (c-x_1)\sin\theta;$$

or we have

$$\frac{x^2}{(c+x_1)^2} + \frac{y^2}{(c-x_1)^2} = 1;$$

viz. the curve on the first plane is an ellipse, the semi-axes of which are $\pm(c+x_1)$, $\pm(c-x_1)$, each taken positively; if $x_1^2+y_1^2=c^2$, viz. if P be on the circle having AB for its diameter, then $y_1^2=(c+x_1)(c-x_1)$, and we have

$$y \div x = -(c-x_1)\left(\sin\theta - \frac{y_1}{c-x_1}\cos\theta\right) \div y_1\left(\sin\theta - \frac{c+x_1}{y_1}\cos\theta\right), = -(c-x_1) \div y_1,$$

viz. as mentioned above, the curve on the fixed plane is a right line.

In the general case, we have

$$x(c-x_1) + \quad yy_1 \quad = (c^2-x_1^2-y_1^2)\cos\theta,$$
$$xy_1 + y(c+x_1) = (c^2-x_1^2-y_1^2)\sin\theta,$$

and thence

$$\{x(c-x_1)+yy_1\}^2 + \{xy_1+y(c+x_1)\}^2 = (c^2-x_1^2-y_1^2)^2;$$

or, what is the same thing,

$$x^2\{(c-x_1)^2+y_1^2\} + 4xycy_1 + y^2\{(c+x_1)^2+y_1^2\} = (c^2-x_1^2-y_1^2)^2.$$

Considering (x_1, y_1) as given, the curve traced out by P on the fixed plane is of the second order; it would be easy to verify from the equation that it is an ellipse, and to obtain for the position and magnitude of the axes the construction already found geometrically.

The same equation, considering therein (x, y) as constant and (x_1, y_1) as current coordinates, gives the curve traced out on the moving plane; the curve is obviously of the fourth order. Transferring the origin to A, we must in place of x_1 write $x_1 - c_1$; the equation thus becomes

$$x^2\{(x_1-2c)^2+y_1^2\} + 4cy_1xy + y^2(x_1^2+y_1^2) = (x_1^2+y_1^2-2cx_1)^2;$$

or, what is the same thing,

$$(x_1^2+y_1^2-2cx_1)^2 - (x^2+y^2)(x_1^2+y_1^2) + 4cx(xx_1-yy_1) - 4c^2x^2 = 0;$$

and if we suppose herein $x=0$, it becomes

$$(x_1^2+y_1^2-2cx_1)^2 - y^2(x_1^2+y_1^2) = 0;$$

or, writing $x_1 = r_1\cos\theta_1$, $y_1 = r_1\sin\theta_1$, where $\theta_1 = \text{angle } QAB$, this is

$$(r_1 - 2c\cos\theta_1)^2 - y^2 = 0,$$

or say it is

$$r_1 = 2c\cos\theta_1 - y,$$

which is the polar equation of the curve described on the moveable plane by the point S, whose coordinates in respect to Ox and Oy are $(0, y)$.

There is no loss of generality in assuming $x=0$. In fact, starting with any point S whatever of the fixed plane, if we draw OS meeting the small circle in A, and through O draw at right angles to this a line meeting the same circle in B, then, as before, the points A and B move along the fixed lines OA_0, OB_0; or as regards the relative motion, taking A, B as fixed points, we have the originally fixed plane now moving in such wise that the two lines OA_0, OB_0 thereof (at right angles to each other) pass always through the points A and B respectively, and the curve is that described by the point S on the line OA; the point O describes the circle on the diameter AB (the small circle), equation $r_1 = 2c\cos\theta_1$; and OQ having a given constant value $= y$, we have for the curve described by the point S the foregoing equation $r_1 = 2c\cos\theta_1 - y$; or writing $y = -f$, that is, taking S on the other side of O at a distance $OS = f$, the equation is $r_1 = 2c\cos\theta_1 + f$; viz. this is a nodal Cartesian or Limaçon, the origin being an acnode or a crunode according as $f >$ or $< 2c$; and if $f = 2c$, then we have the cuspidal curve or cardioid $r_1 = 2c(1 + \cos\theta_1)$, $= 4c\cos^2\frac{1}{2}\theta_1$. The general conclusion is that the centre O of the large circle describes on the moving plane a small circle (centre O_1), and that every other point of the fixed plane describes on the moving plane a Limaçon having for its node a point of the small circle, and being, in fact, the curve obtained by measuring off along the radius vector of the small circle from its extremity a constant distance.

Considering in connexion with the point, coordinates (x_1, y_1), (x, y), a second point, coordinates (X_1, Y_1), (X, Y), in regard to the two sets of axes respectively, we have

$$x = (c + x_1)\cos\theta - y_1\sin\theta, \quad X = (c + X_1)\cos\theta - Y_1\sin\theta,$$

$$y = (c - x_1)\sin\theta - y_1\cos\theta, \quad Y = (c - X_1)\sin\theta - Y_1\cos\theta;$$

from the first two equations we have

$$\cos\theta : \sin\theta : 1 = x(c - x_1) + yy_1 : xy_1 + y(c + x_1) : c^2 - x_1^2 - y_1^2;$$

and substituting these values in the second set, we find

$$\begin{aligned} X : Y : 1 \\ = \ & x\{c^2 + c(X_1 - x_1) - X_1x_1 - Y_1y_1\} + y\{\quad c(y_1 - Y_1) + y_1X_1 - x_1Y_1\} \\ & : x\{\quad c(y_1 - Y_1) - y_1X_1 + x_1Y_1\} + y\{c^2 - c(X_1 - x_1) - X_1x_1 - Y_1y_1\} \\ & : c^2 - x_1^2 - y_1^2; \end{aligned}$$

or the points (x, y), (X, Y), considered as each of them moving on the fixed plane, are homographically related to each other.

To find the curve enveloped on the fixed plane by a given curve of the moving plane, we have only in the equation $f(x_1, y_1) = 0$ of the curve in the moving plane to substitute for x_1, y_1 their values in terms of x, y, θ, and then considering θ as a variable parameter, to find the envelope of the curve represented by this equation. And, similarly, we find the curve enveloped on the moving plane by a given curve of the fixed plane.

Thus, in the particular case of motion above considered, writing, as before,

$$x = (c + x_1)\cos\theta - y_1 \sin\theta,$$
$$y = (c - x_1)\sin\theta - y_1\cos\theta\,;$$

or conversely

$$x_1 = \quad x\cos\theta - y\sin\theta - c\cos 2\theta,$$
$$y_1 = -x\sin\theta - y\cos\theta + c\sin 2\theta\,;$$

the envelope on the moving plane of the line

$$Ax + By + C = 0$$

of the fixed plane is given as the envelope of the line

$$\{A(c + x_1) - By_1\}\cos\theta + \{-A + B(c - x_1)\}\sin\theta + C = 0\,;$$

viz. this is

$$\{A(c + x_1) - By_1\}^2 + \{Ay_1 - B(c - x_1)\}^2 - C^2 = 0\,;$$

that is,

$$(A^2 + B^2)(x_1^2 + y_1^2 + c^2) + 2(A^2 - B^2)cx_1 - 4ABcy_1 = 0,$$

a circle.

But the envelope on the fixed plane of the line

$$Ax_1 + By_1 + C = 0$$

of the moving plane is given as the envelope of the line

$$C + (Ax + By)\cos\theta - (Ay + Bx)\sin\theta - AC\cos 2\theta + BC\sin 2\theta = 0,$$

which can be obtained by equating to zero the discriminant of a quartic function, and is apparently a sextic curve.