731.

ON THE BINOMIAL EQUATION $x^p - 1 = 0$; TRISECTION AND QUARTISECTION.

[From the *Proceedings of the London Mathematical Society*, vol. XI. (1880), pp. 4—17. Read November 13, 1879.]

THE solution of the binomial equation $x^p - 1 = 0$, p a prime number, or, say rather, the equation

$$x^{p-1} + x^{p-2} + \ldots + x + 1 = 0,$$

depends upon the Jacobian function

$$F\alpha = x^1 + \alpha x^g + \ldots + \alpha^{p-2} x^{g^{p-2}},$$

where g is a prime root of p, α any root whatever of the equation $u^{p-1} - 1 = 0$. Taking e a factor of $p-1$, and f for the complementary factor (that is, $p - 1 = ef$), then, if for α we write α^f, or, what is the same thing, taking α^f, $= \beta$, a root of $u^e - 1 = 0$, we have

$$F\beta = X_0 + \beta X_1 + \ldots + \beta^{e-1} X_{e-1},$$

where $X_0, X_1, \ldots, X_{e-1}$ denote each of them a period or sum of f, $= \frac{p-1}{e}$, roots, viz.

$$\begin{aligned} X_0 \;\; &= (1, \quad g^e, \quad \ldots, g^{(f-1)e}), \\ X_1 \;\; &= (g^e, \quad g^{e+1}, \ldots, g^{(f-1)e+1}), \\ &\vdots \\ X_{e-1} &= (g^{e-1}, \quad g^{2e-1}, \ldots, g^{fe-1}) \end{aligned}$$

(read $X_0 = x^1 + x^{g^e} + \ldots + x^{g^{(f-1)e}}$, and so for the other functions).

We have, of course, $F(1)$, $= X_0 + X_1 + \ldots + X_{e-1}$, the sum of all the roots $= -1$; and, further, the general property that any rational and integral function of these periods is expressible as a sum

$$a_0 X_0 + a_1 X_1 + \ldots + a_{e-1} X_{e-1}$$

with known coefficients

$$a_0, a_1, \ldots, a_{e-1}.$$

The several cases $e = 2, 3, 4, \ldots$ may be termed those of the bisection, trisection, quartisection, &c., of the equation; viz.

$e = 2$, there are two periods, X, Y, and $F(-1) = X - Y$;

$e = 3$, three periods, X, Y, Z, and $F\gamma = X + \gamma Y + \gamma^2 Z$, if γ is a root of $u^3 - 1 = 0$;

$e = 4$, four periods, X, Y, Z, W, and $F\delta = X + \delta Y + \delta^2 Z + \delta^3 W$, if δ be a root of $u^4 - 1 = 0$.

It is sufficient to attend to the prime roots γ and δ of the equations

$$u^3 - 1 = 0, \quad u^4 - 1 = 0,$$

respectively; for, if γ or δ be $= 1$, we have simply $F(1)$, $= -1$; and if δ be $= -1$, then the function is $F(-1)$, $= X + Z - (Y + W)$, where $X + Z$ and $Y + W$ are the periods for the bisection. The prime roots δ are of course i and $-i$, and we have

$$F(i) \;\; = X + iY - Z - iW,$$
$$F(-i) = X - iY - Z + iW,$$

respectively.

As regards the bisection, it is known that $(X - Y)^2 = (-)^{\frac{p-1}{4}} p$, which is $+p$ or $-p$, according as p is $\equiv 1$ or 3, mod. 4; and the values of X, Y are thus determined. In what follows, I consider the cases $e = 3$ and $e = 4$ of the trisection and the quartisection respectively.

It is to be remembered that, not the division into periods, but the order of the periods, depends on the choice of g, a prime root at pleasure of p; and, in what follows, I select the prime root used in Reuschle's *Tafeln complexer Primzahlen welche aus Wurzeln der Einheit gebildet sind* (4to, Berlin, 1875): viz. these are

$p = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53,$
$\qquad 59, 61, 67, 71, 73, 79, 83, 89, 97,$

$g = 2, 2, 5, 2, 2, 3, 2, -2, 2, 3, 2, 6, 3, 10, 2,$
$\qquad 2, 2, 2, 62, 5, 3, 2, 30, 10,$

where I quote the whole series, although I am here only concerned with the values of p which are $\equiv 1$ (mod. 3), or $\equiv 1$ (mod. 4).

The periods are consequently those of Reuschle, viz. X, Y, Z are his η_0, η_1, η_2, and X, Y, Z, W his η_0, η_1, η_2, η_3: they can of course, without referring to his work, be easily recalculated, but it is, I think, convenient to have for his values of g the series of residues such as are given (for differently selected values of g) in Jacobi's *Canon Arithmeticus* (4to, Berlin, 1839); and I have accordingly taken out of Reuschle, and annex, such a table.

For instance, $p = 13$, the powers of g are 1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7; and, by writing these down in order in columns of 3 or of 4,

1	8	12	5		1	3	9
2	3	11	10		2	6	5
4	6	9	7		4	12	10
					8	11	7

we have the periods X, Y, Z or X, Y, Z, W, belonging to the trisection and the quartisection of $p = 13$.

I further remark that the equations which I am concerned with are all given in Reuschle, but in a somewhat different form; thus, $p = 13$, quartisection (see p. 13), he has

$$\eta_0{}^2 = \eta_1 + 2\eta_2, \quad \eta_0\eta_1 = -1 - \eta_2, \quad \eta_0\eta_2 = 3 + \eta_1 + \eta_3, \quad \eta_0\eta_3 = -1 - \eta_1,$$

(where observe that here and in every case the value of $\eta_0\eta_3$ is at once obtained from that of $\eta_0\eta_1$ by a mere cyclical interchange of the suffixes, so that the last equation is in fact superfluous); the other equations, using $\eta_0 + \eta_1 + \eta_2 + \eta_3 = -1$ to eliminate any constant term which occurs, give my values

$$\begin{aligned} X^2 &= (\ \ 0,\ \ \ 1,\ \ \ 2,\ \ \ 0)(X, Y, Z, W), \\ XY &= (\ \ 1,\ \ \ 1,\ \ \ 0,\ \ \ 1)(\quad ,, \quad), \\ XZ &= (-3, -2, -3, -2)(\quad ,, \quad). \end{aligned}$$

Similarly, in the case of a trisection, the equation for $\eta_0\eta_2$ is superfluous, and the other equations give my values of X^2 and XY.

Reuschle gives also, and I take from him, the cubic and the quartic equations (such as $p = 13$, $\eta^3 + \eta^2 - 4\eta + 1 = 0$, $\eta^4 + \eta^3 + 2\eta^2 - 4\eta + 3 = 0$), which determine the periods in the trisections and the quartisections respectively.

Many of the results obtained accord with, and furnish exemplifications of general theorems contained in Jacobi's memoir, "Ueber die Kreistheilung und ihre Anwendung auf die Zahlentheorie," *Crelle*, t. XXX. (1846), pp. 166—189; [*Ges. Werke*, t. VI. pp. 254—274].

Trisection, $e = 3$; $p \equiv 1$ (*mod.* 3).

We have three periods X, Y, Z; and we thence obtain

$$\begin{aligned} X^2 &= (a, b, c)(X, Y, Z), \\ XY &= (f, g, h)(\quad ,, \quad), \end{aligned}$$

the coefficients a, b, c, f, g, h being determinate integers. And, by cyclical interchanges, we obtain equations which may be written

$$\begin{aligned} X^2 &= a, b, c, \\ Y^2 &= c, a, b, \\ Z^2 &= b, c, a, \\ XY &= f, g, h, \\ YZ &= h, f, g, \\ ZX &= g, h, f; \end{aligned}$$

viz. here and elsewhere the coefficients a, b, c are written to denote the sum

$$aX + bY + cZ.$$

It is easy to see that

$$f + g + h = \tfrac{1}{3}(p - 1);$$

in fact, a period contains $\frac{1}{3}(p-1)$ terms, and in two consecutive periods X, Y, there are no terms the product of which is unity; hence XY contains $\frac{1}{9}(p-1)^2$ terms, each a power of x, and the sum $XY+YZ+ZX$ contains $\frac{1}{3}(p-1)^2$ such terms, being in fact the sum $X+Y+Z$ taken $\frac{1}{3}(p-1)$ times; whence the relation in question.

Hence also

$$YZ+ZX+XY=-\tfrac{1}{3}(p-1).$$

From the equation $X+Y+Z=-1$, multiplying by X, and for X^2, XY, XZ substituting their values, we obtain an expression

$$(a+f+g+1)X+(b+g+h)Y+(c+h+f)Z,$$

which must identically vanish; viz. the three coefficients must be each of them $=0$; or we must have

$$\begin{aligned} a &= -f-g-1, \\ b &= -g-h, \\ c &= -h-f; \end{aligned}$$

so that, taking f, g, h as known, the other coefficients a, b, c are given in terms of them. The equations give

$$a+b+c=-2(f+g+h)-1.$$

We have $X.YZ=Y.ZX$; that is, $X(h, f, g)=Y(g, h, f)$; or, substituting for X^2, XY, &c. their values,

$$\begin{aligned} h(a, b, c) &= \;\; g(f, g, h) \\ +f(f, g, h) &\;\; +h(c, a, b) \\ +g(g, h, f) &\;\; +f(h, f, g); \end{aligned}$$

that is,

$$\begin{aligned} ah+f^2+g^2 &= gf+ch+fh, \\ bh+fg+gh &= g^2+ah+f^2, \\ ch+fh+fg &= gh+bh+fg, \end{aligned}$$

equations which reduce themselves to the single equation

$$gh+hf+fg+h=f^2+g^2+h^2;$$

and this is the only relation obtainable by consideration of the three equal values

$$X.YZ, \quad Y.ZX, \quad Z.XY.$$

Moreover, this equation being satisfied, the six functions in the three equations become each of them $=fg-h^2$; or we have

$$XYZ=(fg-h^2, \ fg-h^2, \ fg-h^2);$$

that is,

$$XYZ=h^2-fg.$$

We have

$$\begin{aligned} F\gamma . F\gamma^2 &= \;\; X^2+Y^2+Z^2-YZ-ZX-XY \\ &= \;\; (a+b+c-f-g-h)(X+Y+Z) \\ &= -(a+b+c)+(f+g+h) \\ &= \;\; 3(f+g+h)+1; \end{aligned}$$

that is,

$$F\gamma \,.\, F\gamma^2 = p.$$

We have, moreover,

$$\begin{aligned}(F\gamma)^2 = {} & X^2 + 2YZ + \gamma(Z^2+2XY) + \gamma^2(Y^2+2ZX)\\ = {} & [(a,\ b,\ c) + 2(h,\ f,\ g)]\\ & + \gamma\,[(b,\ c,\ a) + 2(f,\ g,\ h)]\\ & + \gamma^2\,[(c,\ a,\ b) + 2(g,\ h,\ f)],\end{aligned}$$

which is

$$= \{(a+2h) + \gamma(b+2f) + \gamma^2(c+2g)\}\,(X+\gamma^2 Y+\gamma Z),$$

as is at once seen by comparing the coefficients of X, Y, Z respectively.

Hence, writing

$$\begin{aligned}& a + 2h + \gamma(b+2f) + \gamma^2(c+2g)\\ & = a + 2h + \gamma(b+2f) - (1+\gamma)(c+2g)\\ & = A + B\gamma,\end{aligned}$$

we have

$$\begin{aligned}A &= a + 2h - c - 2g = 3h - 3g - 1,\\ B &= b + 2f - c - 2g = 3f - 3g.\end{aligned}$$

We have

$$(F\gamma)^2 = (A + B\gamma)\,F\gamma^2,$$

and thence, writing γ^2 for γ,

$$(F\gamma^2)^2 = (A + B\gamma^2)\,F\gamma,$$

equations which give

$$F\gamma \,.\, F\gamma^2,\ = p,\ = (A+B\gamma)(A+B\gamma^2);$$

or, say $p = A^2 - AB + B^2$; viz. p has the complex factor

$$A + B\gamma,\ = 3h - 3g - 1 + \gamma(3f - 3g).$$

Hence also

$$\begin{aligned}(F\gamma\,)^3 &= p\,(A + B\gamma\,),\\ (F\gamma^2)^3 &= p\,(A + B\gamma^2),\end{aligned}$$

and, as before,

$$F\gamma \,.\, F\gamma^2 = p;$$

which equations determine $F\gamma$, $F\gamma^2$, and from these and $F(1) = -1$ we obtain the periods X, Y, Z; we have thus, in fact, the solution of the cubic equation which gives these periods. We have already found the coefficients of this cubic equation, viz.

$$X + Y + Z = -1,\quad YZ + ZX + XY = -\tfrac{1}{3}(p-1),\quad XYZ = h^2 - fg;$$

the equation thus is

$$\eta^3 + \eta^2 - \tfrac{1}{3}(p-1)\,\eta + (fg - h^2) = 0.$$

As already remarked, the values of a, b, c; f, g, h, and the equations in η, are in effect given in Reuschle; the complex factors of p, as given p. 1 ($7 = 2\gamma - 3\gamma^2$, &c.), when reduced to the form $A + B\gamma$, are not identical with the $A + B\gamma$ of the foregoing theory; viz. this $A + B\gamma$ is not Reuschle's selected primary form. I give, in the annexed table for the primes 7, 13, ..., to 97, the values from Reuschle of a, b, c; f, g, h, and of the coefficients of the η-equation; also the values of A and B derived from f, g, h by the foregoing formulæ. It will be seen that all the values are consistent with the theory.

Table for the Trisection.

p	a, f,	b, g,	c h	$\eta^3+\eta^2+$ η^1	η^0	A	B	Page in Reuschle
7	− 2	− 1	− 2	− 2	− 1	2	3	p. 6
	1	0	1					
13	− 4	− 3	− 2	− 4	− 1	− 4	− 3	p. 15
	1	2	1					
19	− 4	− 5	− 4	− 6	− 7	2	− 3	p. 26
	1	2	3					
31	− 7	− 6	− 8	− 10	− 8	5	6	p. 45
	4	2	4					
37	− 8	− 10	− 7	− 12	11	− 4	3	p. 54
	5	4	3					
43	− 11	− 8	− 10	− 14	8	− 1	6	p. 69
	6	4	4					
61	− 14	− 13	− 15	− 20	− 9	− 4	− 9	p. 97
	5	8	7					
67	− 16	− 13	− 16	− 22	5	2	9	p. 105
	9	6	7					
73	− 16	− 18	− 15	− 24	− 27	− 1	− 9	p. 128
	6	9	9					
79	− 20	− 17	− 16	− 26	41	− 10	− 3	p. 138
	9	10	7					
97	− 20	− 23	− 22	− 32	− 79	11	3	p. 168
	10	9	13					

Quartisection, $e = 4$; $p \equiv 1$ (*mod.* 4).

We have four periods X, Y, Z, W; and we obtain

$$X^2 = (a, b, c, d)(X, Y, Z, W),$$
$$XY = (f, g, h, k)(\quad " \quad),$$
$$XZ = (l, m, l, m)(\quad " \quad),$$

the coefficients being determinate integers. It can be shown that $l + m = \frac{1}{8}(p-1)$ or $-\frac{1}{8}(3p+1)$ according as $p \equiv 1$ or 5 (mod. 8). And then, by cyclical interchanges,

$$\begin{aligned} X^2 &= a, b, c, d, \\ Y^2 &= d, a, b, c, \\ Z^2 &= c, d, a, b, \\ W^2 &= b, c, d, a, \\ XY &= f, g, h, k, \\ YZ &= k, f, g, h, \\ ZX &= h, k, f, g, \\ XW &= g, h, k, f, \\ XZ &= l, m, l, m, \\ YW &= m, l, m, l. \end{aligned}$$

We have, in like manner as for the trisection,

$$f + g + h + k = \tfrac{1}{4}(p-1),$$

and so also the expression for

$$\Sigma XY, = XY + XZ + XW + YZ + YW + ZW$$

is

$$= -(f + g + h + k + l + m) = -\tfrac{1}{4}(p-1) - l - m;$$

and, in virtue of the foregoing value of $l + m$, this is $= -\frac{3}{8}(p-1)$ or $\frac{1}{8}(p+3)$ according as $p \equiv 1$ or 5 (mod. 8).

Again, from the equation $X + Y + Z + W = -1$, multiplying by X and reducing,

$$\begin{aligned} a &= -1 - f - g - l, \\ b &= \quad\;\; - g - h - m, \\ c &= \quad\;\; - h - k - l, \\ d &= \quad\;\; - k - f - m, \end{aligned}$$

and thence

$$a + b + c + d = -1 - 2(f + g + h + k) - 2(l + m),$$

and

$$a - b + c - d = -1 + 2(m - l).$$

We have

$$X \,.\, YZ = Y \,.\, ZX = Z \,.\, XY,$$

that is,

$$X\,(k,\ f,\ g,\ h) = Y\,(l,\ m,\ l,\ m) = Z\,(f,\ g,\ h,\ k),$$

and thence

$$\begin{array}{lll} k\,(a,\ b,\ c,\ d) = & l\ (f,\ g,\ h,\ k) = & f\,(l,\ m,\ l,\ m) \\ +f\,(f,\ g,\ h,\ k) & +m\,(d,\ a,\ b,\ c) & +g\,(k,\ f,\ g,\ h) \\ +g\,(l,\ m,\ l,\ m) & +l\ (k,\ f,\ g,\ h) & +h\,(c,\ d,\ a,\ b) \\ +h\,(g,\ h,\ k,\ f) & +m\,(m,\ l,\ m,\ l) & +k\,(h,\ k,\ f,\ g), \end{array}$$

that is,

$$\begin{aligned} ka + f^2 + gl + gh &= lf + md + lk + m^2 = lf + gk + ch + kh, \\ kb + fg + gm + h^2 &= lg + am + lf + ml = fm + fg + hd + k^2, \\ kc + fh + gl + hk &= lh + mb + lg + m^2 = fl + g^2 + ah + kf, \\ kd + fk + gm + fh &= kl + mc + lh + lm = fm + gh + bh + gk, \end{aligned}$$

in which equations $a,\ b,\ c,\ d$ may be regarded as having their foregoing values.

One of these equations is

$$kc + fh + gl + hk = lf + g^2 + ah + kf,$$

that is,

$$-k\,(h+k+l) + fh + gl + hk = lf + g^2 - h\,(f+g+l+1) + kf,$$

or, reducing,

$$l\,(g+h-f-k) = g^2 + k^2 - 2hf - hg - h + kf,$$

which gives l.

Again, another equation is

$$kb + fg + gm + h^2 = fm + fg + hd + k^2,$$

that is,

$$-k\,(g+h+m) + fg + gm + h^2 = fm + fg - h\,(k+f+m) + k^2,$$

or, reducing,

$$m\,(g+h-f-k) = k^2 - h^2 + gk - hf,$$

which gives m.

And we have also

$$md + lk + m^2 = gk + ch + kh,$$

that is,

$$-m\,(k+f+m) + lk + m^2 = gk + kh - h\,(h+k+l),$$

or, reducing,

$$l\,(k+h) - m\,(f+k) = gk - h^2.$$

Substituting herein for $l,\ m$ their values, we have

$$(k+h)\,[g^2 + k^2 - 2hf - hg + kf - h] - (f+k)\,[k^2 - h^2 + gk - hf] + (h^2 - gk)\,[g + h - f - k] = 0.$$

In this equation the only terms of the second order are $-h(h+k)$, which contain the factor h; the terms of the third order contain this same factor h, and throwing it out, and reducing, the equation is found to be

$$(g-k)^2 + (h-f)^2 = h + k,$$

or, as it may also be written,

$$g^2 + k^2 - 2hf - h + (h^2 + f^2 - 2gk - k) = 0\,;$$

and the foregoing values of l, m are

$$l = \frac{(g^2 + k^2 - 2hf - h) - (gh - kf)}{g + h - k - f},$$

$$m = \frac{k^2 - h^2 + gk - hf}{g + h - k - f};$$

and by means of these three equations all the foregoing equations are satisfied.

We have

$$\begin{aligned} FiFi^3 &= (X-Z)^2 + (Y-W)^2 \\ &= X^2 + Y^2 + Z^2 + W^2 - 2(XZ + YW) \\ &= -(a+b+c+d) + 2(l+m)\,; \end{aligned}$$

or, substituting for a, b, c, d, this is

$$= 1 + 2(f+g+h+k) + 4(l+m),$$

viz. it is

$$= \tfrac{1}{2}(p+1) + 4(l+m)\,;$$

or, substituting for $l+m$ its before-mentioned value, then, according as $p \equiv 1$ or 5 (mod. 8), the value is $=p$ or $-p$; that is, we have

$$FiFi^3 = (-)^{\frac{p-1}{4}}\,p.$$

Again, we have

$$\begin{aligned} (Fi)^2 &= (X + iY - Z - iW)^2 \\ &= X^2 - Y^2 + Z^2 - W^2 - 2XZ + 2YW + 2i(XY - YZ + ZW - WX) \\ &= \{a - b + c - d + 2(m-l) + 2(f - g + h - k)\,i\}(X - Y + Z - W) \\ &= (A + Bi)\,F(-1), \end{aligned}$$

where

$$A = a - b + c - d + 2(m-l), = -1 + 4(m-l),$$

$$B = 2(f - g + h - k)\,;$$

or, since $X - Y + Z - W = F(-1)$, this equation is

$$(Fi)^2 = (A + Bi)\,F(-1):$$

and similarly

$$(Fi^3)^2 = (A - Bi)\,F(-1).$$

Moreover

$$[F(-1)]^2 = (-)^{\frac{p-1}{2}} p, \; = p;$$

and we have therefore

$$(\pm p)^2 = (A^2 + B^2) p,$$

that is,

$$A^2 + B^2 = p;$$

or the expression $A + Bi$ determined as above is a complex factor of p.

We may investigate the quartic equation for the determination of the periods X, Y, Z, W. The values of $X + Y + Z + W$ and $XY + XZ + XW + YZ + YW + ZW$ are already known: for the next coefficient $XYZ + XYW + XZW + YZW$, we have $XYZ = (\alpha, \beta, \gamma, \delta)$, where each of the coefficients $\alpha, \beta, \gamma, \delta$ is given under three different forms: the values of YZW, ZWX, WXY are $(\delta, \alpha, \beta, \gamma)$, $(\gamma, \delta, \alpha, \beta)$, $(\beta, \gamma, \delta, \alpha)$; and the required sum therefore is

$$(\alpha + \beta + \gamma + \delta)(X + Y + Z + W), \; = -(\alpha + \beta + \gamma + \delta).$$

Taking the first expressions of these coefficients respectively, we have

$$\begin{aligned} \alpha + \beta + \gamma + \delta = \quad & k(a + b + c + d) \\ & + f(f + g + h + k) \\ & + 2g(l + m) \\ & + h(f + g + h + k), \\ = k\{-1 - \tfrac{1}{2}(p-1) - 2(l+m)\} & + (f + h)\{\tfrac{1}{4}(p-1)\} + 2g(l+m), \\ = 2(g-k)(l+m) + \tfrac{1}{4}(f+h)(p-1) & - \tfrac{1}{2}k(p+1). \end{aligned}$$

We find $XYZW$ most readily as the product of XZ and YW; we thus obtain

$$\begin{aligned} XYZW &= lm(X^2 + Y^2 + Z^2 + W^2 + 2XZ + 2YW) + (l^2 + m^2)(XY + XW + YZ + ZW), \\ &= lm(-a - b - c - d - 2l - 2m) - (l^2 + m^2)(f + g + h + k), \\ &= lm\{1 + 2(f + g + h + k)\} - (l^2 + m^2)(f + g + h + k); \end{aligned}$$

or, substituting for $f + g + h + k$ its value $\frac{1}{4}(p-1)$, this is

$$lm - \tfrac{1}{4}(l-m)^2(p-1), \; = \tfrac{1}{4}\{(l+m)^2 - (l-m)^2 p\}.$$

Hence the required equation, having roots X, Y, Z, W, is

$$\begin{aligned} & \eta^4 + \eta^3 \\ & - \eta^2\{\tfrac{1}{4}(p-1) + l + m\} \\ & + \eta\;\{\tfrac{1}{4}(f+h)(p-1) - \tfrac{1}{2}k(p+1) + 2(g-k)(l+m)\} \\ & + lm - \tfrac{1}{4}(l-m)^2(p-1) \\ & = 0, \end{aligned}$$

where, for the sake of having a single formula, I have retained $l + m$ in place of its value $= -\frac{3}{8}(p-1)$ or $\frac{1}{8}(p+3)$ according as $p \equiv 1$ or 5 (mod. 8).

We thus have the following:—

TABLE FOR THE QUARTISECTION.

p	a / f / l	b / g / m	c / h	d / k	$\eta^4+\eta^3+$ η^2	$\eta^4+\eta^3+$ η^1	$\eta^4+\eta^3+$ η^0	A	B	Page in Reuschle
5	0	1	0	0	1	1	1	−1	−2	p. 2
	0	0	0	1						
	−1	−1								
13	0	1	2	0	2	−4	3	3	−2	p. 13
	1	1	0	1						
	−3	−2								
17	−4	−2	−3	−4	−6	−1	1	−1	4	p. 19
	2	0	1	1						
	1	1								
29	2	3	0	2	4	20	23	−5	−2	p. 36
	1	1	2	3						
	−5	−6								
37	2	1	2	4	5	7	49	−1	6	p. 53
	2	2	4	1						
	−7	−7								
41	−10	−6	−7	−8	−15	18	4	−5	4	p. 61
	4	2	2	2						
	3	2								
53	2	3	6	2	7	−43	47	7	−2	p. 80
	4	4	2	3						
	−11	−9								
61	4	3	2	6	8	42	117	−5	6	p. 96
	3	3	6	3						
	−11	−12								
73	−16	−13	−12	−14	−27	−41	2	3	8	p. 126
	6	5	5	2						
	4	5								
89	−19	−18	−16	−14	−33	39	8	−5	−8	p. 152
	4	8	5	5						
	6	5								
97	−22	−16	−17	−18	−36	91	−61	−9	4	p. 167
	8	6	5	5						
	7	5								

TABLE OF THE POWERS OF REUSCHLE'S SELECTED PRIME ROOTS.

	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97	
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	2	2	5	2	2	3	2	21	2	3	2	6	3	10	2	2	2	2	62	5	3	2	30	10	1
2		4	4	4	4	9	4	4	4	9	4	36	9	6	4	4	4	4	10	25	9	4	10	3	2
3	...	3	6	8	8	10	8	15	8	27	8	11	27	13	8	8	8	8	52	52	27	8	33	30	3
4			2	5	3	13	16	16	16	19	16	25	38	36	16	16	16	16	29	41	2	16	11	9	4
5	...	...	3	10	6	5	13	14	3	26	32	27	28	31	32	32	32	32	23	59	6	32	63	90	5
6				9	12	15	7	18	6	16	27	39	41	28	11	5	3	64	6	3	18	64	21	27	6
7				7	11	11	14	10	12	17	17	29	37	45	22	10	6	61	17	15	54	45	7	76	7
8				3	9	16	9	3	24	20	34	10	25	27	44	20	12	55	60	2	4	7	32	81	8
9	...	...	...	6	5	14	18	17	19	29	31	19	32	35	35	40	24	43	28	10	12	14	70	34	9
10					10	8	17	12	9	25	25	32	10	21	17	21	48	19	32	50	36	28	53	49	10
11	...	...	...	...	7	7	15	22	18	13	13	28	30	22	34	42	35	38	67	31	29	56	77	5	11
12						4	11	2	7	8	26	4	4	32	15	25	9	9	36	9	8	29	85	50	12
13						12	3	19	14	24	15	24	12	38	30	50	18	18	31	45	24	58	58	15	13
14						2	6	8	28	10	30	21	36	4	7	41	36	36	5	6	72	33	49	53	14
15	...	...	...	...	...	6	12	7	27	30	23	3	22	40	14	23	11	5	26	30	58	66	46	45	15
16							5	9	25	28	9	18	23	24	28	46	22	10	50	4	16	49	45	62	16
17	...	...	...	...	...	...	10	5	21	22	18	26	26	5	3	33	44	20	47	20	48	15	15	38	17
18								13	13	4	36	33	35	3	6	7	27	40	3	27	65	30	5	89	18
19								20	26	12	35	34	19	30	12	14	54	13	44	62	37	60	61	17	19
20								6	23	5	33	40	14	18	24	28	47	26	30	18	32	37	50	73	20
21	...	...	...	...	...	...	...	11	17	15	29	35	42	39	48	56	33	52	14	17	17	74	76	51	21
22									5	14	21	5	40	14	43	53	5	37	16	12	51	65	55	25	22
23									10	11	5	30	34	46	33	47	10	7	69	60	74	47	48	56	23
24									20	2	10	16	16	37	13	35	20	14	18	8	64	11	16	75	24
25									11	6	20	14	5	41	26	11	40	28	51	40	34	22	35	71	25
26									22	18	3	2	15	34	52	22	19	56	38	54	23	44	71	31	26
27	...	...	...	...	...	...	...	...	15	23	6	12	2	11	51	44	38	45	13	51	69	5	83	19	27
28										7	12	31	6	16	49	29	15	23	25	36	48	10	87	93	28
29	...	...	...	...	...	...	...	...	...	21	24	22	18	19	45	58	30	46	59	34	68	20	29	57	29
30											11	9	11	2	37	57	60	25	37	24	46	40	69	85	30
31											22	13	33	20	21	55	59	50	22	47	59	80	23	74	31
32											7	37	13	12	42	51	57	33	15	16	19	77	67	61	32
33											14	17	39	26	31	43	53	66	7	7	57	71	52	28	33
34											28	20	31	25	9	27	45	65	8	35	13	59	47	86	34
35	...	...	...	...	...	...	...	...	...	...	19	38	7	15	18	54	29	63	70	29	39	35	75	84	35
36												23	21	9	36	49	58	59	9	72	38	70	25	64	36
37												15	20	43	19	39	55	51	61	68	35	57	38	58	37
38												8	17	7	38	19	49	35	19	48	26	31	72	95	38
39	...	...	...	...	...	...	...	...	...	...	...	7	8	23	23	38	37	3	42	21	78	62	24	77	39
40													24	42	46	17	13	6	48	32	76	41	8	91	40
41	...	...	...	...	...	...	...	...	...	...	...	...	29	44	39	34	26	12	65	14	70	82	62	37	41
42														11	25	9	52	24	45	70	52	81	80	79	42
43														29	50	18	43	28	11	58	77	79	86	14	43
44														8	47	36	25	29	43	71	73	75	88	43	44
45	...	...	...	...	...	...	...	...	...	...	...	...	...	33	41	13	50	58	39	63	61	67	59	42	45
46															29	26	39	49	4	23	25	51	79	32	46
47															5	52	17	31	35	42	75	19	56	29	47
48															10	45	34	62	40	64	67	38	78	96	48
49															20	31	7	57	66	28	43	76	26	87	49
50															40	3	14	47	45	67	50	69	68	94	50

TABLE (*continued*).

	53	59	61	67	71	73	79	83	89	97	
51	27	6	28	27	21	43	71	55	82	67	51
52		12	56	54	24	69	55	27	57	88	52
53		24	51	41	68	53	7	54	19	7	53
54		48	41	15	27	46	21	25	36	70	54
55		37	21	30	41	11	63	50	12	21	55
56		15	42	60	57	55	31	17	4	16	56
57		30	23	53	55	56	14	34	31	63	57
58			46	39	2	61	42	68	40	48	58
59			31	11	53	13	47	53	43	92	59
60				22	20	65	62	23	44	47	60
61				44	33	33	28	46	74	82	61
62				21	58	19	5	9	84	44	62
63				42	46	22	15	18	28	52	63
64				17	12	37	45	36	39	35	64
65				34	34	39	56	72	13	59	65
66					49	49	10	61	34	8	66
67					56	26	30	39	41	80	67
68					64	57	11	78	73	24	68
69					63	66	33	73	54	46	69
70						38	20	63	18	72	70
71						44	60	43	6	41	71
72							22	3	2	22	72
73							26	6	60	26	73
74							40	12	20	66	74
75							41	24	66	78	75
76							44	48	22	4	76
77							53	13	37	40	77
78								26	42	12	78
79								52	14	23	79
80								21	64	36	80
81								42	51	69	81
82									17	11	82
83									65	13	83
84									81	33	84
85									27	39	85
86									9	2	86
87									3	20	87
88										6	88
89										60	89
90										18	90
91										83	91
92										54	92
93										55	93
94										65	94
95										68	95