730.

[ADDITION TO MR SPOTTISWOODE'S PAPER "ON THE TWENTY-ONE COORDINATES OF A CONIC IN SPACE."]

[From the *Proceedings of the London Mathematical Society*, vol. X. (1879), pp. 194—196.]

WRITE

$$U = (a,\ b,\ c,\ d,\ f,\ g,\ h,\ l,\ m,\ n \between x,\ y,\ z,\ t)^2,$$
$$U_0 = (\quad ,, \quad \between \xi,\ \eta,\ \zeta,\ \omega)^2,$$
$$W = (\quad ,, \quad \between x,\ y,\ z,\ t \between \xi,\ \eta,\ \zeta,\ \omega),$$
$$P = (\alpha,\ \beta,\ \gamma,\ \delta \between x,\ y,\ z,\ t),$$
$$P_0 = (\alpha,\ \beta,\ \gamma,\ \delta \between \xi,\ \eta,\ \zeta,\ \omega).$$

Then the equation of the cone, having for its vertex the arbitrary point $(\xi,\ \eta,\ \zeta,\ \omega)$, and passing through the conic $U=0$, $P=0$, is

$$UP_0^2 - 2WPP_0 + U_0P^2 = 0.$$

Or if, to put the coefficients ξ, η, ζ, ω in evidence, we write for a moment

$$A = (a,\ h,\ g,\ l \between x,\ y,\ z,\ t),$$
$$B = (h,\ b,\ f,\ m \between \quad ,, \quad),$$
$$C = (g,\ f,\ c,\ n \between \quad ,, \quad),$$
$$D = (l,\ m,\ n,\ d \between \quad ,, \quad),$$

and therefore

$$W = A\xi + B\eta + C\zeta + D\omega;$$

then the equation is

$$U(\alpha\xi + \beta\eta + \gamma\zeta + \delta\omega)^2 - 2P(\alpha\xi + \beta\eta + \gamma\zeta + \delta\omega)(A\xi + B\eta + C\zeta + D\omega)$$
$$+ P^2(a,\ b,\ c,\ d,\ f,\ g,\ h,\ l,\ m,\ n \between \xi,\ \eta,\ \zeta,\ \omega)^2 = 0.$$

And if we expand first in ξ, η, ζ, ω, and then in x, y, z, t, the final result is

	x^2	y^2	z^2	t^2	yz	zx	xy	xt	yt	zt	
ξ^2		C	B	F	$2A'$				$2L$	$2L'$	$=0.$
$+\eta^2$	C		A	G		$2B'$		$2M'$		$2M$	
$+\zeta^2$	B	A		H			$2C'$	$2N$	$2N'$		
$+\omega^2$	F	G	H		$2F'$	$2G'$	$2H'$				
$+\eta\zeta$	$2A'$			$2F'$	$-2A$	$-2C'$	$-2B'$	$2(Q-R)$	$-2M$	$-2N'$	
$+\zeta\xi$		$2B'$		$2G'$	$-2C'$	$-2B$	$-2A'$	$-2L'$	$2(R-P)$	$-2N$	
$+\xi\eta$			$2C'$	$2H'$	$-2B'$	$-2A'$	$-2C$	$-2L$	$-2M'$	$2(P-Q)$	
$+\xi\omega$		$2M'$	$2N$		$2(Q-R)$	$-2L'$	$-2L$	$-2F$	$-2H'$	$-2G'$	
$+\eta\omega$	$2L$		$2N'$		$-2M$	$-2(R-P)$	$-2M'$	$-2H'$	$-2G$	$-2F'$	
$+\zeta\omega$	$2L'$	$2M$			$-2N'$	$-2N$	$2(P-Q)$	$-2G'$	$-2F'$	$-2H$	

In particular, if $\eta=0$, $\zeta=0$, $\omega=0$, then we have the foregoing equation $X=0$; and the like for the equations $Y=0$, $Z=0$, and $W=0$ respectively.

Take a, b, c, f, g, h for the six coordinates of the line through the points

$$\left|\begin{matrix} x, & y, & z, & t \\ \xi, & \eta, & \zeta, & \omega \end{matrix}\right|;$$

that is, write

$$\mathrm{a}=y\zeta-z\eta,\quad \mathrm{f}=x\omega-t\xi,$$
$$\mathrm{b}=z\xi-x\zeta,\quad \mathrm{g}=y\omega-t\eta,$$
$$\mathrm{c}=x\eta-y\xi,\quad \mathrm{h}=z\omega-t\zeta,$$

where, of course,

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0.$$

Then the foregoing equation of the cone is

$$\left.\begin{matrix} A\mathrm{a}^2 + B\mathrm{b}^2 + C\mathrm{c}^2 + F\mathrm{f}^2 + G\mathrm{g}^2 + H\mathrm{h}^2 \\ -2A'\mathrm{bc}-2B'\mathrm{ca}-2C'\mathrm{ab}+2F'\mathrm{gh}+2G'\mathrm{hf}+2H'\mathrm{fg} \\ +2P\mathrm{af}+2M\mathrm{ag}-2N'\mathrm{ah} \\ -2L'\mathrm{bf}+2Q\mathrm{bg}+2N\mathrm{bh} \\ +2L\mathrm{cf}-2M'\mathrm{cg}+2R\mathrm{ch} \end{matrix}\right\}=0.$$

And this may be regarded as the equation of the *conic* in terms of the twenty-one coordinates of the conic, and of the six coordinates of an arbitrary line meeting the conic. It is, in fact, the general form of the equation given in the paper—Cayley, "On a new Analytical Representation of a Curve in Space," *Quart. Math. Jour.*, vol. III. (1860), [284; this Collection, vol. IV. p. 453].