724.

ON THE DEFORMATION OF A MODEL OF A HYPERBOLOID.

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THE following is a solution of Mr Greenhill's problem set in the Senate-House Examination, January 14, 1878.

"Prove that, if a model of a hyperboloid of one sheet be constructed of rods representing the generating lines, jointed at the points of crossing; then if the model be deformed it will assume the form of a confocal hyperboloid, and prove that the trajectory of a point on the model will be orthogonal to the system of confocal hyperboloids."

Let (x_1, y_1, z_1), (x_2, y_2, z_2) be points on the generating line of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

then

$$\frac{x_1{}^2}{a^2}+\frac{y_1{}^2}{b^2}-\frac{z_1{}^2}{c^2}=1,$$

$$\frac{x_2{}^2}{a^2}+\frac{y_2{}^2}{b^2}-\frac{z_2{}^2}{c^2}=1,$$

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-\frac{z_1z_2}{c^2}=1\,;$$

or, what is the same thing, if

$$\frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c}=p_1,\ q_1,\ r_1\,;\quad \frac{x_2}{a},\ \frac{y_2}{b},\ \frac{z_2}{c}=p_2,\ q_2,\ r_2\,;$$

then

$$p_1{}^2+q_1{}^2-r_1{}^2=1,$$
$$p_2{}^2+q_2{}^2-r_2{}^2=1,$$
$$p_1p_2+q_1q_2-r_1r_2=1.$$

Similarly, if (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) be points on generating line of

$$\frac{\xi^2}{\alpha^2}+\frac{\eta^2}{\beta^2}-\frac{\zeta^2}{\gamma^2}=1,$$

and if

$$\frac{\xi_1}{\alpha},\ \frac{\eta_1}{\beta},\ \frac{\zeta_1}{\gamma}=\mathrm{p}_1,\ \mathrm{q}_1,\ \mathrm{r}_1;\quad \frac{\xi_2}{\alpha},\ \frac{\eta_2}{\beta},\ \frac{\zeta_2}{\gamma}=\mathrm{p}_2,\ \mathrm{q}_2,\ \mathrm{r}_2;$$

then

$$\begin{aligned}\mathrm{p}_1^2 + \mathrm{q}_1^2 - \mathrm{r}_1^2 &= 1,\\ \mathrm{p}_2^2 + \mathrm{q}_2^2 - \mathrm{r}_2^2 &= 1,\\ \mathrm{p}_1\mathrm{p}_2+\mathrm{q}_1\mathrm{q}_2-\mathrm{r}_1\mathrm{r}_2 &= 1.\end{aligned}$$

Hence if (x_1, y_1, z_1), (ξ_1, η_1, ζ_1) be corresponding points on the two surfaces, that is, if

$$\frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c}=\frac{\xi_1}{\alpha},\ \frac{\eta_1}{\beta},\ \frac{\zeta_1}{\gamma},\ =p_1,\ q_1,\ r_1,$$

and similarly, if (x_2, y_2, z_2), (ξ_2, η_2, ζ_2) are corresponding points, that is, if

$$\frac{x_2}{a},\ \frac{y_2}{b},\ \frac{z_2}{c}=\frac{\xi_2}{\alpha},\ \frac{\eta_2}{\beta},\ \frac{\zeta_2}{\gamma}=p_2,\ q_2,\ r_2;$$

then we have, as before, the system of three equations

$$\begin{aligned}p_1^2 + q_1^2 - r_1^2 &= 1,\\ p_2^2 + q_2^2 - r_2^2 &= 1,\\ p_1p_2+q_1q_2-r_1r_2 &= 1.\end{aligned}$$

Then if the two surfaces are confocal, that is, if

$$\alpha^2,\ \beta^2,\ -\gamma^2=a^2+h,\ b^2+h,\ -c^2+h,$$

we shall have

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2.$$

For this equation is

$$a^2(p_1-p_2)^2+b^2(q_1-q_2)^2+c^2(r_1-r_2)^2=\alpha^2(p_1-p_2)^2+\beta^2(q_1-q_2)^2+\gamma^2(r_1-r_2)^2,$$

that is,

$$(p_1-p_2)^2+(q_1-q_2)^2-(r_1-r_2)^2=0,$$

an equation which is obviously true in virtue of the above system of three equations.

Hence, if on confocal surfaces

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad \frac{\xi^2}{a^2+h}+\frac{\eta^2}{b^2+h}-\frac{\zeta^2}{c^2-h}=1,$$

we take two points P_1, P_2 on the first, and Q_1, Q_2 the corresponding points on the second; then P_1, P_2 being on a generating line of the first surface, Q_1, Q_2 will be on a generating line of the second surface, and P_1P_2 will be $= Q_1Q_2$. The same is evidently true for the quadrilaterals $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$, where P_1P_2, P_2P_3, P_3P_4, P_4P_1 are generating lines on the first surface: and therefore Q_1Q_2, Q_2Q_3, Q_3Q_4, Q_4Q_1 are generating lines on the second surface, which proves the theorem.