723.

VARIOUS NOTES.

[From the *Messenger of Mathematics*, vol. VIII. (1879), pp. 45—46, 126, 127.]

An Algebraical Identity: p. 45.

Let a, b, c, f, g, h be the differences of four quantities α, β, γ, δ, say

$$a,\ b,\ c,\ f,\ g,\ h = \beta-\gamma,\ \gamma-\alpha,\ \alpha-\beta,\ \alpha-\delta,\ \beta-\delta,\ \gamma-\delta;$$

then

$$\begin{array}{rrrr} . & h & -g & +a=0, \\ -h & . & +f & +b=0, \\ g & -f & . & +c=0, \\ -a & -b & -c & .=0. \end{array}$$

Now Cauchy's identity

$$(a+b)^7-a^7-b^7=7ab\,(a+b)\,(a^2+ab+b^2)^2,$$

putting therein $a+b=-c$, becomes

$$a^7+b^7+c^7=\ \ 7abc\,(\ \ bc+ca+ab)^2;$$

hence we have

$$\begin{array}{rrrrl} . & h^7 & -g^7 & +a^7 & =-7agh\,(-ga+ah-hg)^2, \\ -h^7 & . & +f^7 & +b^7 & =-7bhf\,(-hb+bf-fh)^2, \\ g^7 & -f^7 & . & +c^7 & =-7cfg\,(-fc+cg-gf)^2, \\ -a^7 & -b^7 & -c^7 & . & =-7abc\,(\ \ bc+ca+ab)^2; \end{array}$$

whence, adding,

$$agh\,(-ga+ah-hg)^2+bhf\,(-hb+bf-fh)^2+cfg\,(-fc+cg-gf)^2+abc\,(bc+ca+ab)^2=0,$$

or, as this may also be written,

$$agh\,(g^2+h^2+a^2)^2+bhf\,(h^2+f^2+b^2)^2+cfg\,(f^2+g^2+c^2)^2+abc\,(a^2+b^2+c^2)^2=0,$$

an identity if a, b, c, f, g, h denote their values in terms of α, β, γ, δ.

Note on a Definite Integral: p. 126.

The integral

$$J=\int_0^1 \frac{k^2x^2dx}{\sqrt{(1-x^2\,.\,1-k^2x^2)}},$$

used by Weierstrass, is at once seen to be $=K-E$; but the proof that the other integral

$$J'=\int_1^{\frac{1}{k}} \frac{k^2x^2dx}{\sqrt{(x^2-1\,.\,1-k^2x^2)}}$$

is $=E'$ is not so immediate.

We have

$$\frac{d}{dy}\,\frac{y\sqrt{(1-y^2)}}{\sqrt{(1-k^2y^2)}}=\frac{1-2y^2+k^2y^4}{(1-y^2)^{\frac{1}{2}}\,(1-k^2y^2)^{\frac{3}{2}}},$$

and thence

$$0=\int_0^1 \frac{(1-2y^2+k^2y^4)\,dy}{(1-y^2)^{\frac{1}{2}}\,(1-k^2y^2)^{\frac{3}{2}}};$$

viz. replacing the numerator by

$$-\frac{k'^2}{k^2}+\frac{1}{k^2}\,(1-k^2y^2)^2,$$

this becomes

$$0=-\frac{k'^2}{k^2}\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}\,(1-k^2y^2)^{\frac{3}{2}}}+\frac{1}{k^2}\int_0^1 \frac{(1-k^2y^2)^{\frac{1}{2}}\,dy}{(1-y^2)^{\frac{1}{2}}},$$

that is,

$$\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}\,(1-k^2y^2)^{\frac{3}{2}}}=\frac{1}{k'^2}\,E;$$

or, writing k' for k,

$$\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}\,(1-k'^2y^2)^{\frac{3}{2}}}=\frac{1}{k^2}\,E'.$$

The integral J' writing therein $x=\dfrac{1}{\sqrt{(1-k'^2y^2)}}$ becomes

$$J'=k^2\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}\,(1-k'^2y^2)^{\frac{3}{2}}},$$

viz. its value is thus $=E'$.

On a Formula in Elliptic Functions: p. 127.

Writing $\text{en}\, u = \dfrac{\text{cn}\, u}{\text{dn}\, u}$, then the formulæ p. 63 of my *Elliptic Functions* give

$$\text{sn}\,(u+v) = \frac{T - T'}{C - C'}, \quad \text{en}\,(u+v) = \frac{B + B'}{C - C'};$$

and, substituting for T, T', B, B', and C, C' their values, we obtain

$$\text{sn}\,(u+v) = \frac{\text{sn}\, u\, \text{en}\, v + \text{sn}\, v\, \text{en}\, u}{1 + k^2\, \text{sn}\, u\, \text{en}\, u\, \text{sn}\, v\, \text{en}\, v},$$

$$\text{en}\,(u+v) = \frac{\text{en}\, u\, \text{en}\, v - \text{sn}\, u\, \text{sn}\, v}{1 - k^2\, \text{sn}\, u\, \text{en}\, u\, \text{sn}\, v\, \text{en}\, v},$$

formulæ which, as regards their numerators, correspond precisely with the formulæ,

$$\sin\,(u+v) = \sin u \cos v + \sin v \cos u$$

and

$$\cos\,(u+v) = \cos u \cos v - \sin u \sin v,$$

of the circular functions, and which in fact reduce themselves to these on putting $k=0$.

The foregoing formulæ, putting therein $k^2 = -1$, are the formulæ given by Gauss, *Werke*, t. III., p. 404, for the lemniscate functions $\sin \text{lemn}\,(a \pm b)$ and $\cos \text{lemn}\,(a \pm b)$; where it is to be observed that these notations do not represent a sine and a cosine, but they are related as the sn and en, viz. that

$$\cos \text{lemn}\, a = \sqrt{(1 - \sin \text{lemn}^2\, a)} \div \sqrt{(1 + \sin \text{lemn}^2\, a)}.$$