721.

FORMULÆ INVOLVING THE SEVENTH ROOTS OF UNITY.

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LET ω be an imaginary cube root of unity, $\omega^2+\omega+1=0$, or say $\omega=\frac{1}{2}\{-1+i\sqrt{(3)}\}$; $\alpha^3=-7(1+3\omega)$, $\beta^3=-7(1+3\omega^2)$, values giving $\alpha^3\beta^3=343$, and the cube roots α, β being such that $\alpha\beta=7$; then $\alpha+\beta$, $=\alpha+\frac{7}{\alpha}$, is a three-valued function $\left(\text{since changing the root } \omega \text{ we merely interchange } \alpha \text{ and } \frac{7}{\alpha}\right)$; and if r be an imaginary seventh root of unity, then

$$\begin{aligned} 3(r+r^6) &= \alpha+\beta-1, \\ 3(r^2+r^5) &= \omega\alpha+\omega^2\beta-1, \\ 3(r^4+r^3) &= \omega^2\alpha+\omega\beta-1. \end{aligned}$$

Any one of these formulæ gives the other two; for observe that we have $\alpha^3=-\alpha\beta(1+3\omega)$, $\beta^3=-\alpha\beta(1+3\omega^2)$, that is, $\alpha^2=-\beta(1+3\omega)$, $\beta^2=-\alpha(1+3\omega^2)$; hence, starting for instance with the first formula, we deduce

$$\begin{aligned} 9(r^2+r^5+2) &= \alpha^2+2\alpha\beta+\beta^2-2\alpha-2\beta+1, \\ &= -\beta(1+3\omega)+14-\alpha(1+3\omega^2)-2\alpha-2\beta+1, \\ &= -\alpha(3+3\omega^2)-\beta(3+3\omega)+15, \\ &= 3\omega\alpha+3\omega^2\beta+15, \end{aligned}$$

that is,

$$3(r^2+r^5)=\omega\alpha+\omega^2\beta-1;$$

and in like manner by squaring each side of this we have the third formula

$$3(r^4+r^3)=\omega^2\alpha+\omega\beta-1.$$

The foregoing formulæ apply to the combinations $r+r^6$, r^2+r^5, r^4+r^3 of the seventh roots of unity, but we may investigate the theory for the roots themselves $r, r^2, r^3, r^4, r^5, r^6$. These depend on the new radical $\sqrt{(-7)}$ or $i\sqrt{(7)}$; introducing instead hereof X, Y, where

$$X=\tfrac{1}{2}\{-1+i\sqrt{(7)}\},$$
$$Y=\tfrac{1}{2}\{-1-i\sqrt{(7)}\},$$

then if

$$A^3=6+3\omega X+(1+3\omega^2)Y,$$
$$B^3=6+3\omega^2 X+(1+3\omega)Y,$$

where

$$AB=i\sqrt{(7)},$$

we have (Lagrange, *Équations Numériques*, p. 294),

$$3r=X+A+B.$$

I found that, in order to bring this into connexion with the foregoing formula, $3(r+r^6)=\alpha+\beta-1$, where as before $\alpha^3=-7(1+3\omega)$, $\beta^3=-7(1+3\omega^2)$, $\alpha\beta=7$, it is necessary that B, A should be linear multiples of α, β respectively, the coefficients being rational functions of ω, X; and that the actual relations are

$$B=\frac{\alpha}{7}\{4-\omega+X(1-2\omega)\},$$

$$A=\frac{\beta}{7}\{5+\omega+X(3+2\omega)\};$$

in verification of which, it may be remarked that these equations give

$$AB=\frac{\alpha\beta}{49}\{(20-\omega-\omega^2)+X(17-4\omega-4\omega^2)+X^2(3-4\omega-4\omega^2)\},$$

viz. in virtue of the equation $\omega^2+\omega+1=0$, the term in $\{\ \}$ is $=21+21X+7X^2$, $=7(X^2+3X+3)$, or since $X^2+X+2=0$, this is $=7(2X+1)$, $=7i\sqrt{(7)}$; the equation thus is $7AB=\alpha\beta\,.\,i\sqrt{(7)}$, which is true in virtue of $AB=i\sqrt{(7)}$ and $\alpha\beta=7$. The same relations may also be written

$$-\alpha=B(\omega^2+X),$$
$$-\beta=A(\omega+X).$$

I found in the first instance

$$\begin{aligned}
3r&=X+A+B,\\
3r^6&=-1-X+A(\omega^2-X)+B(\omega-X),\\
3r^2&=X+\omega^2A+\omega B,\\
3r^5&=-1-X+A(\omega-\omega^2X)+B(\omega^2-\omega X),\\
3r^4&=X+\omega A+\omega^2B,\\
3r^3&=-1-X+A(1-\omega X)+B(1-\omega^2X),
\end{aligned}$$

which in fact gave the foregoing formulæ

$$3(r+r^6)=-1+\ \alpha+\ \beta,$$
$$3(r^2+r^5)=-1+\omega\alpha+\omega^2\beta,$$
$$3(r^4+r^3)=-1+\omega^2\alpha+\ \omega\beta.$$

But there is a want of symmetry in these expressions for r, r^2, &c., inasmuch as the values of r, r^2, r^4 are of a different form from those of r^6, r^5, r^3; to obtain the proper forms, we must for A, B substitute their values in terms of α, β, and we thus obtain

$$3r=\qquad X+\frac{\alpha}{7}\{\ 4-\ \omega+X(\ 1-2\omega)\}+\frac{\beta}{7}\{\ 5+\ \omega+X(\ 3+2\omega)\},$$

$$3r^6=-1-X+\frac{\alpha}{7}\{\ 3+\ \omega+X(-1+2\omega)\}+\frac{\beta}{7}\{\ 2-\ \omega+X(-3-2\omega)\},$$

$$3r^2=\qquad X+\frac{\alpha}{7}\{\ 1+5\omega+X(\ 2+3\omega)\}+\frac{\beta}{7}\{-4-5\omega+X(-1-3\omega)\},$$

$$3r^5=-1-X+\frac{\alpha}{7}\{-1+2\omega+X(-2-3\omega)\}+\frac{\beta}{7}\{-3-2\omega+X(\ 1+3\omega)\},$$

$$3r^4=\qquad X+\frac{\alpha}{7}\{-5-4\omega+X(-3-\ \omega)\}+\frac{\beta}{7}\{-1+4\omega+X(-2+\ \omega)\},$$

$$3r^3=-1-X+\frac{\alpha}{7}\{-2-3\omega+X(\ 3+\ \omega)\}+\frac{\beta}{7}\{\ 1+3\omega+X(\ 2-\ \omega)\};$$

viz. each of the imaginary seventh roots is thus expressed as a linear function of the cubic radicals α, β (involving ω under the radical signs) with coefficients which are functions of ω, X.

Recollecting the equations $\alpha^2=-\beta(1+3\omega)$, $\beta^2=-\alpha(1+3\omega^2)$, $\alpha\beta=7$; $\omega^2+\omega+1=0$, $X^2+X+2=0$; it is clear that, starting for instance from the equation for $3r$, and squaring each side of the equation, we should, after proper reductions, obtain for $9r^2$ an expression of the like form; viz. we thus in fact obtain the expression for $3r^2$; then from the expressions of $3r$ and $3r^2$, multiplying together and reducing, we should obtain the expression for $3r^3$; and so on; viz. from any one of the six equations we can in this manner obtain the remaining five equations.

At the time of writing what precedes I did not recollect Jacobi's paper "Ueber die Kreistheilung und ihre Anwendung auf die Zahlentheorie," *Berliner Monatsber.*, (1837) and *Crelle*, t. XXX. (1846), pp. 166—182; [*Ges. Werke*, t. VI. pp. 254—274]. The starting-point is the following theorem: if x be a root of the equation $\dfrac{x^p-1}{x-1}=0$, p a prime number, and if g is a prime root of p, and

$$F(\alpha)=x+\alpha x^g+\alpha^2x^{g^2}+\ldots+\alpha^{p-1}x^{g^{p-2}},$$

where α is any root of $\dfrac{\alpha^{p-1}-1}{\alpha-1}=0$, we have

$$F(\alpha^m)F(\alpha^n)=\psi(\alpha)F(\alpha^{m+n}),$$

where $\psi(\alpha)$ is a rational and integral function of α with integral coefficients; or, what is the same thing, if α and β be any two roots of the above-mentioned equation, then

$$F(\alpha)F(\beta)=\psi(\alpha,\ \beta)F(\alpha\beta),$$

where $\psi(\alpha,\ \beta)$ is a rational and integral function of α, β with integral coefficients. As regards the proof of this, it may be remarked that, writing x^3 for x, $F(\alpha)$, $F(\beta)$, and $F(\alpha\beta)$ become respectively $\alpha^{-1}F(\alpha)$, $\beta^{-1}F(\beta)$, $(\alpha\beta)^{-1}F(\alpha\beta)$; hence, $F(\alpha)F(\beta)\div F(\alpha\beta)$ remains unaltered, and it thus appears that the function in question is expressible rationally in terms of the *adjoint* quantities α and β. With this explanation the following extract will be easily intelligible:

"The true form (never yet given) of the roots of the equation $x^p-1=0$ is as follows: The roots, as is known, can easily be expressed by mere addition of the functions $F(\alpha)$. If λ is a factor of $p-1$ and $\alpha^\lambda=1$, then it is further known that $\{F(\alpha)\}^\lambda$ is a mere function of α. But it is only necessary to know those values of $F(\alpha)$ for which λ is *the power of a prime number*. For suppose $\lambda\lambda'\lambda''\ldots$ is a factor of $p-1$; further let λ, λ', λ'', ... be powers of different prime numbers, and α, α', α'', ... prime λth, λ'th, λ''th, ... roots of unity, then

$$F(\alpha\alpha'\alpha''\ldots)=\frac{F(\alpha)F(\alpha')F(\alpha'')\ldots}{\psi(\alpha,\ \alpha',\ \alpha'',\ldots)}$$

where $\psi(\alpha,\ \alpha',\ \alpha'',\ldots)$ denotes a rational and integral function of α, α', α'', ... with integral coefficients. Hence, considering always the $(p-1)$th roots of unity as given, there are contained in the expression for x only radicals, the exponents of which are powers of prime numbers, and products of such radicals. But if λ is a power of a prime number, $=\mu^n$, suppose, the corresponding function $F(\alpha)$ can be found as follows: Assume

$$F(\alpha)F(\alpha^i)=\psi_i(\alpha)F(\alpha^{i+1}):$$

then

$$F(\alpha)\ =\sqrt[\mu]{\{\psi_1(\alpha)\ \psi_2(\alpha)\ \ldots\psi_{\mu-1}(\alpha)\ \ F(\alpha^\mu)\}},$$

$$F(\alpha^\mu)=\sqrt[\mu]{\{\psi_1(\alpha^\mu)\psi_2(\alpha^\mu)\ldots\psi_{\mu-1}(\alpha^\mu)F(\alpha^{\mu^2})\}},$$

and so on, up to

$$F(\alpha^{\mu^{n-1}})=\sqrt[\mu]{\{\psi_1(\alpha^{\mu^{n-1}})\psi_2(\alpha^{\mu^{n-1}})\ldots\psi_{\mu-1}(\alpha^{\mu^{n-1}})(-)^{\frac{p-1}{\mu}}\ p\}},"$$

so that the formulæ contain ultimately μth roots only. It is remarked in a foot-note that, when $n=1$, the $\mu-1$ functions can always be reduced to one-sixth part in number, and that by an induction continued as far as $\mu=31$, Jacobi had found that all the functions ψ could be expressed by means of the values of a single one of these functions.

"The $\mu-1$ functions determine, not only the values of all the magnitudes under the radical signs, but also the mutual dependence of the radicals themselves. For replacing α by the different powers of α, one can by means of the values so obtained for these functions rationally express all the μ^n-1 functions $F(\alpha^i)$ by means of the powers of $F(\alpha)$; since all the μ^n-1 magnitudes $\{F(\alpha)\}^i\div F(\alpha^i)$ are each of them equal to a product of several of the functions $\psi(\alpha)$. Herein consists one of the great advantages of the method over that of Gauss, since in this the discovery of the mutual dependency of the different radicals requires a special investigation, which, on account of its laboriousness, is scarcely practicable for even small primes; whereas the introduction of the functions ψ gives simultaneously the quantities under the radical signs, and the mutual dependency of the radicals. The formation of the functions ψ is obtained by a very simple algorithm, which requires only that one should, from the table for the residues of g^m, form another table giving $g^{m'} = 1 + g^m$ (mod. p), [see Table IV. of the Memoir]. According to these rules one of my auditors [Rosenhain] in a Prize-Essay of the [Berlin] *Academy* has completely solved the equations $x^p - 1 = 0$ for all the prime numbers p up to 103."

I am endeavouring to procure the Prize-Essay just referred to. As an example—which however is too simple a one to fully bring out Jacobi's method, and its difference from that of Gauss—consider the equation for the fifth roots of unity, $x^4 + x^3 + x^2 + x + 1 = 0$. According to Gauss, we have $x + x^4$ and $x^2 + x^3$, the roots of the equation $u^2 + u - 1 = 0$; say $x + x^4 = \frac{1}{2}\{-1 + \sqrt{(5)}\}$, $x^2 + x^3 = \frac{1}{2}\{-1 - \sqrt{(5)}\}$. The first of these, combined with $x \,.\, x^4 = 1$, gives $x - x^4 = \sqrt{[-\frac{1}{2}\{5 + \sqrt{(5)}\}]}$; and thence $4x = -1 + \sqrt{(5)} + \sqrt{[-2\{5 + \sqrt{(5)}\}]}$; if from the second of them, combined with $x^2 \,.\, x^3 = 1$, we were in like manner to obtain the values of x^2 and x^3, it would be necessary to investigate the signs to be given to the radicals, in order that the values so obtained for x^2 and x^3 might be consistent with the value just found for x. For the Jacobian process, observing that a prime fourth root of unity is $\alpha = i$, and writing for shortness F_1, F_2, F_3, F_4 to denote $F(\alpha)$, $F(\alpha^2)$, $F(\alpha^3)$, $F(\alpha^4)$ respectively, these functions are

$$F_1 = x - x^4 + i(x^2 - x^3),$$
$$F_2 = x + x^4 - (x^2 + x^3),$$
$$F_3 = x - x^4 - i(x^2 - x^3),$$
$$F_4 = x + x^4 + x^2 + x^3,$$

viz. we have $F_4 = -1$, $F_2^2 = 5$, or say $F_2 = \sqrt{(5)}$, $F_1^2 = -(1 + 2i) F_2$, $= -(1 + 2i)\sqrt{(5)}$; and similarly $F_3^2 = -(1 - 2i) F_2$, $= -(1 - 2i)\sqrt{(5)}$; but also $F_1 F_3 = -5$, so that the values $F_1 = \sqrt{\{-(1 + 2i)\sqrt{(5)}\}}$, $F_3 = \sqrt{\{-(1 - 2i)\sqrt{(5)}\}}$, must be taken consistently with this last equation $F_1 F_3 = \sqrt{(5)}$. The values of F_1, F_2, F_3, F_4 being thus known, the four equations then give simultaneously x, x^4, x^2, x^3, these values being of course consistent with each other. It may be remarked that the form in which x presents itself is

$$4x = -1 + \sqrt{(5)} + \sqrt{\{-(1 + 2i)\sqrt{(5)}\}} + \sqrt{\{-(1 - 2i)\sqrt{(5)}\}},$$

with the before-mentioned condition as to the last two radicals; with this condition we, in fact, have

$$\sqrt{\{-(1 + 2i)\sqrt{(5)}\}} + \sqrt{\{-(1 - 2i)\sqrt{(5)}\}} = \sqrt{[-2\{5 + \sqrt{(5)}\}]},$$

as is at once verified by squaring the two sides.