445.

A MEMOIR ON QUARTIC SURFACES.

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THE present Memoir is intended as a commencement of the theory of the quartic surfaces which have nodes (conical points). A quartic surface may be without nodes, or it may have any number of nodes up to 16. I show that this is so, and I consider how many of the nodes may be given points. Although it would at first sight appear that the number is 8, it is in fact 7; viz., we can, with 7 given points as nodes (but not in a proper sense with 8 or more given points), find a quartic surface; such surface contains in its equation 6 constants, which may be such that the surface has an additional node or nodes. Suppose that the surface has an 8th node:—there are two distinct cases; viz., (1) the 8 nodes are the points of intersection of 3 quadric surfaces, or say they are an octad, and the surface is said to be octadic; (2) the 8th node is any point whatever on a certain sextic surface determined by means of the 7 given nodes, and called the dianodal surface of these 7 points; the quartic surface is said to be a dianome. The two cases are in general exclusive of each other; viz., the 7 given points being any points whatever, the dianodal surface does not pass through the 8th point of the octad; and thus the quartic surface with the 8 nodes is either octadic or else a dianome. Assuming it to be a dianome, the constants may be further determined so that there shall be a 9th node; it is necessary to examine whether this forms with 7 of the 8 nodes an octad. Supposing that it does not (viz., that there are not any 8 nodes in regard to which the surface is octadic), the 9th node is then any point whatever on a certain curve of the order 18, determined by means of the 8 nodes, and called the dianodal curve of these 8 points. And, finally, the constants may be further determined so that there shall be a 10th node; supposing, as before, that this does not form an octad with any 7 of the 9 nodes (viz., that there are not any 8 nodes in regard to which the surface is octadic), the 10th node is then any one of a system of 22 [should be 13] points determined by means of the 9 nodes, and called the dianodal system of these 9 points. But the quartic surface is now completely determined; viz., starting with any 7 given points as nodes, we have a dianome with 8 nodes, 9 nodes, or 10 nodes, say, an octodianome, enneadianome, or decadianome, but not with any greater number of nodes; these can only present themselves when particular conditions are satisfied in regard to the 7 given nodes, and to the 8th and 9th node; and the consideration of the quartic surfaces with more than 10 nodes would thus form a separate branch of the subject.

The case of the decadianome (or quartic surface with 10 nodes formed as above with 7 given points as nodes) is peculiarly interesting. I identify this with the surface which I call a symmetroid; viz., the surface represented by an equation $\Delta = 0$, where Δ is a *symmetrical* determinant of the 4th order the several terms whereof are linear functions of the coordinates (x, y, z, w); this surface is related to the Jacobian surface of 4 quadric surfaces (itself a very remarkable surface), and this theory of the symmetroid and the Jacobian, and of questions connected therewith, forms a large portion of the present Memoir.

The theory of the Jacobian is connected also with the researches in regard to nodal quartic surfaces in general; and, for greater clearness, it has seemed to me proper to commence the Memoir with certain definitions, &c., in regard to this theory. It will be seen in what manner I extend the notion of the Jacobian.

I remark that the present researches on Quartic Surfaces were suggested to me by Professor Kummer's most interesting Memoir "Ueber die algebraischen Strahlensysteme u.s.w.," *Berl. Abh.* 1866, in which, without entering upon the general theory, he is led to consider the quartic surfaces, or certain quartic surfaces, with 16, 15, 14, 13, 12, or 11 nodes; the last of these, or surface with 11 nodes, being in fact a particular case of the symmetroid.

Considerations in regard to the Jacobian of four, or more or less than four, Surfaces.

1. In the case of any four surfaces, $P=0$, $Q=0$, $R=0$, $S=0$, the differential coefficients of P, Q, R, S in regard to the coordinates (x, y, z, w) may be arranged as a square matrix in either of the ways

$$\begin{array}{c|c|c} & P,\ Q,\ R,\ S\ \ ;\ \ \delta_x,\ \delta_y,\ \delta_z,\ \delta_w & \\ \hline \delta_x & & P \\ \delta_y & & Q \\ \delta_z & & R \\ \delta_w & & S \end{array}$$

and with either arrangement we may form one and the same determinant, the Jacobian determinant $J(P, Q, R, S)$, or, equating it to zero, the Jacobian surface $J(P, Q, R, S)=0$, of the four surfaces.

2. In the case of more than four surfaces, adopting the arrangement

$$\begin{array}{c|c} & P,\ Q,\ R,\ S,\ T,\ldots \\ \hline \delta_x & \\ \delta_y & \cdot \\ \delta_z & \\ \delta_w & \end{array}$$

and considering the several determinants which can be formed with any four columns of the matrix, these equated to zero establish a more than one-fold relation between the coordinates; viz., in the case of five surfaces, we have $J(P, Q, R, S, T)\equiv 0$, a twofold relation representing a curve; and in the case of six surfaces, $J(P, Q, R, S, T, U)\equiv 0$, a threefold relation representing a point-system; and (since with four coordinates a relation is at most threefold) these are the only cases to be considered.

3. In the case of fewer than four surfaces, adopting the arrangement

$$\begin{array}{c|c} \delta_x,\ \delta_y,\ \delta_z,\ \delta_w & \\ \hline & P \\ & Q \\ & \cdot \end{array}$$

and considering the several determinants which can be formed with any 3 or 2 columns of the matrix, and equating these to zero, we have in like manner a more than one-fold relation between the coordinates; viz., in the case of three surfaces, we have $J(P, Q, R)\equiv 0$, a twofold relation representing a curve; and in the case of two surfaces $J(P, Q)\equiv 0$, a threefold equation representing a point-system, (viz., this denotes the points $\delta_x P : \delta_y P : \delta_z P : \delta_w P = \delta_x Q : \delta_y Q : \delta_z Q : \delta_w Q$); for a single surface we should have a fourfold relation, and the case is not considered. But observe that if the notation were used, $J(P)\equiv 0$ would denote $\delta_x P=0$, $\delta_y P=0$, $\delta_z P=0$, $\delta_w P=0$, equations which are satisfied simultaneously by the coordinates (x, y, z, w) of any node of the surface $P=0$. Although in what precedes I have used the sign $\equiv$, there is no objection to using, and I shall in the sequel use, the ordinary sign $=$, it being understood that while $J(P, Q, R, S)=0$ denotes a single equation or onefold relation, $J(P, Q, R, S, T)=0$ or $J(P, Q, R)=0$ will each denote a twofold relation, and $J(P, Q, R, S, T, U)=0$ or $J(P, Q)=0$ each of them a threefold relation.

4. It is not asserted that ... $J(P, Q, R)=0$, $J(P, Q, R, S)=0$, $J(P, Q, R, S, T)=0$,... form a continuous series of analogous relations; and there might even be a propriety in using, in regard to four or more surfaces, J, and in regard to four or fewer surfaces an inverted J (viz., in regard to four surfaces, either symbol indifferently); but there is no ambiguity in, and I have preferred to adopt, the use of the single symbol J.

5. Suppose that the orders of the surfaces $P=0$, $Q=0,\ldots$ are $a+1$, $b+1,\ldots$ (so that the orders of the differential coefficients of P, $Q\ldots$ are a, $b,\ldots$), then we have for the orders of the several loci,

$$\begin{array}{rll} J(P,\ Q)=0, & \text{point-system, order} & a^3+a^2b+ab^2+b^3\,; \\ J(P,\ Q,\ R)=0, & \text{curve,} \qquad ,, & a^2+b^2+c^2+bc+ca+ab\,; \\ J(P,\ Q,\ R,\ S)=0, & \text{surface,} \qquad ,, & a+b+c+d\,; \\ J(P,\ Q,\ R,\ S,\ T)=0, & \text{curve,} \qquad ,, & ab+ac\ldots+de\,; \\ J(P,\ Q,\ R,\ S,\ T,\ U)=0, & \text{point-system,} \quad ,, & abc+abd\ldots+def\,; \end{array}$$

see, as to this, Salmon's *Solid Geometry*, Ed. 2, (1865), Appendix IV., "On the Order of Systems of Equations" [not reproduced in the later editions]. In particular, if $a=b=c\ldots=1$, then the orders are 4, 6, 4, 10, 20.

As to the Surface obtained by equating to zero a Symmetrical Determinant.

6. It is also shown (Salmon, Ed. 2, p. 495) that the surface obtained by equating to zero any symmetrical determinant has a determinate number of nodes; viz., if the orders of the terms in the diagonal be a, b, c, &c., then the number of nodes is $=\frac{1}{2}(\Sigma a\,.\,\Sigma ab-\Sigma abc)$, or, as this may also be written, $\frac{1}{2}(\Sigma a^2b+2\Sigma abc)$. In particular, the formula applies to the case of the surface

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix}=0,$$

$(a,\ b,\ c,\ d)$ being here the orders of A, B, C, D respectively, and the orders of F, G, &c., being $\frac{1}{2}(b+c)$, $\frac{1}{2}(a+c)$, &c. If the terms are all of them linear functions of the coordinates, or $a=b=c=d=1$, then the number of nodes is $=10$.

7. That the surface has nodes is, in fact, clear from the consideration that any point for which the minors of the determinant all vanish will be a node; and that (for the symmetrical determinant), by making the minors all of them vanish, we establish only a threefold relation between the coordinates. The expression for the number of the nodes is, I think, obtained most readily as follows:

The nodes will be points of intersection of the curve and surface

$$\left\Vert\begin{matrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \end{matrix}\right\Vert=0,\qquad \begin{vmatrix} B, & F, & M \\ F, & C, & N \\ M, & N, & D \end{vmatrix}=0,$$

these, however, contain in common the points

$$\left\Vert\begin{matrix} H, & B, & F, & M \\ G, & F, & C, & N \end{matrix}\right\Vert=0\,;$$

and not only so, but they touch at the points in question; so that, multiplying together the orders of the curve and surface, and subtracting *twice* the order of the point-system, we obtain the expression for the number of nodes. In the particular case where the functions are all linear, we have a sextic curve and cubic surface intersecting in 18 points; but the curve and surface touch in 4 points, and the number of nodes is $(18-2.4)=10$. And in the same way the formula may be established for the general case.

8. The subsidiary theorem of the contact of the curve and surface requires, however, to be proved. Seeking for the equation of the tangent plane of the surface at any one of the points in question, we have first

$$\begin{vmatrix} \delta B, & \delta F, & \delta M \\ F, & C, & N \\ M, & N, & D \end{vmatrix} + \begin{vmatrix} B, & F, & M \\ \delta F, & \delta C, & \delta N \\ M, & N, & D \end{vmatrix} + \begin{vmatrix} B, & F, & M \\ F, & C, & N \\ \delta M, & \delta N, & \delta D \end{vmatrix} = 0,$$

where, in virtue of the equations

$$\left\|\begin{matrix} H, & B, & F, & M \\ G, & F, & C, & N \end{matrix}\right\| = 0,$$

the last term vanishes. Expanding the other two terms, the equation becomes

$$D(C\delta B + B\delta C - 2F\delta F) - (N^2\delta B - 2MN\delta F + M^2\delta C) + \delta M(FN - CM) + \delta N(BN - MF) = 0;$$

but, in virtue of the same equations, the coefficients of δM and δN each of them vanish, and we have also

$$N^2\delta B + M^2\delta C - 2MN\delta F = \frac{N^2}{C}(C\delta B + B\delta C - 2F\delta F);$$

so that the equation becomes finally $C\delta B + B\delta C - 2F\delta F = 0$. Investigating by a like process the equation of the tangent of the curve

$$\left\|\begin{matrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \end{matrix}\right\| = 0,$$

we find between the differentials δA, δB, &c., a twofold linear relation, expressible by means of the foregoing equation $C\delta B + B\delta C - 2F\delta F = 0$, and one other equation; that is, at each of the points in question the tangent of the curve lies in the tangent plane of the surface, or, what is the same thing, the curve and surface touch at these points.

Surfaces represented by an equation $F(P, Q) = 0$, &c.

9. In the remarks which follow as to the surfaces $F(P, Q) = 0$, $F(P, Q, R) = 0$, &c., the function F is a rational and integral function of (P, Q), (P, Q, R), &c., not in general homogeneous in regard to $P, Q, R, \ldots$. but of such degrees in regard to these functions respectively as to be homogeneous in regard to the coordinates (x, y, z, w).

The surface $F(P, Q) = 0$ has in general a nodal curve $\delta_P F = 0$, $\delta_Q F = 0$; and if it has besides any nodes, these are points of the point-system $J(P, Q) = 0$.

The surface $F(P, Q, R) = 0$ has in general nodes $\delta_P F = 0$, $\delta_Q F = 0$, $\delta_R F = 0$; and if it has besides any nodes, these are points on the curve $J(P, Q, R) = 0$.

The surface $F(P, Q, R, S) = 0$ has not in general, but it may have, nodes $\delta_P F = 0$, $\delta_Q F = 0$, $\delta_R F = 0$, $\delta_S F = 0$; if it has any other nodes, these are points on the surface $J(P, Q, R, S) = 0$.

Nodes of a Quartic Surface; Circumscribed Cone having its vertex at a Node.

10. A quartic surface may be without nodes; or it may have any number of nodes up to 16. Consider a quartic surface having a node or nodes; and take the single node, or (if more nodes than one) any one of the nodes, as the vertex of a circumscribed cone; then, considering any plane through the vertex, the section will be a quartic curve having a node at the vertex, and the generating lines in the plane will be the tangents from the node to the quartic curve; the number of them is therefore 6, and the order of the circumscribed cone is thus $= 6$. Each tangent intersects the quartic curve in the node counting as two intersections, and in the point of contact counting as two intersections; there are consequently no singular tangents; and therefore in the circumscribed cone no singular lines arising from a singular tangency of the generating line. Hence, in the case of a single node on the surface, the circumscribed cone is a cone of the order 6 without nodal or stationary lines; and the class is $= 30$. But in the case of more than one node, say k nodes, the circumscribed cone passes through the remaining $k - 1$ nodes, and the generating line through each of these nodes is a nodal line of the cone; that is, the cone has $k - 1$ nodal lines, and its class is $= 30 - 2k + 2$. The cone is not of necessity a proper cone; the maximum number of nodal lines is when it breaks up into 6 planes, and we have then $k - 1 = 15$; that is, the number of nodes of the surface is at most $= 16$.

11. It is easy to form a table of the different *primâ facie* possible forms of the sextic cone, according to the number of nodes of the surface; viz., writing 6 for a proper sextic cone without nodal lines, 6_1, $6_2 \ldots 6_{10}$ for the proper sextic cone with 1, 2, ... or 10 nodal lines; and so 5, $5_1 \ldots 5_6$ for the proper quintic cones, 4, 4_1, 4_2, 4_3, 3, 3_1, 2 for the quartic, cubic, and quadric cones, and 1 for the plane, the table is

Circumscribed Sextic Cone.

Nodes of Surface.									
1	6								
2	6_1								
3	6_2								
4	6_3								
5	6_4								
6	6_5	5, 1							
7	6_6	5_1, 1							
8	6_7	5_2, 1							
9	6_8	5_3, 1	4, 2						
10	6_9	5_4, 1	4_1, 2	4, 1, 1	3, 3				
11	6_{10}	5_5, 1	4_2, 2	4_1, 1, 1	3_1, 3				
12	...	5_6, 1	4_3, 2	4_2, 1, 1	3_1, 3_1	3, 2, 1			
13	...	...	...	4_3, 1, 1	...	3_1, 2, 1	3, 1, 1, 1	2, 2, 2	
14	...	...	...	...	...	...	3_1, 1, 1, 1	2, 2, 1, 1	
15	...	...	...	...	...	...	...		2_1, 1, 1, 1, 1
16	...	...	...	...	...	...	...		1, 1, 1, 1, 1, 1;

and moreover, in the cases where there are two or more forms of the sextic cone, then the k sextic cones may be of the different forms in various combinations. The total number of cases *primâ facie* possible is thus very great; but only a comparatively small number of them actually exist.

12. In the case where there is a plane 1, the sextic cone breaks up into this plane, and into a (proper or improper) quintic cone intersecting the plane in 5 lines; that is, there will be in the plane 6 nodes; the plane is, in fact, a singular tangent plane meeting the surface in a conic twice repeated; and the 6 nodes lie on this conic. Taking any one of these nodes as vertex, the corresponding sextic cone breaks up into the plane, and into a (proper or improper) quintic cone.

13. In the cases $k=1$, 2, 3, 4, 5, and $k=15$, 16, there is only one form of sextic cone; so that each node (at least so far as appears) stands in the same relation to the surface. Considering the last mentioned two cases; $k=16$,—each of the 16 nodes gives 6 singular tangent planes, but each of these passes through 6 nodes; therefore the number of planes is $=16$: similarly, $k=15$, the number of singular tangent planes is $15 \times 4 \div 6$, $=10$.

For $k=14$, the cones are 3_1, 1, 1, 1, or 2, 2, 1, 1: it is easy to see that we have only the three cases

Cones 3_1, 1, 1, 1 : 2, 2, 1, 1			Singular tangent planes
No. may be	14 , 0	gives	$(14 . 3 + 0 . 2) \div 6, = 7$
„	8 , 6	„	$(8 . 3 + 6 . 2) \div 6, = 6$
„	2 , 12	„	$(2 . 3 + 12 . 2) \div 6, = 5$

and we may in the like manner limit the number of possible cases, for other values of k. But I do not at present further pursue the inquiry.

As to the Number of Constants contained in a Surface.

14. We say that a surface $P=0$ contains or depends upon a certain number of constants; viz., this is the number of constants contained in the equation $P=0$ of the surface, taking the coefficient of any one term to be equal to unity; thus the general quadric surface contains 9 constants; the surface can in fact be determined so as to satisfy 9 conditions; or, as we might express it, the *Postulation* of the surface is $=9$. [I have elsewhere said *Postulandum* and *Capacity*: I prefer this last expression.] And if, in the general equation so containing 9 constants, k of these are given, or, what is the same thing, if the quadric surface be made to satisfy any k conditions, then the number of constants, or postulation of the surface, is $=9-k$.

15. But a different form of expression is sometimes convenient; the conditions to be satisfied are frequently such that, being satisfied by the surfaces $P=0$, $Q=0,\ldots$, they will be satisfied by the surface $\alpha P+\beta Q+\ldots=0$, where α, β, ... are any constant multipliers whatever. When this is so, there will be a certain number of solutions $P=0$, $Q=0,\ldots$ not connected by any such relation, or say of asyzygetic solutions, such that the general surface satisfying the conditions in question is $\alpha P+\beta Q+\ldots=0$; and hence, taking one of these coefficients as unity, the number of constants, or postulation of the surface, is equal to the number of the remaining coefficients, or, what is the same thing, it is less by unity than the number of the asyzygetic solutions $P=0$, $Q=0\ldots$. Instead of considering the number of constants, or postulation, we may consider the number of solutions (that is, asyzygetic solutions) or surfaces $P=0$, $Q=0,\ldots$ which satisfy the conditions in question.

16. Thus, for the quadric not subjected to any conditions, there are 10 surfaces (for example, these may be taken to be the surfaces $x^2=0$, $y^2=0$, $z^2=0$, $w^2=0$, $yz=0$, $zx=0$, $xy=0$, $xw=0$, $yw=0$, $zw=0$); and the general quadric surface is by means of these expressed linearly in the form $(a,\ldots\between x, y, z, w)^2=0$. So for the quadric surfaces through k given points, the number of these is $=10-k$; thus for the surfaces through 4 given points, say the points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), the 6 given surfaces may be taken to be $yz=0$, $zx=0$, $xy=0$, $xw=0$, $yw=0$, $zw=0$, and every other quadric surface through the 4 points is by means of these expressed linearly in the form $(a,\ldots\between yz, zx, xy, xw, yw, zw)=0$; for the quadric surfaces through 8 points there are two surfaces $P=0$, $Q=0$; and every quadric surface through the

8 points is by means of these expressed linearly in the form $\alpha P+\beta Q=0$; and (as the extreme case) if the quadric surface passes through 9 given points, then there is the single quadric surface $P=0$.

17. In the questions in which such quadric surfaces present themselves, it is in general quite immaterial what particular surfaces are selected as the surfaces $P=0$, $Q=0, \ldots$; the selection may be made at pleasure and, being so made, the surfaces are to be regarded as completely determinate; viz., there would be no gain of generality if these were replaced by any other surfaces $\alpha P+\beta Q \ldots=0$. For instance, in the theory of the quartic surfaces with 6 given points as nodes, we have through the 6 given points the 4 quartic surfaces $P=0$, $Q=0$, $R=0$, $S=0$, and we consider the quartic functions $(a, \ldots \between P, Q, R, S)^2$ and $J(P, Q, R, S)$: each of these is unaltered as to its form when P, Q, R, S are replaced each of them by any linear function of these quantities; viz., $(a, \ldots \between P, Q, R, S)^2$ is changed into a new quadric function $(a', \ldots \between P, Q, R, S)^2$, and $J(P, Q, R, S)$ into a mere constant multiple of its original value. We have herein a justification of the expressions in question, through 6 given points there are 4 quadric surfaces, &c.

General theory of the Quartic Surface with a given Node or Nodes.

18. A quartic surface contains 34 constants; and the number of conditions to be satisfied in order that a given point may be a node is $=4$. Hence, if the surface has k given points as nodes, the number of constants is $=34-4k$; and it would at first sight appear that k might be $=8$, and that with the 8 given points as nodes we should have a quartic surface containing 2 constants. But this is not so in a proper sense; for through the 8 given points we have 2 quadric surfaces $P=0$, $Q=0$; and we can by means of these form a quartic surface $(a, b, c \between P, Q)^2=0$, containing 2 constants, and having in a sense the 8 points as nodes. This, however, is no proper quartic surface, but is a system of 2 quadric surfaces, each of them passing through the 8 points, and the two quadric surfaces therefore intersecting in a quadri-quadric curve through the 8 points; which curve is therefore a nodal curve on the compound surface; and it is only as points on this nodal curve, and not in a proper sense, that the 8 given points are nodes of the quartic surface. The greatest value of k is thus $k=7$.

19. Of course, if $k=0$, we have the general quartic surface $U=0$, containing 34 constants. The cases $k=1$, $k=2$, $k=3$ (viz., a single given node, 2 given nodes, 3 given nodes), may be at once disposed of; taking for instance the 1st node to be the point (1, 0, 0, 0), the 2nd node the point (0, 1, 0, 0), the 3rd node the point (0, 0, 1, 0), we find at once an equation $U=0$, with 30, 26, or 22 constants, having the given node or nodes.

Four given Nodes.

20. The case of 4 given nodes is just as easy; but in reference to what follows, it is proper to consider it more in detail. The equation should contain 18 constants; we have through the 4 given points 6 quadric surfaces, $P=0$, $Q=0$, $R=0$, $S=0$, $T=0$,

$U=0$, and we can by means of them form a quartic equation $(a, \ldots \between P, Q, R, S, T, U)^2=0$, having the 4 given points as nodes; this contains, however, $(21-1=)$ 20 constants; the reduction to the right number 18 occurs by reason that the functions (P, Q, R, S, T, U), although linearly independent, are connected by two quadric equations

$$(*\between P, Q, R, S, T, U)^2=0, \quad (*\between P, Q, R, S, T, U)^2=0;$$

hence writing the equation of the quartic surface in the form

$$(a, \ldots \between{}_{,,})^2 - \lambda\,(*\between{}_{,,})^2 - \mu\,(*\between{}_{,,})^2 = 0,$$

the coefficients λ, μ may be so determined as to reduce to zero the coefficients of any two terms of the equation, and the number of constants really is $20-2=18$, as it should be.

21. In proof, observe that, taking the 4 given nodes to be the points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (1, 0, 0, 0), the quadric surfaces may be taken to be $yz=0$, $zx=0$, $xy=0$, $xw=0$, $yw=0$, $zw=0$; the equation of the quartic surface will thus be

$$(a, \ldots \between yz, zx, xy, xw, yw, zw)^2=0;$$

but we have between the functions xy, &c., the two identical relations

$$xy\,.\,zw - xz\,.\,yw = 0, \quad xy\,.\,zw - xw\,.\,yz = 0;$$

and the number of constants is thus $=18$.

Five given Nodes.

22. In the case of 5 given nodes, the number of constants should be $=14$. We have through the 5 given points, 5 quadric surfaces $P=0$, $Q=0$, $R=0$, $S=0$, $T=0$, and we form herewith the quartic equation $(a, \ldots \between P, Q, R, S, T)^2=0$, containing the right number 14 of arbitrary constants. The functions P, Q, &c. are in this case not connected by any quadric relation, and the equation just written down is in fact the general equation of the quartic surface with the 5 given nodes.

23. In verification, take the first 4 nodes to be as above, and the 5th node to be the point (1, 1, 1, 1); we may write

$$(P, Q, R, S, T) = \{x(y-z),\ x(y-w),\ y(x-z),\ y(x-w),\ xy-zw\};$$

and if from the 5 equations $P=x(y-z)$, &c., we eliminate (x, y, z, w), we obtain one, and only one, relation between the functions P, Q, R, S, T; this is found to be

$$PS(Q+R-T) - QR(P+S-T) = 0,$$

or, what is the same thing,

$$R(P-Q)(S-T) - P(R-S)(Q-T) = 0;$$

viz., it is a cubic relation, and there is consequently no quadric relation between the 5 functions.

Six given Nodes.

24. In the case of 6 given nodes, the quartic surface should contain 10 constants. We have through the 6 given points 4 quadric surfaces $P=0$, $Q=0$, $R=0$, $S=0$; but if we form herewith the quartic surface $(a, \ldots \between P, Q, R, S)^2=0$, this contains only 9 constants. It is to be shown that the Jacobian surface $J(P, Q, R, S)=0$ of the 4 quadric surfaces (or say of the 6 points) is a quartic surface having the 6 given points as nodes, and not included in the foregoing form $(a, \ldots \between P, Q, R, S)^2=0$; this being so, we have the quartic surface

$$(a, \ldots \between P, Q, R, S)^2 + \theta J(P, Q, R, S) = 0,$$

having the 6 given points as nodes, and containing the complete number of constants, viz., 10.

25. The 6 given nodes being any points whatever, their coordinates may be taken to be $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 1, 1, 1)$, and $(\alpha, \beta, \gamma, \delta)$. I proceed to find the Jacobian of these 6 points. For this purpose, let (a, b, c, f, g, h) be the 6 coordinates of the line through the points $(1, 1, 1, 1)$ and $(\alpha, \beta, \gamma, \delta)$, viz.,

$$\begin{aligned} a &= \beta - \gamma, & f &= \alpha - \delta, \\ b &= \gamma - \alpha, & g &= \beta - \delta, \\ c &= \alpha - \beta, & h &= \gamma - \delta, \end{aligned}$$

whence $af + bg + ch = 0$, and also

$$\begin{array}{rrrrl} \cdot & h & -g & +a & = 0, \\ -h & \cdot & +f & +b & = 0, \\ g & -f & \cdot & +c & = 0, \\ -a & -b & -c & \cdot & = 0, \end{array}$$

we have through the 6 points the plane pairs

$$\begin{aligned} x\,(\ \ \cdot\ - hx - gz + aw) &= 0, \\ y\,(-hx\ \ \cdot\ + fz + bw) &= 0, \\ z\,(\ \ gx - fy\ \ \cdot\ + cw\,) &= 0, \\ w\,(-ax - by - cz\ \ \cdot\ ) &= 0, \end{aligned}$$

where, adding the four equations, we have identically $0=0$. For this reason, we cannot take these to be the equations of the 4 quadric surfaces, but we may take the first 3 of them for the surfaces $P=0$, $Q=0$, $R=0$; and for the 4th surface $S=0$, I take the quadric cone having its vertex at the point $(0, 0, 0, 1)$; viz., the equation is

$$a\alpha yz + b\beta zx + c\gamma xy = 0;$$

that is, I write

$$(P, Q, R, S) = \{x(hy - gz + aw),\ y(-hx + fz + bw),\ z(gx - fy + cw),\ (a\alpha yz + b\beta zx + c\gamma xy)\}.$$

26. The Jacobian is then easily found to be

$$\begin{aligned}&(b\beta zx + c\gamma xy)\,(-agh,\ bhf,\ cfg,\ abc,\ -af^2,\ -gB,\ hC,\ aA,\ b^2g,\ -c^2h \between x,\ y,\ z,\ w)^2\\ &+ (c\gamma xy + a\alpha yz)\,(agh,\ -bhf,\ cfg,\ abc,\ fA,\ -bg^2,\ -hC,\ -a^2f,\ bB,\ c^2h \between x,\ y,\ z,\ w)^2\\ &+ (a\alpha yz + b\beta zx)\,(agh,\ bhf,\ -cfg,\ abc,\ -fA,\ gB,\ -ch^2,\ a^2f,\ -b^2g,\ cC \between x,\ y,\ z,\ w)^2 = 0\,;\end{aligned}$$

where for the moment A, B, C denote $bg-ch$, $ch-af$, $af-bg$ respectively. Collecting and reducing, the whole divides by $2abc$; and if finally we replace a, b, c, f, g, h by their values, the result is

$$J = \left\{\begin{aligned}&(\beta-\gamma)\,yz\,(\alpha w^2-\delta x^2) + (\alpha-\delta)\,xw\,(\beta z^2-\gamma y^2)\\ &+(\gamma-\alpha)\,zx\,(\beta w^2-\delta y^2) + (\beta-\delta)\,yw\,(\gamma x^2-\alpha z^2)\\ &+(\alpha-\beta)\,xy\,(\gamma w^2-\delta z^2) + (\gamma-\delta)\,zw\,(\alpha y^2-\beta x^2)\end{aligned}\right\} = 0.$$

27. It may be shown *à posteriori* that J is not a quadric function of P, Q, R, S. For, attempting to express it in this form, J does not contain the terms x^2w^2, y^2w^2, z^2w^2, and it thence at once appears that the coefficients of P^2, Q^2, R^2 each of them vanish. Hence, introducing for convenience the factor 2, I assume $(0, 0, 0, D, F, G, H, L, M, N \between P, Q, R, S)^2 = 2J$. Comparing the terms in $w^2(yz,\ zx,\ xy)$, we obtain

$$bcF = a\alpha,\quad caG = b\beta,\quad abH = c\gamma\,;$$

and comparing the coefficients of $w\,(y^2z,\ z^2x,\ x^2y,\ yz^2,\ zx^2,\ xy^2)$, we obtain

$$-Ff + a\alpha M = \frac{h\alpha}{b},\qquad Ff + a\alpha N = -\frac{g\alpha}{c},$$

$$-Gg + b\beta N = \frac{f\beta}{c},\qquad Gg + b\beta L = -\frac{h\beta}{a},$$

$$-Hh + c\gamma L = \frac{g\gamma}{a},\qquad Hh + c\gamma M = -\frac{f\gamma}{b}\,;$$

substituting for F, G, H their values, we obtain from the first 3 equations L, M, N $= \dfrac{-f}{bc}$, $\dfrac{-g}{ca}$, $\dfrac{-h}{ab}$, and from the second 3 equations, L, M, $N = \dfrac{f}{bc}$, $\dfrac{g}{ca}$, $\dfrac{h}{ab}$; that is, the equations are inconsistent, and the function J is not expressible in the form in question.

Jacobian Surface of Six given Points.

28. The equation $J = 0$ is the locus of the vertices of the quadric cones which pass through the given 6 points; calling these 1, 2, 3, 4, 5, 6, we see at once that the surface passes through the 15 lines 12, 13, ... 56, and also through the ten lines 123 . 456 (viz., line of intersection of the planes through 1, 2, 3, and through 4, 5, 6), &c. In fact, taking the vertex at any point O in the line 1, 2, the lines drawn to the six points are $O1 = O2$, $O3$, $O4$, $O5$, $O6$; viz., there are only five lines, so that these lie in a quadric cone. And taking the vertex at any point in the line 123 . 456,

the lines to the 6 points lie in these planes 123 and 456 respectively, and the quadric cone is in fact this plane-pair. Moreover, the surface containing the lines 12, 13, 14, 15, 16, must have the point 1 for a node; and similarly, the points 2, 3, 4, 5, 6 are each of them a node on the surface. It is to be added that the surface contains the skew cubic through the 6 points, or say the skew cubic 123456. See, as to this, *post* No. 108.

29. The surface in question (the Jacobian of the 6 points) is a particular case of the Jacobian of any 4 quadric surfaces. This more general surface will be considered in the sequel; I only remark here that it contains 10 lines, corresponding to the 10 lines 123.456, &c., but it has not any other lines, or any nodes.

Jacobian Curve of Seven given Points, or of an Octad of Points.

30. In connexion with what precedes, we may here consider a curve which presents itself in the sequel; viz., the curve which is the locus of the vertices of the quadric cones which pass through seven given points. The general case is when no one of the points is the vertex of a quadric cone through the other 6 points. We have through the 7 points the three quadric surfaces $P=0$, $Q=0$, $R=0$; hence, forming the equation $\alpha P+\beta Q+\gamma R=0$ of the general quadric surface through the 7 points, and making this a cone, we find as the locus of the vertex $J(P, Q, R)=0$; the analytical form shows that this is a sextic curve. It appears, moreover, that the curve is symmetrically related to all the 8 points $P=0$, $Q=0$, $R=0$; and instead of calling it the Jacobian of the 7 points, we may call it the Jacobian of the octad. But in further explanation, take the points to be 1, 2, 3, 4, 5, 6, 7; the vertex will lie on each of the Jacobian surfaces 123456 and 123457; and it is at present assumed that 7 is not a point on the first surface, nor 6 a point on the second surface. The two surfaces have in common the lines 12, 13, ... 45, and they consequently besides intersect in a curve of the 6th order, or sextic curve, which is the locus in question. At the point 1 there is on the first surface a tangent cone through the lines 12, 13, 14, 15, 16, and on the second surface a tangent cone through the lines 12, 13, 14, 15, 17; these two cones have for their complete intersection the lines 12, 13, 14, 15, which lines belong to the complete intersection of the two surfaces, but not to the sextic curve. It thus appears, *à posteriori*, that the sextic curve does not pass through the point 1; and similarly, that it does not pass through any of the points 2, 3, 4, or 5. As to the points 6 and 7, each of these is on only one of the quartic surfaces, and therefore the curve of intersection does not pass through either of these points.

31. Suppose, however, that one of the seven points is the vertex of a cone through the other six; it is of course the same thing whether we take this to be one of the points 1, 2, 3, 4, 5, or one of the points 6 and 7, but the result comes out more easily in the latter case; viz., in the former case, taking 1 to be the point in question, the two tangent cones at 1 are one and the same cone, and all that appears is that there is nothing to hinder a branch or branches of the sextic curve from passing through the point 1. But in the latter case, taking 7 for the point in question, then 7 lies on the surface 123456, being a simple point on this surface, but a node on the surface 123457; and it thus appears that there are through 7 two

branches of the sextic curve; so that any one of the seven points, being the vertex of a cone through the other six, is an actual double point on the sextic curve.

32. In the case where two of the points are each of them the vertex of a cone through the other six points, then the seven points lie on a skew cubic; and the sextic curve of the general case becomes this skew cubic twice repeated.

Seven given Nodes.

33. In the case of 7 given nodes, the number of constants should be $=6$; the 7 given points determine 3 quadric surfaces $P=0$, $Q=0$, $R=0$: and we have hence the quartic surface $(a, \ldots \between P, Q, R)^2=0$, containing 5 constants only. That this is not the general quartic surface with the 7 given nodes, is also clear from the consideration that the surface in question has 8 nodes; viz., the 8 points of intersection of the three quadric surfaces. Suppose that a particular quartic surface, having the 7 given nodes, but not of the last mentioned form, is $\Delta=0$; then a quartic surface having the 7 given nodes is

$$(a, \ldots \between P, Q, R)^2+\theta\Delta=0;$$

and this, as containing 6 constants, will be the general quartic surface with the 7 given nodes.

34. It follows that, if $\Delta'=0$ be another quartic surface having the 7 given nodes, we must have identically $\Delta'-p\Delta=(*\between P, Q, R)^2$, where p is a determinate constant and $(*\between P, Q, R)^2$ a determinate quadric function of (P, Q, R). The formula extends to the case where $\Delta'=0$ has the 8 nodes $(P=0, Q=0, R=0)$, but we have then $p=0$, and the meaning is simply that the general quartic surface having the 8 nodes is $(*\between P, Q, R)^2=0$.

35. A particular quartic surface having (in an improper sense) the 7 given nodes, but not having the 8th node, is $M\Omega=0$, where $M=0$ in the plane through any 3 of the 7 points and $\Omega=0$ is the cubic surface through these same 3 points, and having the remaining 4 points as nodes. The equation of the cubic surface, if the 4 points are taken to be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), is obviously of the form

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{w}=0, \text{ (that is, } ayzw+bzxw+cxyw+dxyz=0),$$

and by making the surface pass through the 3 points we determine linearly the coefficients (a, b, c, d), that is, their ratios. The equation of the quartic surface thus is

$$(a, \ldots \between P, Q, R)^2+\theta M\Omega=0,$$

the 7 given points being here proper nodes; and the formula being precisely equivalent to the preceding one containing Δ.

36. We can with the 7 given points form 35 such combinations $M\Omega=0$ of a plane and a cubic surface, and so present the equation of the quartic surface under 35 different forms; these are of course equivalent in virtue of the before mentioned formula for $\Delta'-p\Delta$; viz., we must have identically $M\Omega-pM'\Omega'=(*\between P, Q, R)^2$: a theorem of some interest, which it might be difficult to verify *à posteriori*.

Investigation of the cases of 8 *Nodes.*

37. It has already been shown that a quartic surface cannot in a proper sense have 8 given nodes. In regard to the quartic surfaces with 8 nodes, we start from the surface with 7 given nodes; viz.,

$$(a, \ldots \between P, Q, R)^2 + \theta \nabla = 0,$$

or, what is the same thing,

$$(a, \ldots \between P, Q, R)^2 + \theta M\Omega = 0;$$

and we inquire in what cases this surface has an 8th node. Obviously if $\theta = 0$, that is, if the surface is $(a, \ldots \between P, Q, R)^2 = 0$, the surface will have an 8th node, the remaining intersection of the quadric surfaces $P = 0$, $Q = 0$, $R = 0$ (observe that this is a point in no wise depending on the particular quadric surfaces, but uniquely determined by means of the 7 given points); and we have thus one kind, say the "octadic" surface, of the quartic surfaces with 8 nodes; viz., the nodes are the 8 points of intersection of any 3 quadric surfaces, or they are an octad of points. By what precedes, 7 of the nodes may be given points, and the remaining node is then a uniquely determinate point, the 8th point of the octad.

38. But if θ be not $= 0$, there may still be an 8th node; viz., this must then be a point on the Jacobian surface $J(P, Q, R, \nabla) = 0$, which is of the order 6. It is clear *à priori* that this must be a surface depending only on the 7 points, but independent of the particular surfaces $P = 0$, $Q = 0$, $R = 0$, $\nabla = 0$; to verify this, observe that, substituting for ∇ the function ∇', $= p\nabla + (* \between P, Q, R)^2$, we in fact leave the Jacobian unaltered; I call it the dianodal surface of the 7 points.

39. I say that the 8th node may be any point whatever on the dianodal surface; in fact, regarding for a moment the coordinates of the node as given, and expressing that the point is a node on the quartic surface, we have 4 equations containing

$$aP_0 + hQ_0 + gR_0, \quad hP_0 + bQ_0 + fR_0, \quad gP_0 + fQ_0 + cR_0,$$

(P_0, Q_0, R_0 the values of P, Q, R at the node,) but which, if only the point be on the dianodal surface, reduce themselves to three equations; viz., we have between the coefficients (a, b, c, f, g, h) and θ three equations which being satisfied, the point in question will be a node. And it thus appears that, taking the 8th node to be a given point on the dianodal surface, the equation $(a, \ldots \between P, Q, R)^2 + \theta\nabla = 0$ of the quartic surface will contain 3 constants. Observe that we may through the 8 nodes draw 2 quadric surfaces $P = 0$, $Q = 0$; and this being so if $\Delta = 0$ be a particular quartic surface with the 8 nodes, then the general quartic surface will be

$$(a, b, c \between P, Q)^2 + \theta\Delta = 0,$$

containing the right number 3 of constants. But there is not here any simple form of the surface $\Delta = 0$, such as the form $M\Omega = 0$ for the surface through 7 given points.

40. It is clear *à priori* that the relation between the 8 nodes is a symmetrical one; so that the 8th point being situate anywhere on the dianodal surface of the 7 points, each of the points will be situate on the dianodal surface of the remaining 7 points. This is a remarkable property of the dianodal surface, which will have to be again considered.

41. In what precedes, we have the second kind of quartic surfaces with 8 nodes, say the "dianome"; viz., each node is a point on the dianodal surface of the remaining 7 nodes; any 7 of the nodes may be taken to be given points, and the remaining node to be any point whatever on the dianodal surface of the 7 points.

The Dianodal Surface.

42. Consider the seven points 1, 2, 3, 4, 5, 6, 7. As already mentioned, through three of these, say 1, 2, 3, we may draw a plane $M=0$; and through the same three points, with the remaining points 4, 5, 6, 7 as nodes ($3+4.4=19$ conditions), a cubic surface $\Omega=0$; this surface passing through the six lines, 45, 46, ... 67. Hence we have $\Delta, =M\Omega, =0$, a quartic surface with the seven points as nodes. And using this form of Δ, it may be shown that the dianodal $J(P, Q, R, \Delta)=0$ passes through the 21 lines 12, 13, ... 67, and through 35 plane cubics such as $M=0$, $\Omega=0$; viz., this is a cubic in the plane 123 passing through the points 1, 2, 3, and through the intersections of the plane with each of the six lines 45, 46, ... 67 (nine points determining the cubic); the complete intersection by the plane 123 being therefore composed of this cubic and of the three lines 12, 13, 23. For the passage through the cubic, we have only to observe that

$$J(P, Q, R, M\Omega) = J(P, Q, R, \Omega)M + J(P, Q, R, M)\Omega = 0$$

is satisfied by $M=0$, $\Omega=0$; and for the passage through the lines, taking $x=0$, $y=0$, $z=0$, $w=0$ for the equations of the planes 567, 674, 745, and 456 respectively, each of the functions P, Q, R is of the form $ayz+bzx+cxy+fxw+gyw+hzw$, and the function Ω is of the form $Ayzw+Bzwx+Cwxy+Dxyz$. Hence, writing in the derived functions for instance $z=0$, $w=0$, the first and second lines of the determinant $J(P, Q, R, \Omega)$ will be of the form

$$\begin{vmatrix} cy, & c'y, & c''y, & 0 \\ cx, & c'x, & c''x, & 0 \end{vmatrix},$$

or the determinant vanishes for $z=0$, $w=0$; that is, for any point of the line 45 we have $\Omega=0$ and also $J(P, Q, R, \Omega)=0$; consequently $J(P, Q, R, M\Omega)=0$, and the like for the other lines. The theorem is thus proved.

43. I say that the dianodal surface passes through each of the 7 skew cubics, such as 123456. To prove this, it is only necessary to show that the skew cubic

123456 lies on the dianodal surface. For this purpose it will be enough to show that the skew cubic meets the plane 712 in a point of the surface; for then it will, in like manner, meet each of the 15 planes 712, 713, ... 756 in a point of the surface; that is, we shall have 15 intersections of the curve and surface, and there are, besides, the intersections 1, 2, 3, 4, 5, 6, in all 21 intersections; that is, the skew cubic must lie on the surface.

44. The plane 712 meets the surface in three lines and in a plane cubic determined by the points 7, 1, 2 and the six intersections of the plane with the lines 34, 35, ... 56. We have therefore to show that this plane cubic meets the skew cubic 123456. Consider for a moment the points 1, 2, 3, 4, 5, 6 and another point 7′. As seen above, we have in general, through the points 1, 2, 7′ and with the points 3, 4, 5, 6 as nodes, a determinate cubic surface, which surface passes through the lines 34, 35, ... 56. But the cubic surface becomes indeterminate if the points 1, 2, 7′, 3, 4, 5, 6 are on the same skew cubic; that is, if 7′ is any point whatever on the skew cubic 123456 (the proof presently). Taking, then, 7′ as the intersection of the skew cubic by the plane 712, we have in this plane the points 7′, 1, 2, and the intersections of the plane by the lines 34, 35, ... 56, nine points through which there pass an infinity of plane cubics; that is, the plane cubic determined by the points 7, 1, 2 and the six intersections will pass through the point 7′; viz., it meets the skew cubic 123456.

45. For the subsidiary theorem, taking X, Y, Z, W as current coordinates, viz., $X=0$, $Y=0$, $Z=0$, $W=0$ as the equations of the planes 456, 563, 634, 345 respectively, (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) as the coordinates of the points 1 and 2 respectively, and (x, y, z, w) for those of 7′; the equation of the cubic surface passing through 7′, 1, 2, and having the nodes 3, 4, 5, 6, is

$$\begin{vmatrix} \frac{1}{X}, & \frac{1}{Y}, & \frac{1}{Z}, & \frac{1}{W} \\ \frac{1}{x}, & \frac{1}{y}, & \frac{1}{z}, & \frac{1}{w} \\ \frac{1}{x_1}, & \frac{1}{y_1}, & \frac{1}{z_1}, & \frac{1}{w_1} \\ \frac{1}{x_2}, & \frac{1}{y_2}, & \frac{1}{z_2}, & \frac{1}{w_2} \end{vmatrix} = 0;$$

and this ceases to be a determinate function if only

$$\begin{Vmatrix} \frac{1}{x}, & \frac{1}{y}, & \frac{1}{z}, & \frac{1}{w} \\ \frac{1}{x_1}, & \frac{1}{y_1}, & \frac{1}{z_1}, & \frac{1}{w_1} \\ \frac{1}{x_2}, & \frac{1}{y_2}, & \frac{1}{z_2}, & \frac{1}{w_2} \end{Vmatrix} = 0;$$

viz., considering (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) as given, this is a twofold relation between the coordinates (x, y, z, w) of the point 7′. The relation may be represented by the four equations $(yzw)=0$, $(zwx)=0$, $(wxy)=0$, $(xyz)=0$, if for shortness

$$(yzw) = \begin{vmatrix} yz\ , & zw\ , & wy \\ y_1z_1, & z_1w_1, & w_1y_1 \\ y_2z_2, & z_2w_2, & w_2y_2 \end{vmatrix}$$

and the like as to the other symbols. The four equations represent quadric surfaces, each two intersecting in a line [e.g., $(yzw)=0$, $(zwx)=0$ in the line $z=0$, $w=0$], and the four surfaces besides intersecting in a skew cubic, which is the required locus of the point 7′, and which, as is seen at once, passes through the points 1, 2, 3, 4, 5, 6.

46. By what precedes, we have on the dianodal surface through the point 1 the lines 12, 13, 14, 15, 16, 17, and the skew cubics 123456, &c. The six lines are not on the same quadric cone, and it thus appears that the point 1 must be a cubic-node (point where, instead of the tangent plane, we have a cubic cone) on the surface. It is to be remarked that the lines 12, 13, 14, 15, 16, and the tangent at 1 to the skew cubic 123456, lie in a quadric cone; viz., this tangent is given as the sixth intersection of the cubic cone with the quadric cone through the lines 12, 13, 14, 15, 16.

47. I revert to the equation of the dianodal surface as given in the form $J=J(P, Q, R, M\Omega)=0$, where $M=0$ is the plane through the points 1, 2, 3, and $\Omega=0$ the cubic surface through these points, and having the points 4, 5, 6, 7, as nodes. We can find the orders of the several functions P, Q, R, M, Ω in the coordinates (x_1, y_1, z_1, w_1), &c., of the several points; viz., writing for shortness x_1^2 to denote the order 2 in regard to (x_1, y_1, z_1, w_1), and so in other cases, we have

$$\begin{aligned} P=Q=R&=x^2(x_5,\ x_6,\ x_7)^2(x_1,\ x_2,\ x_3,\ x_4)^2, \\ M&=x\ (x_5,\ x_6,\ x_7), \\ \Omega&=x^3(x_5,\ x_6,\ x_7)^3(x_1,\ x_2,\ x_3,\ x_4)^9; \end{aligned}$$

{where, of course, the x^2, x, x^3 show in like manner the orders in regard to the current coordinates (x, y, z, w); the proof in regard to Ω is easily supplied.} The order of J is equal that of $PQRM\Omega$, less 4 as regards the current coordinates, by reason of the differentiations; that is, we have $J=x^6(x_1x_2x_3)^{10}(x_4x_5x_6x_7)^{15}$; and we thus see that the equation of the dianodal surface as above obtained is encumbered with a constant factor of the form $(x_1x_2x_3)^4(x_4x_5x_6x_7)^9$. In fact, the relation between the 7 points and the current point (x, y, z, w), or say the point 8, as expressing that the 8 points are the nodes of a dianome, should be a symmetrical one in regard to the coordinates of the several points; and being of the order 6 in regard to the coordinates (x, y, z, w), it should be of the same order in regard to the other coordinates; that is, the true form would be $J=(xx_1x_2x_3x_4x_5x_6x_7)^6=0$.

48. It is possible that taking the 4 points, say 1, 2, 3, 4, to be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), and the 3 points, say 5, 6, 7, to be (1, 1, 1, 1), $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$, the extraneous factor might exhibit itself, and that the

equation divested of this factor might be of a tolerably simple form. I have not, however, worked this out, but I have, by an independent process, obtained in regard to the dianodal surface of the 7 points a result which may be interesting.

49. The dianodal surface, *quà* surface having the first-mentioned 4 points for cubic nodes, has its equation of the form

$$yzw\,(y,\ z,\ w)^3 + zxw\,(z,\ x,\ w)^3 + xyw\,(x,\ y,\ w)^3 + xyz\,(x,\ y,\ z)^3 + xyzw\,(x,\ y,\ z,\ w)^2 = 0\,;$$

where in the cubic functions the terms x^3, y^3, z^3, w^3 none of them appear. If for instance $w = 0$, the equation becomes $(x,\ y,\ z)^3 = 0$; which, by what precedes, is a known cubic curve, viz., the curve through the points 1, 2, 3 and the intersections of the plane 123 by the lines 45, 46, 47, 56, 57, 67; and we can by this consideration find the cubic function $(x,\ y,\ z)^3$, and thence by symmetry the other cubic functions. I take

$$\left.\begin{matrix}(a,\ b,\ c,\ f,\ g,\ h)\\ (a',\ b',\ c',\ f',\ g',\ h')\\ (\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h})\end{matrix}\right\}\ \text{for coordinates of line through}\ \left\{\begin{matrix}(1,\ 1,\ 1,\ 1), & (\alpha,\ \beta,\ \gamma,\ \delta)\\ (1,\ 1,\ 1,\ 1), & (\alpha',\ \beta',\ \gamma',\ \delta')\\ (\alpha,\ \beta,\ \gamma,\ \delta), & (\alpha',\ \beta',\ \gamma',\ \delta')\end{matrix}\right.$$

respectively; viz., I write

$$\begin{array}{ll|ll|ll} a = \beta - \gamma, & f = \alpha - \delta & a' = \beta' - \gamma', & f' = \alpha' - \delta' & \mathrm{a} = \beta\gamma' - \beta'\gamma, & \mathrm{f} = \alpha\delta' - \alpha'\delta \\ b = \gamma - \alpha, & g = \beta - \delta & b' = \gamma' - \alpha', & g' = \beta' - \delta' & \mathrm{b} = \gamma\alpha' - \gamma'\alpha, & \mathrm{g} = \beta\delta' - \beta'\delta \\ c = \alpha - \beta, & h = \gamma - \delta & c' = \alpha' - \beta', & h' = \gamma' - \delta' & \mathrm{c} = \alpha\beta' - \alpha'\beta, & \mathrm{h} = \gamma\delta' - \gamma'\delta \end{array}$$

and I write moreover

$$\begin{aligned} \lambda &= \quad \cdot \qquad \mathrm{h} - \mathrm{g} + \mathrm{a}, \\ \mu &= -\mathrm{h} \qquad \cdot \quad + \mathrm{f} + \mathrm{b}, \\ \nu &= \quad \mathrm{g} - \mathrm{f} \qquad \cdot \quad + \mathrm{c}, \\ \varpi &= -\mathrm{a} - \mathrm{b} - \mathrm{c} \qquad \cdot \end{aligned}$$

50. This being so, the cubic curve through the last-mentioned six points has its equation of the form

$$\frac{A}{ax + by + cz} + \frac{B}{a'x + b'y + c'z} + \frac{C}{\mathrm{a}x + \mathrm{h}y + \mathrm{g}z} + \frac{D}{\lambda x + \mu y + \nu z} = 0\,;$$

and to make this pass through the points 1, 2, 3, we write therein successively $(y = 0,\ z = 0)$, $(z = 0,\ x = 0)$, $(x = 0,\ y = 0)$; viz., we have for the ratios $A : B : C : D$ the three equations

$$\frac{A}{a} + \frac{B}{a'} + \frac{C}{\mathrm{a}} + \frac{D}{\lambda} = 0,$$

$$\frac{A}{b} + \frac{B}{b'} + \frac{C}{\mathrm{h}} + \frac{D}{\mu} = 0,$$

$$\frac{A}{c} + \frac{B}{c'} + \frac{C}{\mathrm{g}} + \frac{D}{\nu} = 0.$$

In eliminating, for instance, B for the first and second equations, the resulting equation divides by $ab' - a'b$, $= \mathrm{a} + \mathrm{b} + \mathrm{c}$, and we thus obtain, between A, C, D, the three equations (equivalent to two)

$$\frac{A}{bc} + \frac{C\alpha'}{\mathrm{bc}} + \frac{Df'}{\mu\nu} = 0,$$

$$\frac{A}{ca} + \frac{C\beta'}{\mathrm{ca}} + \frac{Dg'}{\nu\lambda} = 0,$$

$$\frac{A}{ab} + \frac{C\gamma'}{\mathrm{ab}} + \frac{Dh'}{\lambda\mu} = 0,$$

from which the ratios $A : C : D$ may be obtained by actual calculation. After all reductions, we have

$$\begin{aligned} A &= \quad a\,b\,c\ \{(\alpha'\delta' + \beta'\gamma')\,\mathrm{af} + (\beta'\delta' + \gamma'\alpha')\,\mathrm{bg} + (\gamma'\delta' + \alpha'\beta')\,\mathrm{ch}\}, \\ B &= -\,a'b'c'\ \{(\alpha\,\delta + \beta\,\gamma)\,\mathrm{af} + (\beta\,\delta + \gamma\,\alpha)\,\mathrm{bg} + (\gamma\,\delta + \alpha\,\beta)\,\mathrm{ch}\}, \\ C &= \quad \mathrm{a\,b\,c}\ \{(\alpha\alpha'\lambda + \beta\beta'\mu + \gamma\gamma'\nu + \delta\delta'\varpi\}, \\ D &= -\,\lambda\,\mu\,\nu\ \{(\alpha\alpha'\mathrm{a} + \beta\beta'\mathrm{b} + \gamma\gamma'\mathrm{c}\}\,; \end{aligned}$$

viz., A, B, C, D are proportional to these values respectively. Multiplying by the product of the denominators, I find without much difficulty that the resulting cubic function is divisible by $\mathrm{a} + \mathrm{b} + \mathrm{c}$; hence, introducing the factor xyz, and an indeterminate multiplier l, I write

$$xyz\,(x,\ y,\ z)^3 = \frac{l}{\mathrm{a}+\mathrm{b}+\mathrm{c}}\,xyz\,(ax+by+cz)\,(a'x+b'y+c'z)\,(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\,(\lambda x+\mu y+\nu z)$$
$$\times\left\{\frac{A}{ax+by+cz} + \frac{B}{a'x+b'y+c'z} + \frac{C}{\mathrm{a}x+\mathrm{b}y+\mathrm{c}z} + \frac{D}{\lambda x+\mu y+\nu z}\right\},$$

where A, B, C, D have the values above written down.

51. Considering the orders in regard to $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$, and observing that a, b, c and a', b', c' are linear functions of the two sets respectively, but that $\mathrm{a}, \mathrm{b} \ldots \mathrm{h}, \lambda \ldots \varpi$, are linear in the two sets conjointly, or say

$$a, \ldots = \alpha,\ a', \ldots = \alpha'\,;\ \mathrm{a}, \ldots = \alpha\alpha'\,;$$

we have

$$Aa'\mathrm{a}\lambda = \alpha^5\alpha'^4 \,.\, \alpha^2\alpha'^3 = \alpha^7\alpha'^7,$$

so that after the division by $\mathrm{a} + \mathrm{b} + \mathrm{c}$, $= \alpha\alpha'$, the order will be $\alpha^6\alpha'^6$. Hence l will be a mere numerical factor, and the last-mentioned equation gives, without any extraneous factor, the terms $xyz\,(x, y, z)^3$ in the equation of the dianodal surface of the seven points.

Octadic Surfaces with 9 or 10 Nodes.

52. In regard to the surfaces with 9 and 10 nodes, I consider first the octadic surfaces. Starting as before with the given points 1, 2, 3, 4, 5, 6, 7, we have a determinate point 8 completing the octad, and the surface with the 8 nodes is

$$(a, \ldots \between P,\ Q,\ R)^2 = 0,$$

(5 constants). Suppose that there is another node 9; this must be a point on the Jacobian curve $J(P, Q, R) = 0$, which (as was seen) is a sextic curve not passing through any of the 8 points; the node 9 may be *any* point on this curve, viz., taking its coordinates as given, the condition of its being a node gives 4 equations, and these for the very reason that the point is on the Jacobian curve, reduce themselves to 2 equations, which can be satisfied by means of the constants $(a, \ldots)$; the resulting equation should therefore contain 3 constants.

53. In order to find it, taking as above 9 a given point on the Jacobian curve, this will be the vertex of a quadric cone, say $P = 0$, through the 8 points; we may draw through the 9 points another quadric surface $Q = 0$, and through the 8 points a quadric surface $R = 0$; this being so, we have the quartic surface $(a, b, 0, 0, g, h \between P, Q, R)^2 = 0$, having the 9 nodes, and containing, as it should do, 3 constants; this may be written

$$(aP + 2hQ + 2gR)P + bQ^2 = 0;$$

viz., if $bR' = aP + 2hQ + 2gR$, that is, if $R' = 0$ be the general quadric surface through the 8 points, then the equation is $Q^2 - PR' = 0$, where observe that R' is considered as containing implicitly 3 constants.

54. If there is a 10th node, say 10, this is also a point on the Jacobian curve $J(P, Q, R) = 0$, and it may be any point whatever on the curve; taking it as a given point on the curve, the resulting equation should contain 1 constant. We may take $P = 0$ to be the quadric cone, vertex 9, through the 8 points, $R = 0$ the quadric cone, vertex 10, through the 8 points, $Q = 0$ the quadric surface through the 8 points and the points 9 and 10 (viz., the surface through 9, 10 and any 7 of the 8 points will pass through the remaining 8th point). The equation of the quartic surface then is

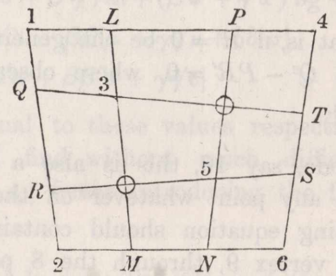
$$(0, b, 0, 0, g, 0 \between P, Q, R)^2 = 0;$$

that is, $bQ^2 + 2gPR = 0$, containing 1 constant; we may reduce this to $Q^2 - PR = 0$, the constant being considered as contained implicitly in one of the functions. It is clear that the constant cannot be so determined as to give rise to an 11th node, nor indeed to any other singularity in the surface.

55. In the case of the surface with 9 nodes, it is clear that this is octadic in one way only; the node 9 cannot form an octad with any 7 of the remaining nodes. But in the case of the surface with 10 nodes, the question arises whether the nodes 9 and 10 may not be such as to form an octad with some six, say with the nodes 1, 2, 3, 4, 5, 6 of the remaining 8 nodes; that is, whether we can have 1, 2, 3, 4, 5, 6, 7, 8 forming an octad, and also 1, 2, 3, 4, 5, 6, 9, 10 forming an octad. I will show that this is impossible if only the points 1, 2, 3, 4, 5, 6 are given points, that is, points assumed at pleasure and not specially related to each other. For this purpose, assuming that the points form 2 octads as above, take through 1, 2, 3, 4, 5, 6, 7, 9 the quadric surfaces $P = 0$, $Q = 0$, then each of these passes through 8, 10; take $R = 0$ any other quadric surface through 1, 2, 3, 4, 5, 6, 7, 8, and $S = 0$ any other quadric surface through 1, 2, 3, 4, 5, 6, 9, 10. Then $P = 0$, $Q = 0$, $R = 0$ intersect in the 1st octad,

and $P=0$, $Q=0$, $S=0$ intersect in the 2nd octad; the quartic surface (if it exists) must be simultaneously of the forms $(*\between P, Q, R)^2=0$, $(*\between P, Q, S)^2=0$; and this implies an identical equation $(*\between P, Q, R, S)^2=0$. The quadric surfaces are surfaces through the points 1, 2, 3, 4, 5, 6, and taking through these six points any other quadric surfaces $A=0$, $C=0$, $E=0$, $H=0$, we have P, Q, R, S each of them a linear function of A, C, E, H; and the relation between P, Q, R, S gives a like relation $(*\between A, C, E, H)^2=0$ between A, C, E, H. I assume $A=123\,.\,456$, $E=134\,.\,256$, $H=145\,.\,236$, $C=152\,.\,346$; viz., $A=0$ is the plane-pair formed by the planes through 1, 2, 3 and 4, 5, 6 respectively; and so for the others: we have to show that there is not any such identical relation $(*\between A, C, E, H)^2=0$.

56. We may through 3 draw the lines LM, QT to meet 14, 26 and 12, 46 respectively; and through 5 the lines RS, NP to meet 14, 26 and 12, 46 respectively. Observe that the points O in the figure are apparent intersections only; viz., NP does



not meet QT, *nor* LM meet RS. In fact, if NP met QT it would be a line in the series of lines meeting 14, QT, 26; or 5 would be situate in a hyperboloid, determined by means of the points 1, 2, 4, 6, 3; viz., 5 would not be an arbitrary point: and so LM does not meet RS. Now the quadrics E, H meet in the lines 14, 26, LM, NP, and the quadrics A, C in the lines 12, 46, QT, RS. Suppose that we had identically $(*\between A, C, E, H)^2=0$; putting therein $E=0$, $H=0$, we should have $(*\between A, C)^2=0$, viz., $(A+\lambda C)(A+\mu C)=0$; or there would exist quadrics of the forms $A+\lambda C=0$ containing the lines 14, 26, LM, NP. Now there is no quadric surface $A+\lambda C=0$ containing the line NP; for $A+\lambda C=0$ is a quadric containing the sides of the quadrilateral $QRST$; the generating lines of the one kind meet each of the lines RS, QT; those of the other kind neither. Hence NP, which meets RS but not QT, cannot be a generating line of either kind; and we have no identical relation $(A, C, E, H)^2=0$.

57. In the octadic surface with 9 nodes; starting with any 7 nodes of the octad, 9 is not the 8th point of the octad, and hence (by the theory of the dianome) it must lie in the dianodal surface of the 7 points; that is, the dianodal surface of the 7 points must pass through 9, viz., through any point whatever of the Jacobian curve of the 7 points, that is, of the octad; or (what is the same thing) the dianodal surface of the 7 points passes through the Jacobian curve of the octad. This is an obvious property of the dianodal surface, the surface $J(P, Q, R, \nabla)=0$ contains the Jacobian curve $J(P, Q, R)=0$. But it further appears that, starting with any 6 points of the octad and with the point 9 (that is, any point whatever of the Jacobian curve), the

dianodal surface of these 7 points must contain the remaining 2 points of the octad. And in the octadic surface with 10 nodes, starting with any 5 points of the octad and with the points 9 and 10 (that is, any two points on the Jacobian curve) the dianodal surface of these 7 points must contain the remaining three points of the octad. I have not attempted to verify these last properties of the dianodal surface.

Dianomes with 9 *or* 10 *Nodes.*

58. I now consider the dianomes with 9 and 10 nodes. Starting from the general form

$$(a, b, c \between P, Q)^2 + \theta\Delta = 0,$$

where $\Delta = 0$ is a particular quartic surface having the 8 nodes, it at once appears that if there is a 9th node, say 9, this must be a point on the Jacobian curve $J(P, Q, \Delta) = 0$, or say on the dianodal curve of the 8 points, viz. ($a = b = 1$, $c = 3$, in the formula No. 5), this is a curve of the order 18; the node may be any point whatever on this curve, and taking it to be a given point on the curve, the number of constants in the resulting equation should be 1. Hence if $P = 0$ be the quadric surface through the 9 points, and $\Delta = 0$ a particular quartic surface having the 9 points as nodes, the general equation is $aP^2 + \theta\Delta = 0$.

59. But we may consider the question somewhat differently. Starting with the 7 given points 1, 2, 3, 4, 5, 6, 7 and with 8 a given point on the dianodal surface of the 7 points; it is clear that 9 must be on the dianodal surface 1234567, and also on the dianodal surface 1234568; the complete intersection is of the order 36, and we have to consider how this breaks up so as to contain as part of itself the dianodal curve of the order 18.

Dianodal Curve of 8 *Points.*

60. Consider first any 8 points whatever 1, 2, 3, 4, 5, 6, 7, 8; where 8 is not on the dianodal surface 1234567, nor 7 on the dianodal surface 1234568. The two surfaces have in common the 15 lines 12, 13, ... 56 and the skew cubic 123456, they therefore besides intersect in a curve of the order 18. At the point 1 the tangent cubic cones of the two surfaces intersect in the lines 12, 13, 14, 15, 16 and the tangent to the skew cubic 123456, 6 lines lying in a quadric cone; they therefore besides intersect in 3 lines lying in a plane; that is, the point 1 is on the curve of the order 18 an actual triple point, the 3 tangents lying *in plano;* and the like of course in regard to each of the points 2, 3, 4, 5, 6. But as 7, 8 lie each of them on only one of the two surfaces, the curve of the order 18 does not pass through 7 or 8.

61. If, however, 8 lies on the dianodal surface 1234567, then each of the 8 points will lie on the dianodal surface of the other 7; and in particular 7 will lie on the dianodal surface 1234568. The surfaces intersect as before in a residual curve of the order 18; the only difference is that 7 and 8 are now points on each surface; viz., each of them is on one of the surfaces an ordinary point, and on the other a cubic node; the points 7 and 8 are thus each of them an actual triple point on the curve; and at each of them the 3 tangents are *in plano.* We thus see that the dianodal

curve 12345678 is a curve of the order 18, such that each of the 8 points is a triple point on the curve, the tangents at each of them being *in plano*.

Ten Nodes.

62. Suppose there is a 10th node, say 10; starting from the equation $aP^2 + \theta\Delta = 0$ ($P = 0$ the quadric surface through the 9 points, $\Delta = 0$ a particular quartic surface having the 9 points as nodes), it at once appears that the node must be one of the points $J(P, \Delta) = 0$; hence, taking it to be one of these points, we have 4 equations, which, in virtue of the node being one of the points in question, reduce themselves to a single equation determining the ratio $a : \theta$; we have thus a completely determinate surface, say $\square = 0$ having the 10 points as nodes. The number of points $J(P, \Delta)$, writing in the formula No. 5, $a = 1$, $b = 3$, is obtained as $1 + 3 + 9 + 27 = 40$, but it is to be observed that the surface $P = 0$ passes through each of the 9 nodes of the surface $\Delta = 0$; these count twice among the points $J(P, \Delta) = 0$, and the number of residual points (or say the dianodal centres of the 9 points) is $40 - 18 = 22$; viz., this is the number of positions of the node 10. [The nine points count each *three* times and the number of residual points, or positions of the node 10, is thus not $40 - 18 = 22$, but $40 - 27, = 13$.]

Dianodal Centres of 9 *Points.*

63. In further explanation, observe that 9 is any point on the dianodal curve 12345678; the node 10 must lie on this same curve, and also on the dianodal surface 1234569. Take $P = 0$ the quadric through all the 9 points, $Q = 0$ a quadric through all but the point 9, $R = 0$ through all but the point 8, $S = 0$ through all but the point 7. The dianodal curve 12345678 is $J(P, Q, \nabla) = 0$, and the dianodal surface 1234569 is $J(P, R, S, \nabla) = 0$; the total number of intersections is $6 \times 18 = 108$; these include the $4 \times 18 = 72$ points of intersection of the dianodal curve $J(P, Q, \Delta) = 0$ with the Jacobian surface $J(P, Q, R, S) = 0$, except the four points $J(P, Q) = 0$, which are the vertices of the 4 quadric cones through 1, 2, 3, 4, 5, 6, 7, 8 (which 4 points are not situate on the curve $J(P, R, S) = 0$), and there are besides 40 points $\{108 = (72 - 4) + 40\}$ which are the before mentioned points $J(P, \Delta) = 0$; viz., these are the 9 points each twice [three times], and the residual 22 [13] points which are the dianodal centres of the 9 points.

General result as to the Dianomes.

64. We have thus established the theory of the dianome quartic surfaces; viz., we have

The octodianome, 8 nodes, 7 of them arbitrary, and the 8th an arbitrary point on the dianodal surface (order 6) of the 7 points.

The enneadianome, 9 nodes, the 9th an arbitrary point on the dianodal curve (order 18) of the 8 points.

The decadianome, 10 nodes, the 10th any one of the 22 [13] dianodal centres of the 9 points.

And as already mentioned, so long as the first 7 nodes are arbitrary, there cannot be more than 10 nodes in all.

THE SYMMETROID.

The Lineolinear Correspondence of Quartic Surfaces.

65. I consider four equations $S=0$, $T=0$, $U=0$, $V=0$, lineolinear in regard to the two sets of coordinates (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$; viz., each of these equations is of the form

$$(*\between x, y, z, w\between \alpha, \beta, \gamma, \delta)=0.$$

This implies that the point (x, y, z, w) lies on a certain quartic surface $\Theta=0$, and the point $(\alpha, \beta, \gamma, \delta)$ on a certain quartic surface $\Delta=0$, and that the two surfaces correspond point to point to each other. In fact, writing the four equations in the form

$$\begin{aligned}
L\alpha + M\beta + N\gamma + P\delta &= 0,\\
L'\alpha + M'\beta + N'\gamma + P'\delta &= 0,\\
L''\alpha + M''\beta + N''\gamma + P''\delta &= 0,\\
L'''\alpha + M'''\beta + N'''\gamma + P'''\delta &= 0,
\end{aligned}$$

where L, &c., are linear functions of (x, y, z, w), then eliminating $(\alpha, \beta, \gamma, \delta)$, we obtain the equation

$$\Theta = \begin{vmatrix} L, & M, & N, & P \\ L', & M', & N', & P' \\ L'', & M'', & N'', & P'' \\ L''', & M''', & N''', & P''' \end{vmatrix} = 0;$$

and similarly, writing the four equations in the form

$$\begin{aligned}
Ax + By + Cz + Dw &= 0,\\
A'x + B'y + C'z + D'w &= 0,\\
A''x + B''y + C''z + D''w &= 0,\\
A'''x + B'''y + C'''z + D'''w &= 0,
\end{aligned}$$

where A, &c., are linear functions of $(\alpha, \beta, \gamma, \delta)$, then eliminating (x, y, z, w), we obtain the equation

$$\Delta = \begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix} = 0.$$

Moreover, Θ being $=0$, the four linear equations in $(\alpha, \beta, \gamma, \delta)$ are equivalent to three equations, and give for instance $(\alpha, \beta, \gamma, \delta)$ proportional to the determinants formed with the matrix

$$\begin{vmatrix} L', & M', & N', & P' \\ L'', & M'', & N'', & P'' \\ L''', & M''', & N''', & P''' \end{vmatrix};$$

and similarly, Δ being $=0$, the four linear equations in (x, y, z, w) are equivalent to three equations, and give for instance (x, y, z, w) proportional to the determinants formed with the matrix

$$\left|\begin{array}{llll} A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{array}\right|;$$

which establishes the point-to-point correspondence of the two surfaces.

66. It would at first sight appear that any quartic surface $(*\between\alpha, \beta, \gamma, \delta)^4=0$ whatever might have its equation expressed in the foregoing determinant form $\Delta = 0$. This equation seems, in fact, to contain homogeneously as many as 64 constants. But if we multiply the determinant line into line by a constant determinant

$$\left|\begin{array}{llll} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \\ a''', & b''', & c''', & d''' \end{array}\right|$$

and then column into column by another constant determinant, the coefficients, all but one of them, of these constant determinants may be used to specialize the form of the resulting equation, [say they are apoclastic constants]; this equation will really contain $64-(2.16-1)=33$ constants; and in order that the quartic surface $(*\between\alpha, \beta, \gamma, \delta)^2=0$ may have its equation expressible in the form $\Delta=0$, a single relation must hold good among the coefficients: but this in passing (1).

67. Returning to the quartic surface

$$\Delta = \left|\begin{array}{llll} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{array}\right| = 0,$$

we may connect this not only with the foregoing surface $\Theta=0$, but in a similar manner with another quartic surface $\Phi=0$; viz., taking the current coordinates $(\xi, \eta, \zeta, \omega)$, we may form the lineolinear equations

$$A\xi + A'\eta + A''\zeta + A'''\omega = 0,$$

$$B\xi + B'\eta + B''\zeta + B'''\omega = 0,$$

$$C\xi + C'\eta + C''\zeta + C'''\omega = 0,$$

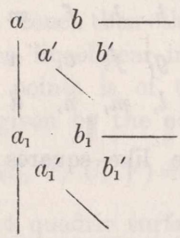
$$D\xi + D'\eta + D''\zeta + D'''\omega = 0,$$

1 Applying the same reasoning to a cubic determinant $\Delta=0$, the number of constants is $36-(2.9-1)=19$; so that a cubic surface is expressible in the form in question. And so for the quadric determinant $\Delta=0$, the number of constants is $16-(2.4-1)=9$; so that a quadric surface is expressible in the form in question, as is otherwise obvious.

which, by the elimination of $(\xi, \eta, \zeta, \omega)$, give $\Delta = 0$, and by the elimination of $(\alpha, \beta, \gamma, \delta)$ a determinant quartic equation $\Phi = 0$ between the coordinates $(\xi, \eta, \zeta, \omega)$; and of course the two surfaces $\Delta = 0$, $\Phi = 0$ have a point-to-point correspondence such as exists between the surfaces $\Theta = 0$, $\Delta = 0$. The relation of the point $(\alpha, \beta, \gamma, \delta)$ on the surface $\Delta = 0$ to the point (x, y, z, w) on the surface $\Theta = 0$, and to the point $(\xi, \eta, \zeta, \omega)$ on the surface $\Phi = 0$, may be conveniently indicated by means of the diagram

$$\begin{array}{cccc|c}
\multicolumn{4}{c}{\overbrace{x\ ,\quad y\ ,\quad z\ ,\quad w}^{\Theta}} & \\
\hline
A\ , & B\ , & C\ , & D & \xi \\
A'\ , & B'\ , & C'\ , & D' & \eta \\
A''\ , & B''\ , & C''\ , & D'' & \zeta \\
A'''\ , & B'''\ , & C'''\ , & D''' & \omega
\end{array}\left.\vphantom{\begin{array}{c}A\\A\\A\\A\end{array}}\right\}\Phi.$$

68. It is to be observed that, writing for $A, B, \ldots$ their values as linear functions of $(\alpha, \beta, \gamma, \delta)$, we have in all 64 constant coefficients, which we may conceive arranged in the form of a cube, thus:



and taking these in fours height-wise, (a, a_1, a_2, a_3), &c., we compose with them the linear functions $a\alpha + a_1\beta + a_2\gamma + a_3\delta$, &c., which enter into the equation $\Delta = 0$; taking them in fours length-wise, (a, b, c, d), &c., we compose the linear functions $ax+by+cz+dw$, &c., which enter into the equation $\Theta = 0$; and taking them in fours breadth-wise (a, a', a'', a'''), &c., we compose the linear functions $a\xi + a'\eta + a''\zeta + a'''\omega$, &c., which enter into the equation $\Phi = 0$.

69. The process may be indefinitely repeated; we obtain always the same three surfaces over and over again, but on them an indefinite series of corresponding points; viz., we may write

$$\begin{array}{l}
\ldots\Theta\ ,\ \Delta\ ,\ \Phi,\ \ \Theta,\ \Delta,\ \Phi,\ \ \Theta,\ \Delta,\ \Phi\ldots \\
\ldots P_1,\ Q_1,\ R_1,\ \ P,\ Q,\ R,\ \ P',\ Q',\ R'\ldots
\end{array}$$

viz., a point Q on Δ corresponds to a point P on Θ and to a point R on Φ; R corresponds to Q on Δ and to a new point P' on Θ; P' to R on Φ and to a new point Q' on Δ, and so on. And in the opposite direction P corresponds to Q on Δ, and to a new point R_1 on Φ; R_1 to P on Θ and to a new point Q_1 on Δ; and so on. And of course the correspondence of any two points of the series, whether belonging to the same surface or to different surfaces, is a one-to-one correspondence.

The Symmetrical Case; Symmetroid and Jacobian.

70. I have established the foregoing general theory; but it is only a particular case of it which connects itself with the theory of nodal quartics; viz., the cube of coefficients is a symmetrically arranged cube

$$\begin{array}{lllll} a & h & g & l & \\ \vdots & h & b & f & m \\ \vdots & g & f & c & n \\ \vdots & l & m & n & d \end{array}$$

$$\begin{array}{lll} a_1 & h_1 \ldots & \\ \vdots & h_1 & b_1 \\ \vdots & \cdot & \end{array}$$

or say its upper face is the symmetrical square matrix

$$\left|\begin{array}{llll} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{array}\right|$$

and the other horizontal planes, the like squares with the several terms affected by suffixes.

The surface $\nabla = 0$ is here a surface of the form

$$\nabla = \left|\begin{array}{llll} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & P \end{array}\right| = 0$$

$\{A, B,$ &c. linear functions of $(\alpha, \beta, \gamma, \delta)\}$ viz., ∇ is a symmetrical determinant; I call this a symmetroid; the surfaces $\nabla = 0$, $\Phi = 0$ are one and the same surface, the Jacobian of 4 quadric surfaces; moreover the points P and R are one and the same point, and the correspondence R to P' is a reciprocal one; so that, instead of the indefinite series of points, we have only 2 points Q, Q' on the surface ∇, and 2 points P, P' on the surface $\Theta\ (= \Phi)$; viz., the diagram is

$$\ldots \Delta,\ \Theta,\ \Theta,\ \Delta,\ \Theta,\ \Theta,\ \Delta \ldots$$

$$\ldots Q',\ P',\ P,\ Q,\ P,\ P',\ Q' \ldots$$

moreover the symmetroid surface $\nabla = 0$ is a surface with 10 nodes, which is clearly not octadic, and which is therefore the decadianome.

71. Consider the quadric surfaces

$$S = (a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2 = 0,$$

$$T = (a_1, \ldots \quad \between \quad ,, \quad)^2 = 0,$$

$$U = (a_2, \cdots \quad \between \quad ,, \quad)^2 = 0,$$

$$V = (a_3, \ldots \quad \between \quad ,, \quad)^2 = 0,$$

and a point $(\alpha, \beta, \gamma, \delta)$ in the same or in a different space, such that the surface $\alpha S + \beta T + \gamma U + \delta V = 0$ is a cone, or say for shortness,

$$\alpha S + \beta T + \gamma U + \delta V = \text{cone};$$

$(\alpha, \beta, \gamma, \delta)$ is said to be the determining point, or determinator of the cone. And if we establish the equations

$$\delta_x(\alpha S + \beta T + \gamma U + \delta V) = 0,$$

$$\delta_y(\quad ,, \quad) = 0,$$

$$\delta_z(\quad ,, \quad) = 0,$$

$$\delta_w(\quad ,, \quad) = 0,$$

which express that the surface is a cone, then the point (x, y, z, w) is the vertex of the cone. We have thus 4 equations lineolinear in (x, y, z, w) and also in $(\alpha, \beta, \gamma, \delta)$, so that the relation between the 2 points is of the nature of that above considered. The relation between (x, y, z, w) is given by the equation

$$J(S, T, U, V) = 0;$$

viz., the locus is the Jacobian of the 4 quadric surfaces. The relation between $(\alpha, \beta, \gamma, \delta)$ is given by the equation

$$\nabla = \begin{vmatrix} a\alpha + a_1\beta + a_2\gamma + a_3\delta, & h\alpha + \ldots, & g\alpha + \ldots, & l\alpha + \ldots \\ h\alpha + \ldots & , b\alpha + \ldots, & f\alpha + \ldots, & m\alpha + \ldots \\ g\alpha + \ldots & , f\alpha + \ldots, & c\alpha + \ldots, & n\alpha + \ldots \\ l\alpha + \ldots & , m\alpha + \ldots, & n\alpha + \ldots, & d\alpha + \ldots \end{vmatrix} = 0,$$

so that the locus is (by the foregoing definition) the symmetroid. And the determinator point on the symmetroid thus corresponds to the cone-vertex on the Jacobian.

72. But the Jacobian may be obtained in a different manner; viz., if we establish the equations

$$(\xi\delta_x + \eta\delta_y + \zeta\delta_z + \omega\delta_w)\, S = 0,$$

$$(\quad ,, \quad)\, T = 0,$$

$$(\quad ,, \quad)\, U = 0,$$

$$(\quad ,, \quad)\, V = 0,$$

then the elimination of $(\xi, \eta, \zeta, \omega)$ leads to the equation $J(S, T, U, V) = 0$ of the Jacobian surface. And since each of the equations is symmetrical in regard to (x, y, z, w)

and $(\xi, \eta, \zeta, \omega)$, it appears that the point $(\xi, \eta, \zeta, \omega)$ is also a point on the Jacobian surface. We have on the symmetroid a point related to $(\xi, \eta, \zeta, \omega)$ in the same way that $(\alpha, \beta, \gamma, \delta)$ on the symmetroid is related to the point (x, y, z, w); and this completes the system of the 4 points, Q on the symmetroid, P and P' on the Jacobian, Q' on the symmetroid; but in what follows I make no use of this last point Q'.

73. The points (x, y, z, w), $(\xi, \eta, \zeta, \omega)$ on the Jacobian correspond in such wise that, taking the polar planes of either of them in regard to the quadrics $S=0$, $T=0$, $U=0$, $V=0$, these intersect in a single point, viz., in the other of the two corresponding points. Or, what is the same thing, the line joining the two points cuts each of the four quadrics harmonically, whence also it cuts harmonically any quadric surface whatever of the series $\alpha S+\beta T+\gamma U+\delta V=0$, ($\alpha$, β, γ, δ being here arbitrary multipliers); viz., this property is an immediate interpretation of the equation

$$(\xi\delta_x+\eta\delta_y+\zeta\delta_z+\omega\delta_w)(\alpha S+\beta T+\gamma U+\delta V)=0,$$

or, as this is more conveniently written,

$$(a, \ldots \between \xi, \eta, \zeta, \omega \between x, y, z, w)=0,$$

if for a moment $(a, \ldots)$ denote the coefficients of the quadric function $\alpha S+\beta T+\gamma U+\delta V$.

74. Consider any 6 pairs of points (x_1, y_1, z_1, w_1), $(\xi_1, \eta_1, \zeta_1, \omega_1)$, &c., related as above; the quartic surfaces $S=0$, $T=0$, $U=0$, $V=0$ are surfaces cutting harmonically the lines joining the two pairs of points respectively; or say they are quadrics cutting harmonically 6 given segments; and the general quadric surface which cuts harmonically the 6 given segments is $\alpha S+\beta T+\gamma U+\delta V=0$. We thus see that the Jacobian surface $J(S, T, U, V)=0$ is in fact the locus of the vertices of the quadric cones which cut harmonically 6 given segments. The surface so defined by M. Chasles (*Comptes Rendus*, tom. LII., 1861, pp. 1157—62), and shown by him to be a quartic surface, is thus identified with the Jacobian of any 4 quartic surfaces; and included herein we have the particular case, also considered by him, of the locus of the vertices of the quadric cones which pass through 6 given points, or Jacobian of the 6 given points.

75. It is to be shown that there are 10 systems of values $(\alpha, \beta, \gamma, \delta)$, or, what is the same thing, 10 points on the symmetroid, for each of which the quartic surface $\alpha S+\beta T+\gamma U+\delta V=0$ is a plane-pair. For any such system of values the plane-pair may be regarded as a cone, having its vertex at any point whatever on the line which is the axis of the plane-pair; that is, each point of this line is the vertex of a cone of the system of surfaces $\alpha S+\beta T+\gamma U+\delta V=0$; or, what is the same thing, the axis of the plane-pair lies on the Jacobian surface; viz., there will be on the Jacobian surface 10 lines. Moreover, to the point $(\alpha, \beta, \gamma, \delta)$ on the symmetroid there corresponds indifferently any point whatever on the axis of the plane-pair. The analytical expressions for (x, y, z, w) in terms of $(\alpha, \beta, \gamma, \delta)$ must therefore, for the values in question of $(\alpha, \beta, \gamma, \delta)$, become indeterminate; and this can only happen if for the values in question the first minors of the determinant ∇ all of them vanish. But a point $(\alpha, \beta, \gamma, \delta)$, for which the minors of ∇ all of them vanish, is obviously a node on the symmetroid; and it thus appears that there are on the symmetroid 10 nodes,

each corresponding to a line on the Jacobian, and that the condition for determining these is

$$\alpha S + \beta T + \gamma U + \delta V = \text{plane-pair};$$

viz., the values of $(\alpha, \beta, \gamma, \delta)$, which satisfy this condition, belong to a node of the symmetroid, and the line on the Jacobian is the axis of the plane-pair.

76. Reverting to the equation $\nabla = 0$ of the symmetroid, where ∇ is a symmetrical determinant the terms of which are linear functions of the coordinates $(\alpha, \beta, \gamma, \delta)$, it has already been shown, *ante* No. 7, that this is a surface with 10 nodes; but this may be also proved as follows. Writing as before

$$\alpha S + \beta T + \gamma U + \delta V = (A, B, C, D, F, G, H, L, M, N \between x, y, z, w)^2 = 0,$$

the condition that this shall be a plane-pair implies a threefold relation between the coefficients A, B, &c., and the required number of nodes is equal to the order of this threefold relation. Establishing between the coefficients A, B, &c., any 6 linear relations whatever, we should have a ninefold relation to determine the ratios of the 10 quantities; and the number of solutions would be equal to the order of the threefold relations. But taking the 6 linear relations to be of the form $(A, \ldots \between x_1, y_1, z_1, w_1)^2 = 0$, the question is in fact to find the number of the plane-pairs which pass through 6 given points; and this is clearly $= 10$.

77. Applying the conclusion to the system of quadric surfaces $\alpha S + \beta T + \gamma U + \delta V = 0$, we see that there are in the system 10 plane-pairs; and that the lines of intersection, or axes of the plane-pairs, are lines upon the Jacobian surface.

78. The equation $\nabla = 0$ of the symmetroid seems to contain homogeneously 40 constants. But starting with any given symmetrical determinant, we may multiply it line into line by a constant determinant, and then column into column by the same constant determinant, in such wise that the resulting product is still a symmetrical determinant; and the coefficients of the constant determinant may then be used to specialise the form of the equation. The equation $\nabla = 0$ of the symmetroid thus really contains $40 - 16 = 24$ constants; this is as it should be, for the symmetroid, *quà* quartic surface with 10 nodes, contains $34 - 10 = 24$ constants.

Symmetroid with given Nodes.

79. A symmetroid can be formed with 7 given points as nodes; but there is no proper symmetroid with 8 given points as nodes. If we endeavour to form such a symmetroid, we obtain a system of 2 quadric cones, each of them passing through the 8 points; viz., these are any 2 out of the 4 quadric cones which pass through the 8 points. This will be shown in a moment; for the complete *à posteriori* identification with the decadianome, it would be necessary to show that a symmetroid could be found having for nodes 7 given points, an 8th point anywhere on the dianodal surface, and a 9th point anywhere on the dianodal curve; but this I have not succeeded in effecting.

80. We have for any node $(\alpha, \beta, \gamma, \delta)$ of the symmetroid,

$$\alpha S + \beta T + \gamma U + \delta V = \text{plane-pair}.$$

If, then, 4 of the given nodes are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), we must have S, T, U, V each of them a plane-pair. We may without loss of generality assume $S = x^2 + y^2$, $T = z^2 + w^2$; this, however, does not determine the signification of the coordinates (x, y, z, w), for S will remain unaltered if we write therein

$$x\cos\theta + y\sin\theta,\quad x\sin\theta - y\cos\theta \text{ for } x, y;$$

and similarly T will remain unaltered if we write therein

$$z\cos\theta_1 + w\sin\theta_1,\quad z\sin\theta_1 - w\cos\theta_1 \text{ for } z, w.$$

Hence, if we go on to assume

$$U = k\,(x + m\,y + n\,z + p\,w)\,(x + m'\,y + n'z + p'\,w),$$
$$V = k_1\,(x + m_1 y + n_1 z + p_1 w)\,(x + m_1' y + n_1' z + p_1' w),$$

we may imagine the θ, θ_1 so determined that, for instance,

$$m + m' = 0,\quad p_1 + p_1' = 0;$$

we have thus

$$S = x^2 + y^2,$$
$$T = \qquad z^2 + w^2,$$
$$U = k\,(x + m\,y + n\,z + p\,w)\,(x - m\,y + n'\,z + p'w),$$
$$V = k_1\,(x + m_1 y + n_1 z + p_1 w)\,(x + m_1' y + n_1' z - p_1 w);$$

formulæ which contain the 12 constants

$$(k, m, n, p, n', p', k_1, m_1, n_1, p_1, m_1', n_1').$$

This is right, for the symmetroid containing 24 constants, the symmetroid with 4 given nodes should contain $(24 - 4\,.\,3 =)$ 12 constants. And each additional given node will determine 3 constants: hence for 4 new given nodes the expressions become determinate (not of necessity uniquely so).

81. But for any 4 new nodes, the equations may be satisfied by writing therein $n = n'$, $p = -p'$, $m = -m_1'$, $n_1 = n_1'$; viz., they then assume the form

$$S = \;x^2 + \qquad y^2,$$
$$T = \qquad z^2 \qquad + w^2,$$
$$U = (a\,x + c\,z)^2 + (b\,y + d\,w)^2,$$
$$V = (a'x + c'z)^2 + (b'y + d'w)^2,$$

containing 8 constants, which may be determined so that the nodes shall be the 4 given points. If now with the last mentioned values we form the value of $\alpha S + \beta T + \gamma U + \delta V$, this will consist of two terms $(*\between x, z)^2$ and $(*\between y, w)^2$, the first of which will be a square if

$$(\alpha + \gamma a^2 + \delta a'^2)(\beta + \gamma c^2 + \delta c'^2) - (\gamma ac + \delta a'c')^2 = 0, \text{ say this is } \Lambda = 0,$$

and the second will be a square if

$$(\alpha+\gamma b^2+\delta b'^2)(\beta+\gamma d^2+\delta d'^2)-(\gamma bd+\delta b'd')^2=0, \text{ say this is } \Lambda'=0;$$

so that the condition

$$\alpha S+\beta T+\gamma U+\delta V=\text{cone}$$

will be satisfied if $\Lambda=0$, or if $\Lambda'=0$; that is, the equation of the symmetroid will be $\Lambda\Lambda'=0$, or the symmetroid breaks up into the 2 quadric surfaces $\Lambda=0$, $\Lambda'=0$, each of which is a cone.

82. It is to be further observed that, considering the first mentioned 4 points (1, 0, 0, 0), &c., and any other 4 given points whatever, the equation of any one of the 4 quadric cones through these 8 points will be of the form

$$(*\between\beta\gamma,\ \gamma\alpha,\ \alpha\beta,\ \alpha\delta,\ \beta\delta,\ \gamma\delta)=0;$$

viz., any equation of this form, being a cone, will admit of being expressed, and that in one way only, in the form $\Lambda=0$. Consider then any one of the 4 cones through the 8 points, and let its equation be thus expressed; we have the values of the coefficients a, c, a', c', which enter into the expressions of S, T, U, V; and similarly, considering any other of the 4 cones, and expressing its equation in the like form, we have the values of the coefficients b, d, b', d' which enter into the expressions of S, T, U, V.

83. If instead of taking 2 different cones through the 8 points, we take in each case the same cone, the expressions for S, T, U, V would be

$$\begin{aligned} S &= x^2+y^2,\\ T &= z^2+w^2,\\ U &= (ax+cz)^2+(ay+cw)\\ V &= (a'x+c'z)^2+(a'y+c'w)^2; \end{aligned}$$

and we have identically

$$(ac'-a'c)(aa'S-cc'T)-a'c'U+acV=0.$$

This solution may be disregarded.

84. Instead of the assumption $S=x^2+y^2$, $T=z^2+w^2$, we may take $x=0$, $y=0$, $z=0$, $w=0$ to be planes of the plane-pairs S, T, U, V respectively; it is then easy to fix the remaining constants so that the 5th and 6th nodes of the symmetroid shall be given points. Suppose that the coordinates of the 5th node are (1, 1, 1, 1); to obtain the result in the most simple manner, I take for the moment Ω an arbitrary quadric function $(x, y, z, w)^2$, and I write

$$\begin{aligned} S &= x\,(\delta_x\Omega \quad\; + hy - gz + aw),\\ T &= y\,(\delta_y\Omega - hx \quad\; + fz + bw),\\ U &= z\,(\delta_z\Omega + gx - fy \quad\; + cw),\\ V &= w\,(\delta_w\Omega - ax - by - cz \quad\;), \end{aligned}$$

where the coefficients are arbitrary. We have identically $S+T+U+V=2\Omega$; wherefore the given point (1, 1, 1, 1) will be a node of the symmetroid if only $\Omega=0$ be a plane-pair; and it is easy to see that we may without loss of generality take one factor to be $x+y+z+w$, and write

$$\Omega=(x+y+z+w)(lx+my+nz+pw);$$

viz., Ω having this value, the symmetroid, $\alpha S+\beta T+\gamma U+\delta V=$ cone, will have the 5 given nodes; the equation contains, as it should do, 9 constants.

85. In order that the symmetroid may have a 6th given node $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, I observe that the constants may be determined so that $\alpha_1 S+\beta_1 T+\gamma_1 U+\delta_1 V$ shall be equal to an arbitrary quadric function, say

$$\alpha_1 S+\beta_1 T+\gamma_1 U+\delta_1 V=(\mathrm{a, b, c, d, f, g, h, l, m, n})(x, y, z, w)^2;$$

this in fact gives

$$(\mathrm{l, m, n, p})=\left(\frac{\mathrm{a}}{\alpha_1}, \frac{\mathrm{b}}{\beta_1}, \frac{\mathrm{c}}{\gamma_1}, \frac{\mathrm{d}}{\delta_1}\right);$$

and then, completing the comparison,

$$S=x\left\{\left[\frac{\mathrm{a}}{\alpha_1}\right]x+\left[\frac{2\mathrm{h}}{\alpha_1-\beta_1}-\frac{\beta_1}{\alpha_1-\beta_1}\left(\frac{\mathrm{a}}{\alpha_1}+\frac{\mathrm{b}}{\gamma_1}\right)\right]y\right.$$
$$\left.+\left[\frac{2\mathrm{g}}{\alpha_1-\gamma_1}-\frac{\gamma_1}{\alpha_1-\gamma_1}\left(\frac{\mathrm{a}}{\alpha_1}+\frac{\mathrm{c}}{\gamma_1}\right)\right]z+\left[\frac{2\mathrm{l}}{\alpha_1-\delta_1}-\frac{\delta_1}{\alpha_1-\delta_1}\left(\frac{\mathrm{a}}{\alpha_1}+\frac{\mathrm{d}}{\delta_1}\right)\right]w\right\},$$

$$T=y\left\{\left[\frac{2\mathrm{h}}{\beta_1-\alpha_1}-\frac{\alpha_1}{\beta_1-\alpha_1}\left(\frac{\mathrm{b}}{\beta_1}+\frac{\mathrm{a}}{\alpha_1}\right)\right]x+\left[\frac{\mathrm{b}}{\beta_1}\right]y\right.$$
$$\left.+\left[\frac{2\mathrm{f}}{\beta_1-\gamma_1}-\frac{\gamma_1}{\beta_1-\gamma_1}\left(\frac{\mathrm{b}}{\beta_1}+\frac{\mathrm{c}}{\gamma_1}\right)\right]z+\left[\frac{2\mathrm{m}}{\beta_1-\delta_1}-\frac{\delta_1}{\beta_1-\delta_1}\left(\frac{\mathrm{b}}{\beta_1}+\frac{\mathrm{d}}{\delta_1}\right)\right]w\right\},$$

$$U=z\left\{\left[\frac{2\mathrm{g}}{\gamma_1-\alpha_1}-\frac{\alpha_1}{\gamma_1-\alpha_1}\left(\frac{\mathrm{c}}{\gamma_1}+\frac{\mathrm{a}}{\alpha_1}\right)\right]x+\left[\frac{2\mathrm{f}}{\gamma_1-\beta_1}-\frac{\beta_1}{\gamma_1-\beta_1}\left(\frac{\mathrm{c}}{\gamma_1}+\frac{\mathrm{b}}{\beta_1}\right)\right]y\right.$$
$$\left.+\left[\frac{\mathrm{c}}{\gamma_1}\right]z+\left[\frac{2\mathrm{n}}{\gamma_1-\delta_1}-\frac{\delta_1}{\gamma_1-\delta_1}\left(\frac{\mathrm{c}}{\gamma_1}+\frac{\mathrm{d}}{\delta_1}\right)\right]w\right\},$$

$$V=w\left\{\left[\frac{2\mathrm{l}}{\delta_1-\alpha_1}-\frac{\alpha_1}{\delta_1-\alpha_1}\left(\frac{\mathrm{d}}{\delta_1}+\frac{\mathrm{a}}{\alpha_1}\right)\right]x+\left[\frac{2\mathrm{m}}{\delta_1-\beta_1}-\frac{\beta_1}{\delta_1-\beta_1}\left(\frac{\mathrm{d}}{\delta_1}+\frac{\mathrm{b}}{\beta_1}\right)\right]y\right.$$
$$\left.+\left[\frac{2n}{\delta_1-\gamma_1}-\frac{\gamma_1}{\delta_1-\gamma_1}\left(\frac{\mathrm{d}}{\delta_1}+\frac{\mathrm{c}}{\gamma_1}\right)\right]z+\left[\frac{d}{\delta_1}\right]w\right\};$$

viz., these values give

$$S+T+U+V=(x+y+z+w)\left(\frac{\mathrm{a}}{\alpha_1}x+\frac{\mathrm{b}}{\beta_1}y+\frac{\mathrm{c}}{\gamma_1}z+\frac{\mathrm{d}}{\delta_1}w\right),$$

$$\alpha_1 S+\beta_1 T+\gamma_1 U+\delta_1 V=(\mathrm{a, b, c, d, f, g, h, l, m, n})(x, y, z, w);$$

hence, taking the function $(\mathrm{a}, \ldots)(x, y, z, w)^2$ to be a plane-pair equal to $(x+iy+jz+kw)$ $(x+i_1y+j_1z+k_1w)$ suppose, or considering the coefficients (a, ...) as given functions of (i, j, k, i_1, j_1, k_1), we have the symmetroid having the 6 given nodes and containing the last mentioned 6 constants.

The Jacobian with given Lines.

86. The Jacobian contains 24 constants; obviously it is uniquely determined if 4 of the plane-pairs thereof are given; and it is also determined, but not uniquely, if 6 of the lines thereof are given. We may enquire how many given nodes of the symmetroid may be considered as corresponding to given plane-pairs, or lines of the Jacobian. Take as given any 4 nodes of the symmetroid; the corresponding 4 plane-pairs may be taken to be given plane-pairs; and we may besides take as given a 5th node of the symmetroid. For let the first 4 nodes of the symmetroid be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1); the given plane-pairs $P_1Q_1=0$, $P_2Q_2=0$, $P_3Q_3=0$, $P_4Q_4=0$; (l_1, l_2, l_3, l_4) any system of values such that we have

$$l_1P_1Q_1+l_2P_2Q_2+l_3P_3Q_3+l_4P_4Q_4=\text{plane-pair};$$

and (1, 1, 1, 1) the 5th node of the symmetroid; we have only to assume

$$(S,\ T,\ U,\ V)=(l_1P_1Q_1,\ l_2P_2Q_2,\ l_3P_3Q_3,\ l_4P_4Q_4).$$

87. Suppose, however, that on the Jacobian we have given, not the 4 plane-pairs, but only the 4 axes of the plane-pairs; the plane-pairs may be taken to be

$$(1,\ b_1,\ c_1 \between P_1,\ Q_1)^2=0, \ldots\ldots (1,\ b_4,\ c_4 \between P_4,\ Q_4)^2=0,$$

where the 8 constants $(b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4)$ are in the first instance undetermined. If we attempt to find l_1, l_2, l_3, l_4, so that

$$l_1(1,\ b_1,\ c_1 \between P_1,\ Q_1)^2 \ldots\ldots + l_4(1,\ b_4,\ c_4 \between P_4,\ Q_4)^2=\text{plane-pair of given axis},$$

we have between the coefficients (b, c) 4 equations; and similarly, if we attempt to find m_1, m_2, m_3, m_4 such that

$$m_1(1,\ b_1,\ c_1 \between P_1,\ Q_1)^2 \ldots\ldots + m_4(1,\ b_4,\ c_4 \between P_4,\ Q_4)^2=\text{plane-pair of another given axis},$$

we have 4 more equations between the coefficients (b, c); viz., these will be determined by the 8 equations (this is in fact the before mentioned property that 6 lines of the Jacobian may be taken to be given lines). But considering only the first system of equations; in order that to the given axis may correspond a given node on the symmetroid, say the node (1, 1, 1, 1), we have only to write

$$S=l_1(1,\ b_1,\ c_1 \between P_1,\ Q_1)^2, \ldots\ldots V=l_4(1,\ b_4,\ c_4 \between P_4,\ Q_4)^2;$$

that is, we may take as given 5 nodes of the symmetroid, and the corresponding 5 lines of the Jacobian; the formulæ will contain 4 constants; we may by means of them make the Jacobian have a 6th given line, thus determining the constants; or we may make the symmetroid have a 6th given node, leaving in this case one constant arbitrary.

Correspondence on the Jacobian: Lines and Skew Cubics.

88. I consider the correspondence of two points on the Jacobian; it is to be shown that when one of the points is on a line of the Jacobian, the corresponding point will be on a skew cubic; that is, that corresponding to each line of the Jacobian we have (on the Jacobian) a skew cubic. Call the plane-pairs of the system of quadric surfaces 1, 2, 3, ... 10; selecting any 4 of these, say 1, 2, 3, 4, the polar planes of any point of the Jacobian in regard to these 4 plane-pairs will meet in a point which will be the required corresponding point. And observe that, in regard to any one of the plane-pairs, say 1, the polar plane of a point P is the plane through the axis harmonic to the plane through the axis and the point P. Hence, for a point on the axis of 1, the polar plane in regard to 1 is indeterminate; the polar planes in regard to the plane-pairs 2, 3, 4 respectively meet in a point which is the required corresponding point. We may for any point whatever take the polar planes in regard to the plane-pairs 2, 3, 4 respectively, and call the intersection of these planes the corresponding point; this being so, if the first mentioned point moves along a line, the corresponding point moves along a curve, which is easily shown to be a skew cubic cutting the axis of each plane-pair twice; that is, in regard to the plane-pairs 2, 3, 4, the locus corresponding to any line whatever is a skew cubic cutting the axis of each plane-pair twice. In particular, the corresponding curve of the axis of 1, is a skew cubic cutting the axis of the plane-pairs 2, 3, 4 each twice; but the axis of 1 does not stand in any special relation to the plane-pairs 2, 3, 4, as distinguished from the remaining plane-pairs 5, 6...10; we have therefore the more complete theorem, that the skew cubic cuts the axes of the plane-pairs 2, 3,...10 each twice; or, instead of the plane-pairs, speaking of the line 1, 2, 3,...10, we may say that corresponding to any one of the lines we have a skew cubic meeting the other 9 lines each of them twice.

89. I stop for a moment to prove the subsidiary theorem assumed in the foregoing demonstration. Let the 3 plane-pairs be $PQ=0$, $RS=0$, $TU=0$, and let the line be that joining the points (x_0, y_0, z_0, w_0) and (x_1, y_1, z_1, w_1); the coordinates of any point in the line may be taken to be $\lambda x_0+\mu x_1$, $\lambda y_0+\mu y_1$, $\lambda z_0+\mu z_1$, $\lambda w_0+\mu w_1$; and hence for the polar plane in regard to the plane-pair $PQ=0$ we have

$$\{(\lambda x_0+\mu x_1)\,\delta_x \ldots + (\lambda w_0+\mu w_1)\,\delta_w\}\, PQ=0;$$

viz., this equation may be written

$$\lambda\,(PQ_0+P_0Q)+\mu\,(PQ_1+P_1Q)=0;$$

forming the like equations in regard to the other 2 plane-pairs respectively, and eliminating λ, μ, we obtain for the required locus

$$\left\| \begin{matrix} PQ_0+P_0Q, & RS_0+R_0S, & TU_0+T_0U \\ PQ_1+P_1Q, & RS_1+R_1S, & TU_1+T_1U \end{matrix} \right\| = 0,$$

a skew cubic; and on writing herein $P=0$, $Q=0$, the equations become

$$\begin{vmatrix} RS_0+R_0S, & TU_0+T_0U \\ RS_1+R_1S, & TU_1+T_1U \end{vmatrix}=0;$$

viz., the line $(P=0,\ Q=0)$ meets the skew cubic in the points where the line meets the quadric surface determined by this last equation, that is in 2 points.

90. We have thus on the Jacobian the 10 lines 1, 2,... 9, 10, and corresponding thereto respectively the 10 skew cubics 1′, 2′,... 9′, 10′, where each line meets twice each of the skew cubics except that denoted by the same number; a relation similar to that which exists between the lines 1, 2, 3, 4, 5, 6 and 1′, 2′, 3′, 4′, 5′, 6′, which compose a double-sixer on a cubic surface.

Suppose that there are given on the Jacobian the lines 1, 2, 3, 4, 5, 6; meeting each of these twice, we have the skew cubics 7′, 8′, 9′, 10′; and then

$$\text{the lines}\ \begin{array}{c}7\\8\\9\\10\end{array}\ \text{meet twice each of the cubics}\ \begin{array}{c}8',\ 9',\ 10'\\9',\ 10',\ 7'\\10',\ 7',\ 8'\\7',\ 8',\ 9'\end{array}$$

so that the determination of the remaining 4 lines depends upon that of the skew cubics 7′, 8′, 9′, 10′, which meet each of the given lines twice.

91. To determine a skew cubic cutting twice each of 6 given lines, I proceed as follows. Let the lines be 1, 2, 3, 4, 5, 6; take $U=0$ the general quadric surface through the lines 1 and 2, $V=0$ the general quadric surface through the lines 1, 3 (the equations contain each of them homogeneously 4 constants). The 2 surfaces intersect in the line 1, and in a skew cubic cutting twice each of the lines 1, 2, 3; we have therefore to determine the constants so that the 2 surfaces may meet the line 4 in the same 2 points, the line 5 in the same 2 points, the line 6 in the same two points. Imagine for a moment the equations of any one of the lines 4, 5, 6 to be $z=0$, $w=0$; the equations of the 2 surfaces, substituting therein these values, would assume the forms

$$(a,\ b,\ c\between x,\ y)^2=0,\quad (a',\ b',\ c'\between x,\ y)^2=0;$$

and the conditions for the intersection in the same 2 points would be $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'},=p$ suppose. This is in fact the form of the conditions, understanding a, b, c to be linear functions of the coefficients of U, and a', b', c' to be linear functions of the coefficients of V. We have in this manner 3 sets of equations involving respectively the indeterminate quantities p, q, r; viz., these may be represented by

$$a=pa',\ b=pb',\ c=pc';\quad d=qd',\ e=qe',\ f=qf';\quad g=rg',\ h=rh',\ i=ri';$$

where the unaccented letters $a, b, \dots i$ are linear functions of the coefficients of U, and the accented letters $a', b', \dots i'$ linear functions of the coefficients of V. Eliminating the coefficients of U, V, we have between p, q, r a twofold relation, which may be represented as follows:

$$\left\| \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ p & p & p & q & q & q & r & r & r \\ p & p & p & q & q & q & r & r & r \\ p & p & p & q & q & q & r & r & r \\ p & p & p & q & q & q & r & r & r \end{matrix} \right\| = 0,$$

it being understood that the 1's represent constants, and the p's, q's, and r's linear functions of these variables respectively. The several equations of the system, regarding therein p, q, r as coordinates, represent each of them a quartic curve; any 2 of these intersect in 16 points; but the number of points common to all the curves is $=10$. But each of the curves passes through the 3 points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; these are consequently included among the 10 points, but they do not give a proper solution of the question; and the number of solutions is thus reduced to $10-3=7$. There is yet another solution to be rejected; viz., $U=0$ being a quadric surface through the lines 1, 2, and $V=0$ the quadric surface through the lines 1, 3, it is possible to determine the coefficients of U, V so that each of these surfaces shall be the quadric surface through the lines 1, 2, 3; and if we then have identically $U=\theta V$, it is clear that corresponding values of p, q, r are $p=q=r\,(=\theta)$. We have thus the point $p=q=r$ common to all the curves of the system; this solution counts, I believe, once only, and the number of relevant solutions is $7-1=6$.

92. It may be observed, in regard to the foregoing solution, that if we take $123=0$ as the equation of the quadric surface through the lines 1, 2, 3, and so in other cases, then the equation of the surfaces $U=0$ and $V=0$ may be taken to be

$$\lambda \,.\, 123 + \mu \,.\, 124 + \nu \,.\, 125 + \rho \,.\, 126 = 0,$$
$$\lambda' \,.\, 132 + \mu' \,.\, 134 + \nu' \,.\, 135 + \rho' \,.\, 136 = 0,$$

respectively, the coefficients of the two surfaces being here put in evidence. And it is clear that for $\mu=\nu=\rho=0$, $\mu'=\nu'=\rho'=0$, the surfaces become each of them the surface through the lines 1, 2, 3.

93. The conclusion is, that touching twice each of the six lines 1, 2, 3, 4, 5, 6, we have six skew cubics; it would appear that any four of these may be taken for the skew cubics $7'$, $8'$, $9'$, $10'$ (so that there are 15 such tetrads of cubics). I am not, however, able to verify that we then have the remaining 4 lines each cutting twice 3 of the 4 skew cubics; assuming that for each system of 4 skew cubics there is one and only one, such system of lines, then of course to the given system of lines 1, 2, 3, 4, 5, 6, there will belong 15 systems of lines 7, 8, 9, 10, and therefore also 15 Jacobian surfaces.

Further Investigations as to the Jacobian, &c.

94. Taking $(\xi, \eta, \zeta, \omega)$ as plane-coordinates, two quadric surfaces

$$(a, b, c, d, f, g, h, l, m, n \between \xi, \eta, \zeta, \omega)^2 = 0$$

and

$$(A, B, C, D, F, G, H, L, M, N \between x, y, z, w)^2 = 0$$

are said to be interverts (or interverse) one of the other, when we have between the coefficients the relation

$$(a, b, c, d, f, g, h, l, m, n \between A, B, C, D, F, G, H, L, M, N) = 0,$$

that is

$$aA + \ldots + 2fF + \ldots = 0.$$

The condition that the two surfaces may be interverts of each other is linear in regard to the coefficients of each surface separately; hence, using a before explained locution, we may say—interverse to a given quadric surface we have 9 quadrics; interverse to two given quadrics 8 quadrics; or generally, that interverse to k given quadrics we have $10-k$ quadrics. And, moreover, if the quadrics of the two systems be $L=0$, $M=0$, &c., and $S=0$, $T=0$, $U=0$, &c., then every quadric $\lambda L + \mu M + \ldots = 0$ is interverse to each of the quadrics $\alpha S + \beta T + \gamma U + \ldots = 0$.

If the quadric $(a, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0$ be an intervert of the plane-pair

$$(lx + my + nz + pw \between l'x + m'y + n'z + p'w) = 0,$$

the condition is

$$(a, \ldots \between l, m, n, p \between l', m', n', p') = 0;$$

viz., this expresses that the two planes are harmonics in regard to the pair of planes drawn through the axis of the plane-pair to touch the quadric surface; or say, that the plane-pair is harmonic in regard to the quadric.

95. To apply this to the Jacobian surface, I recall that, starting with the given quadric surfaces $S=0$, $T=0$, $U=0$, $V=0$, and taking $(\alpha, \beta, \gamma, \delta)$ to be such that

$$\alpha S + \beta T + \gamma U + \delta V = \text{plane-pair},$$

there are 10 such plane-pairs, and that the axes of these are the lines of the Jacobian. If instead of the given quadric surfaces, we consider the six interverse surfaces $(a_1, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0, \ldots (a_6, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0$, then the condition is that the plane-pair shall be harmonic in regard to each of these surfaces. Let the quadric surfaces be called 1, 2, 3, 4, 5, 6; then, attending to any three of these, say 1, 2, 3, the plane-pair is harmonic in regard to these three surfaces. Through the axis of the plane-pair draw tangent planes to 1, 2, and 3 respectively; each of these pairs of planes is harmonic in regard to the planes of the plane-pair; that is, the three pairs of tangent planes are in involution; or, as we may also express it, the axis is (*quoad* its planes) in involution in regard to the three quadric surfaces. Conversely, when the axis is thus in involution in regard to the surfaces 1, 2, and 3, we may by means of the surfaces 1 and 2 determine the two planes of the plane-pair, and then these will be harmonics in regard to the surface 3. It thus appears that the axis is given as a line which is (*quoad* its planes) in involution in regard to the surfaces 1, 2, 3, to the surfaces 1, 2, 4, the surfaces 1, 2, 5, and the surfaces 1, 2, 6, respectively; or, as we may express it, as a line which is (*quoad* its planes) in involution in regard to the surfaces 1, 2, 3, 4, 5, 6.

96. It is substantially the same thing, but it is rather easier, to consider the whole question under the reciprocal form; viz., instead of a plane-pair and a quadric surface represented by an equation in plane-coordinates, to take a point-pair and a quadric surface represented by an equation in point-coordinates; we have thus a line which is (*quoad* its points) in involution in regard to three given quadric surfaces, or as we may more simply express it, which cuts in involution the three given surfaces; and we thus arrive at the problem of finding a line which cuts in involution six given quadric surfaces; viz., this is equivalent to the above problem where the line has to satisfy (*quoad* its planes) the like condition; and in each problem the number of solutions should be $=10$.

97. Consider a line which cuts in involution the three given surfaces $(a_1, \ldots \between x, y, z, w)=0$, $(a_2, \ldots \between x, y, z, w)^2=0$, $(a_3, \ldots \between x, y, z, w)^2=0$. I will presently show that this implies a cubic relation $(* \between \mathrm{a, b, c, f, g, h})^3$ between the six coordinates of the line. But assuming it for the moment, suppose that the line cuts in involution the three surfaces and a fourth quadric surface $(a_4, \ldots \between x, y, z, w)^2=0$. Considering the line as cutting in involution the surfaces 1, 2, 4, we have between the six coordinates a second cubic relation; there is, however, a reduction, and the order of the resulting twofold relation between the coordinates is $3 \,.\, 3-4=5$. To explain this, observe that every line which cuts in the same two points the surfaces 1 and 2 respectively (that is, which cuts the curve of intersection twice) will in an improper sense cut in involution the surfaces 1, 2, 3, and also the surfaces 1, 2, 4. There is thus a reduction equal to the order in the six coordinates of the twofold relation which expresses that the line cuts twice the curve of intersection of the surfaces 1 and 2. Join hereto the relations that the line meets each of two given lines; the coordinates of the line are determined by the twofold relation (say its order is $=\lambda$) two linear equations, and the universal equation $\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0$; the number of solutions is $=2\lambda$. But the number of solutions is equal to that of the lines which meet the quadri-quadric curve of intersection twice, and meet also each of two given lines; or what is the same thing, it is equal to the order of the scroll generated by the lines which meet the curve twice, and also a given line. We have for the curve of intersection (m the order, h the number of apparent double points) $m=4$, $h=2$; whence order of the scroll is $2+\frac{1}{2} \,.\, 4 \,.\, 3=8$; that is, $2\lambda=8$, or $\lambda=4$, which is the required reduction.

98. If the line cut in involution 5 given quadric surfaces {say the 5th surface is $(a_5, \ldots \between x, y, z, w)^2=0$}; then we have between the 6 coordinates a threefold relation, the order of which is $3.5-$reduction. This should be $=10$, and consequently the reduction $=5$; for admitting the value to be 10, the order (in the ordinary sense) of the scroll generated by the lines which cut in involution the 5 given quadrics should be $=20$;

and conversely. But the value 20 may be verified without difficulty. For the question may be transformed as follows:—If a point-pair be harmonic in regard to each of 5 given quadrics, how many of the axes (or lines through the 2 points of a point-pair) cut a given line. Take (x, y, z, w), (x', y', z', w') as the coordinates of the 2 points of a point-pair; the harmonic condition in regard to a quadric surface $U=0$ is $x'\delta_x U + y'\delta_y U + z'\delta_z U + w'\delta_w U = 0$ {where U is regarded as a function of the (x, y, z, w) belonging to a point of the point-pair}; the condition for the intersection with a given line is a lineolinear equation in the coordinates (x, y, z, w) and (x', y', z', w'), or say it is $Lx' + My' + Nz' + Pw' = 0$, where L, M, N, P are linear functions of the coordinates; we have thence for (x, y, z, w) the threefold relation

$$\left\|\begin{array}{cccccc} L, & \delta_x U_1, & \delta_x U_2, & \delta_x U_3, & \delta_x U_4, & \delta_x U_5 \\ M, & \delta_y U_1 & . & . & & \\ N, & \delta_z U_1 & . & & & \\ P, & \delta_w U_1 & & & & \end{array}\right\| = 0,$$

which denotes a system of $\frac{1}{6}.6.5.4 = 20$ points.

It would seem that if the line cuts in involution 6 given quadrics, there should be between the 6 coordinates a fourfold relation of the order $\frac{1}{2}.10 = 5$; this would imply a reduction 25, viz. we should have $5 = 3.10 - 25$. I do not understand this, and I drop the question.

99. I return to the question to find the relation between the coordinates (a, b, c, f, g, h) of a line which cuts in involution the 3 quadric surfaces

$$(a_1, b_1, c_1, d_1, f_1, g_1, h_1, l_1, m_1, n_1 \between x, y, z, w)^2 = 0,\ (a_2, \ldots \between x, y, z, w)^2 = 0,\ (a_3, \ldots \between x, y, z, w)^2 = 0.$$

Writing down any two of the equations of the line, for instance

$$\begin{aligned} \mathrm{h}y - \mathrm{g}z + \mathrm{a}w &= 0, \\ -\mathrm{h}x \qquad + \mathrm{f}z + \mathrm{b}w &= 0, \end{aligned}$$

if we substitute the values of (x, y) in the equation of the first surface, it becomes

$$(a_1, \ldots \between \mathrm{f}z + \mathrm{b}w,\ \mathrm{g}z - \mathrm{a}w,\ \mathrm{h}z,\ \mathrm{h}w)^2 = 0;$$

or if we write for shortness

$$\Pi = (\mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{o}),\quad \Pi' = (\mathrm{b}, -\mathrm{a}, \mathrm{o}, \mathrm{h}),$$

then the equation is

$$(a_1, \ldots \between \Pi)^2 . z^2 + 2(a_1, \ldots \between \Pi \between \Pi') . zw + (a_1, \ldots \between \Pi')^2 . w^2 = 0,$$

and forming the like equations for the other two surfaces, the condition of involution is at once found to be

$$\left|\begin{array}{ccc} (a_1, \ldots \between \Pi)^2, & (a_1, \ldots \between \Pi \between \Pi'), & (a_1, \ldots \between \Pi')^2 \\ (a_2, \ldots \between \Pi)^2, & (a_2, \ldots \between \Pi \between \Pi'), & (a_2, \ldots \between \Pi')^2 \\ (a_3, \ldots \between \Pi)^2, & (a_3, \ldots \between \Pi \between \Pi'), & (a_3, \ldots \between \Pi')^2 \end{array}\right| = 0.$$

100. It is convenient, in working this out, to consider Π, Π' as standing, in the first instance, for (x, y, z, w), (x', y', z', w'), these symbols being ultimately replaced by the above-mentioned values. Writing also, for shortness, (abc) to denote the determinant $a_1(b_2c_3 - b_3c_2) + \&c.$, and so in other cases, it is at once seen that the function on the right-hand side is a sum of such determinants each into a proper factor, containing the coordinates (a, b, c, f, g, h), originally of the order 6, but where each term contains the factor h^3, which may be omitted; or finally the result is of the order 3 in the coordinates. Thus we have a term

$$(abc)\begin{vmatrix} x^2, & xx', & x'^2 \\ y^2, & yy', & y'^2 \\ z^2, & zz', & z'^2 \end{vmatrix}$$

where the second factor is

$$\begin{aligned} x^2y'z'(yz' - y'z) + y^2z'x'(zx' - z'x) + z^2x'y'(xy' - x'y), \quad &= z^2x'y',(xy' - x'y), \\ &= \mathrm{h}^2(-\mathrm{ab})(-\mathrm{af} - \mathrm{bg}), \quad = -\mathrm{abch}^3, \end{aligned}$$

or, omitting the factor $-\mathrm{h}^3$, the term is $(abc)\,\mathrm{abc}$.

101. There are in all 120 terms, but 16 of these are found to vanish (viz., these are the terms in agh, bhf, cfg; ahl, bfm, cgn; agl, bhm, cfn; dmn, dnl, dlm; fgn, ghl, hfm). The final result contains therefore 104 terms; viz., as a further abbreviation writing abc &c., instead of (abc) &c., to denote the above-mentioned determinants, the equation is

$$\begin{aligned}
& abc\ .\ \mathrm{abc} - bcd\ .\ \mathrm{agh} - cad\ .\ \mathrm{bhf} - abd\ .\ \mathrm{cfg} \\
& + bcf\ .\ \mathrm{a}^3 + cag\ .\ \mathrm{b}^3 + abh\ .\ \mathrm{c}^3 + adl\ .\ \mathrm{f}^3 + bdm\ .\ \mathrm{g}^3 + cdn\ .\ \mathrm{h}^3 \\
& + abn\ .\ \mathrm{c}(\mathrm{bg} - \mathrm{af}) + adf\ .\ \mathrm{f}(\mathrm{ch} - \mathrm{bg}) \\
& + bcl\ .\ \mathrm{a}(\mathrm{ch} - \mathrm{bg}) + bdg\ .\ \mathrm{g}(\mathrm{af} - \mathrm{ch}) \\
& + cam\ .\ \mathrm{b}(\mathrm{af} - \mathrm{ch}) + cdh\ .\ \mathrm{h}(\mathrm{bg} - \mathrm{af}) \\
& - bcg\ .\ \mathrm{a}^2\mathrm{b} - bch\ .\ \mathrm{a}^2\mathrm{c} + bcm\ .\ \mathrm{a}^2\mathrm{g} - bcn\ .\ \mathrm{a}^2\mathrm{h} \\
& - cah\ .\ \mathrm{b}^2\mathrm{c} - caf\ .\ \mathrm{b}^2\mathrm{a} + can\ .\ \mathrm{b}^2\mathrm{h} - cal\ .\ \mathrm{b}^2\mathrm{f} \\
& - abf\ .\ \mathrm{c}^2\mathrm{a} - abg\ .\ \mathrm{c}^2\mathrm{b} + abl\ .\ \mathrm{c}^2\mathrm{f} - abm\ .\ \mathrm{c}^2\mathrm{g} \\
& - adg\ .\ \mathrm{bf}^2 + adh\ .\ \mathrm{cf}^2 + adm\ .\ \mathrm{f}^2g + adn\ .\ \mathrm{f}^2\mathrm{h} \\
& - bdh\ .\ \mathrm{cg}^2 + bdf\ .\ \mathrm{ag}^2 + bdn\ .\ \mathrm{g}^2\mathrm{h} + bdl\ .\ \mathrm{g}^2\mathrm{f} \\
& - cdf\ .\ \mathrm{ah}^2 + cdg\ .\ \mathrm{bh}^2 + cdl\ .\ \mathrm{h}^2\mathrm{f} + cdm\ .\ \mathrm{h}^2\mathrm{g} \\
& + 2\left\{\begin{aligned} & afg\ .\ \mathrm{b}^2\mathrm{c} - afh\ .\ \mathrm{bc}^2 + afl\ .\ \mathrm{bcf} - afm\ .\ \mathrm{c}^2\mathrm{h} - afn\ .\ \mathrm{b}^2\mathrm{g} \\ & + bgh\ .\ \mathrm{c}^2\mathrm{a} - bgf\ .\ \mathrm{ca}^2 + bgm\ .\ \mathrm{cag} - bgl\ .\ \mathrm{a}^2\mathrm{f} - bgl\ .\ \mathrm{c}^2\mathrm{h} \\ & + chf\ .\ \mathrm{a}^2\mathrm{b} - chg\ .\ \mathrm{ab}^2 + chn\ .\ \mathrm{abh} - chm\ .\ \mathrm{b}^2\mathrm{g} - chm\ .\ \mathrm{a}^2\mathrm{f} \end{aligned}\right\}
\end{aligned}$$

$$+2\begin{Bmatrix} agm\,.\,\mathrm{bcf} - agn\,.\,\mathrm{b^2f} - ahm\,.\,\mathrm{c^2f} + ahn\,.\,\mathrm{bcf} \\ + bhn\,.\,\mathrm{cag} - bfl\,.\,\mathrm{c^2g} - bfn\,.\,\mathrm{a^2g} + bfl\,.\,\mathrm{cag} \\ + cfl\,.\,\mathrm{abh} - chm\,.\,\mathrm{a^2h} - cgl\,.\,\mathrm{b^2h} + cgm\,.\,\mathrm{abh} \end{Bmatrix}$$

$$+2\begin{Bmatrix} - amn\,.\,\mathrm{af^2} - anl\,.\,\mathrm{bf^2} - alm\,.\,\mathrm{cf^2} + dfg\,.\,\mathrm{ch^2} \\ - bnl\,.\,\mathrm{bg^2} - blm\,.\,\mathrm{cg^2} - bmn\,.\,\mathrm{ag^2} + dgh\,.\,\mathrm{af^2} \\ - clm\,.\,\mathrm{ch^2} - cmn\,.\,\mathrm{ah^2} - cnl\,.\,\mathrm{bh^2} + dhf\,.\,\mathrm{bg^2} \end{Bmatrix}$$

$$+2\begin{Bmatrix} - dfl\,.\,\mathrm{fgh} - dfm\,.\,\mathrm{g^2h} - dfn\,.\,\mathrm{gh^2} \\ - dgm\,.\,\mathrm{fgh} - dgn\,.\,\mathrm{h^2f} - dgl\,.\,\mathrm{hf^2} \\ - dhn\,.\,\mathrm{fgh} - dhl\,.\,\mathrm{f^2g} - dhm\,.\,\mathrm{fg^2} \end{Bmatrix}$$

$$+4\begin{Bmatrix} fgh\,.\,\mathrm{bch} - fgm\,.\,\mathrm{ach} - fmn\,.\,\mathrm{agh} - fnl\,.\,\mathrm{bgh} - flm\,.\,\mathrm{cgh} \\ + ghm\,.\,\mathrm{caf} - ghn\,.\,\mathrm{baf} - gnl\,.\,\mathrm{bhf} - glm\,.\,\mathrm{chf} - gmn\,.\,\mathrm{ahf} \\ + hfn\,.\,\mathrm{abg} - hfl\,.\,\mathrm{cbg} - hlm\,.\,\mathrm{cfg} - hmn\,.\,\mathrm{afg} - hnl\,.\,\mathrm{bfg} \end{Bmatrix}$$

$$-4fgh\,.\,\mathrm{abc} \qquad\qquad = 0.$$

And observe, by what precedes, this triple system of lines contains each of the following double systems: viz., the lines which meet the quadriquadric curve (2, 3) twice, those which meet the curve (3, 1) twice, those which meet the curve (1, 2) twice.

Persymmetrical Case: the Hessian of a Cubic.

102. Reverting to the general equation

$$\alpha S + \beta T + \gamma U + \delta V = \text{cone},$$

which connects the symmetroid and Jacobian, it is evident that if S, T, U, V are the derivatives, in regard to the coordinates, of a single cubic function U, $= (*\between x, y, z, w)^3$, then the symmetroid and the Jacobian become one and the same surface; viz., this is the Hessian surface $H = 0$ derived from the given cubic surface. The two corresponding points on the symmetroid and the Jacobian respectively, and the two corresponding points on the Jacobian, become one and the same pair of corresponding points on the Hessian; viz., either of these points is such that its first polar surface in regard to the cubic is a quadric cone having for its vertex the other corresponding point. And the Hessian surface unites the properties of the Jacobian and the symmetroid, viz., it has 10 nodes and 10 lines. It is, in fact, known that there are five planes such that the intersection of every two of them is a line on the Hessian surface, and the intersection of every three of them a node on the surface; viz., if the equations of the five planes are $x = 0$, $y = 0$, $z = 0$, $w = 0$, $u = 0$, then the equation of the Hessian surface is

$$xyzwu\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} + \frac{d}{w} + \frac{e}{u}\right) = 0,$$

a form which puts in evidence the properties just referred to.

Quartics with 11 *or more Nodes.*

103. I mention two results which, although they relate to quadric surfaces with more than 10 nodes, present themselves in such immediate connexion with the present Memoir, that it is natural to speak of them. If, in the equation

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix} = 0,$$

of the symmetroid ($A, B, \ldots$ linear functions of the coordinates), we have identically $A = 0$, then the surface has evidently a node $H = 0$, $G = 0$, $L = 0$; viz., this is a node in addition to the usual 10 nodes, or the surface has in all 11 nodes. And so also if (identically in every case) B is $= 0$, there are 12 nodes; if C is $= 0$, there are 13 nodes; and if D is $= 0$, there are 14 nodes. These are, in fact, quartic surfaces with 11, 12, 13, and 14 nodes respectively, mentioned in Kummer's Memoir.

104. We may consider the symmetroid derived from the quadric surfaces which pass through 6 given points; viz., taking as before (see No. 25) the coordinates of the 6 points to be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 1), $(\alpha, \beta, \gamma, \delta)$, and (a, b, c, f, g, h) as the coordinates of the line joining the last-mentioned two points; and, to avoid confusion, taking for the present purpose (X, Y, Z, W) instead of $(\alpha, \beta, \gamma, \delta)$ for the coordinates of a point on the symmetroid, the equation is obtained by arranging in the form of a determinant the coefficients of the quadric form

$$\begin{aligned} &Xx\,(\qquad hy - \quad gz + aw) \\ &+ Yy\,(-hx \qquad + \quad fz + bw) \\ &+ Zz\,(\;\; gx - \quad fy \qquad + cw) \\ &+ W\,(a\alpha\, yz + b\beta\, zx + c\gamma\, xy \qquad); \end{aligned}$$

viz., the equation in question is

$$\begin{vmatrix} \cdot & h(X-Y)+c\gamma W, & g(Z-X)+b\beta W, & aX \\ h(X-Y)+c\gamma W, & \cdot & f(Y-Z)+a\alpha W, & bY \\ g(Z-X)+b\beta W, & f(Y-Z)+a\alpha W, & \cdot & cZ \\ aX & bY & cZ & \cdot \end{vmatrix} = 0;$$

or, as it may be more simply written,

$$\sqrt{aX\,\{f(Y-Z)+a\alpha W\}} + \sqrt{bY\,\{g(Z-X)+b\beta W\}} + \sqrt{cZ\,\{h(X-Y)+c\gamma W\}} = 0.$$

This is, in fact, a surface with 16 nodes. It would appear that additional nodes correspond to the six common intersections of the quadric surfaces, or nodes of the Jacobian; and it would seem that for four quadric surfaces having in common 1, 2, 3, 4, 5, or 6 points, the corresponding symmetroid would have 11, 12, 13, 14, 15, or 16 nodes. But I reserve this for future consideration.

I take the opportunity of mentioning some results which have a connexion, although not an immediate one, with the subject of the present Memoir.

Quadric Surface through three given Lines.

105. To find the equation to the quadric surface through the three lines $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$. Take on one of the lines the points $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$; then the equation of a quadric surface through this line will be of the form

$$\left|\begin{array}{cccccccccc} x^2 & y^2 & z^2 & w^2 & yz & zx & xy & xw & yw & zw \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \beta\gamma & \gamma\alpha & \alpha\beta & \alpha\delta & \beta\delta & \gamma\delta \\ 2\alpha\alpha' & 2\beta\beta' & 2\gamma\gamma' & 2\delta\delta' & \beta\gamma'+\beta'\gamma & \gamma\alpha'+\gamma'\alpha & \alpha\beta'+\alpha'\beta & \alpha\delta'+\alpha'\delta & \beta\delta'+\beta'\delta & \gamma\delta'+\gamma'\delta \\ \alpha'^2 & \beta'^2 & \gamma'^2 & \delta'^2 & \beta'\gamma' & \gamma'\alpha' & \alpha'\beta' & \alpha'\delta' & \beta'\delta' & \gamma'\delta' \\ \vdots &&&&&&&&& \end{array}\right| = 0;$$

and if we form thus a determinant with three of its lines relating to the line 1, three of them to the line 2, and three to the line 3, we have the equation of the quadric surface through the three lines. But considering in the determinant the three lines which refer to the line 1, it is clear that the determinant is a function of the order 3 of the coordinates $(a_1, b_1, c_1, f_1, g_1, h_1)$ of the line in question; and the like as regards the other two lines respectively. Now observe that if two of the lines intersect, the problem becomes indeterminate (in fact, the plane of the intersecting lines, and any plane whatever through the third line, constitute a solution); the condition for the intersection of the lines 1 and 2 is $a_1f_2+a_2f_1+b_1g_2+b_2g_1+c_2h_1+c_1h_2=0$; hence, if this condition be satisfied, the determinant must vanish; it therefore divides by the factor a_1f_2+&c.; but, similarly, it divides by the factors a_2f_3+ &c. and a_3f_1+&c.; and throwing out the three factors, the result should be of the order 1, that is linear, in regard to the three sets of coordinates respectively. I have obtained this reduced result in my "Memoir on the Six Coordinates of a Line" (*Camb. Phil. Trans.*, t. XI., 1869, p. 311 [435]); viz., writing (abc) to denote the determinant $a_1(b_2c_3-b_3c_2)+$&c., and so for the other like determinants, the result is

$$\begin{aligned} &(agh)\,x^2+(bhf)\,y^2+(cfg)\,z^2+(abc)\,w^2 \\ &+[(abg)-(cah)]\,xw+[(bfg)+(chf)]\,yz \\ &+[(bch)-(abf)]\,yw+[(cgh)+(afg)]\,zx \\ &+[(caf)-(bcg)]\,zw+[(ahf)+(bgh)]\,xy=0. \end{aligned}$$

Condition that five given lines may lie in a Cubic Surface.

106. Taking the lines to be $(a_1, b_1, c_1, f_1, g_1, h_1), \ldots (a_5, b_5, c_5, f_5, g_5, h_5)$, and $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ the coordinates of any two points on one of the lines, the equation of a cubic surface through this line would be

$$\left|\begin{array}{llll} x^3, \ldots\ldots & x^2y, \ldots\ldots & xyz, & \ldots\ldots \\ \alpha^3, & \alpha^2\beta, & \alpha\beta\gamma, & \\ 3\alpha^2\alpha', & 2\alpha\alpha'\beta+\alpha^2\beta', & \alpha'\beta\gamma+\alpha\beta'\gamma+\alpha\beta\gamma', & \\ 3\alpha\alpha'^2, & 2\alpha\alpha'\beta'+\alpha'^2\beta, & \alpha\beta'\gamma'+\alpha'\beta\gamma'+\alpha\beta'\gamma', & \\ \alpha'^3, & \alpha'^2\beta', & \alpha'\beta'\gamma', & \end{array}\right| = 0;$$

and hence it at once appears that, forming a determinant of 20 lines, wherein four lines relate to the line 1, four to the line 2, ..., four to the line 5, and equating this to zero, we have the required condition. But the condition so obtained is of the order ($\frac{1}{2}4 . 3 =$) 6 in regard to the coordinates of each line; and, as for the quadric, it is satisfied identically if we have any such equation as $a_1f_2 + \&c. = 0$; it consequently contains the several factors $a_1f_2 + \&c.$, which can be formed with the coordinates of any two of the five lines; and throwing out these factors, the condition should be of the order 2 in regard to the coordinates of each line. We in fact know that the required relation between the five lines is that they shall all of them be cut by a sixth line; and moreover that, writing $a_1f_2 + a_2f_1 + b_1g_2 + b_2g_1 + c_1h_2 + c_2h_1 = 12$, &c., then that the condition for this is

$$\left|\begin{array}{ccccc} . , & 12, & 13, & 14, & 15 \\ 21, & . , & 23, & 24, & 25 \\ 31, & 32, & . , & 34, & 35 \\ 41, & 42, & 43, & . , & 45 \\ 51, & 52, & 53, & 54, & . \end{array}\right| = 0,$$

being, as it should be, of the order 2 in regard to the coordinates of each line.

Condition that 7 *given lines shall lie on a Quartic Surface.*

107. Taking the lines to be $(a_1, b_1, c_1, f_1, g_1, h_1), \ldots (a_7, b_7, c_7, f_7, g_7, h_7)$, then in precisely the same way we form a determinant of the order ($\frac{1}{2}5 . 4 =$) 10 in regard to the coordinates of each line; this determinant however divides out by the several factors $a_1f_2 + \&c.$, which can be formed with the seven lines; or throwing these out and equating the quotient to zero, we have an equation of the order 4 in regard to the coordinates of each line. It would not be practicable to obtain the reduced equation in this manner, and I do not know how to obtain it otherwise, but the material conclusion is that the order is $= 4$.

The Jacobian of 6 *points.*

108. Any 6 points whatever may be regarded as points on a skew cubic; and the coordinates (x, y, z, w) may be taken so that the equations of the skew cubic shall be $\left\|\begin{array}{ccc} x, & y, & z \\ y, & z, & w \end{array}\right\| = 0$. This being so, the coordinates of the 6 given points may be taken to be $(1, t_1, t_1^2, t_1^3), \ldots (1, t_6, t_6^2, t_6^3)$; and the equation of the Jacobian surface of

the 6 points can then be expressed in a very simple form, putting in evidence the passage of the surface through the skew cubic; viz. writing

$$p_1=\Sigma t_1,\quad p_2=\Sigma t_1t_2,\quad p_3=\Sigma t_1t_2t_3,\quad p_4=\Sigma t_1t_2t_3t_4,\quad p_5=\Sigma t_1t_2t_3t_4t_5,\quad p_6=t_1t_2t_3t_4t_5t_6\,;$$

moreover,

$$\square=\tfrac{1}{2}(6xyzw-4xz^3-4y^3w+3y^2z^2-x^2w^2),$$

and therefore

$$\begin{aligned}
\delta_x\square &= -\ \ xw^2-2z^3\ \ +3yzw,\\
\delta_y\square &= \ \ \ 3yz^2-6y^2w+3xzw,\\
\delta_z\square &= \ \ \ 3y^2z-6xz^2+3xyz,\\
\delta_w\square &= -\ \ x^2w-2y^3\ \ +3xyz\,;
\end{aligned}$$

then the equation of the Jacobian surface is

$$\begin{aligned}
&3\,(\,xp_3 \qquad\quad + zp_1-2w\,)\,\delta_x\square\\
+&\ (\qquad\qquad\quad 2zp_2-wp_1)\,\delta_y\square\\
+&\ (\,xp_5-2yp_4 \qquad\qquad\ )\,\delta_z\square\\
+&3\,(2xp_6-\ \ yp_5 \qquad - wp_3)\,\delta_w\square=0.
\end{aligned}$$

There is not much difficulty in the direct investigation; but a simple verification may be obtained by showing that the surface contains upon it the 15 lines 12, 13,... 56. Write in the equation

$$(x,\ y,\ z,\ w)=(\lambda+\mu,\ \lambda s+\mu t,\ \lambda s^2+\mu t^2,\ \lambda s^3+\mu t^3),$$

the values $\delta_x\square$ &c. are found to contain the factor $\lambda\mu\,(s-t)^3$, and omitting this common factor the values are as

$$\tfrac{1}{3}(\lambda s^3-\mu t^3),\quad -(\lambda s^2-\mu t^2),\quad (\lambda s-\mu t),\quad -\tfrac{1}{3}(\lambda-\mu)\,;$$

the equation thus becomes

$$\begin{aligned}
&\{\lambda(-2s^3\ +\ s^2p_1+\ p_3)+\mu(-2t^3\ +\ t^2p_1+\ p_3)\}\,(\lambda s^3-\mu t^3)\\
-&\{\lambda(-\ s^3p_1+2s^2p_2\qquad)+\mu(-\ t^3p_1+2t^2p_2\qquad)\}\,(\lambda s^2-\mu t^2)\\
+&\{\lambda(-2sp_4+\quad p_5\qquad)+\mu(-2tp_4+\quad p_5\qquad)\}\,(\lambda s-\mu t\,)\\
-&\{\lambda(-\ s^3p_3-\ \ sp_5+2p_6)+\mu(-\ t^3p_3-\ \ tp_5+2p_6)\}\,(\lambda\ -\mu\ )=0,
\end{aligned}$$

viz., collecting the terms, the coefficient of $\lambda\mu$ vanishes, and the whole is

$$\begin{aligned}
&-2\lambda^2\,(1,\ p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ p_6\between s,\ -1)^6\\
&+2\mu^2\,(1,\ p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ p_6\between t,\ -1)^6=0\,;
\end{aligned}$$

viz., this equation is satisfied if s denote any one of the quantities $(t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6)$, and t any one of the same 6 quantities; that is, the equation of the surface is satisfied when $(x,\ y,\ z,\ w)$ are the coordinates of a point on the line joining any 2 of the 6 points.

Locus of the vertex of a Quadric Cone which touches each of Six given Lines.

109. Representing as before each line by means of its six coordinates, let (x, y, z, w) be the coordinates of the vertex, and (X, Y, Z, W) current coordinates. Suppose that (a, b, c, f, g, h) are the coordinates of any one of the lines, the equation of the plane through this line and the vertex is

$$\begin{aligned} a(xW - wX) + b\,(yW - wY) + c\,(zW - wZ) \\ +f\,(yZ - zY) + g(zX \;\; - xZ\;) + h(xY \;\; - yX) = 0\,; \end{aligned}$$

or, what is the same thing, writing for shortness

$$\begin{aligned} P &= \quad . \qquad hy - gz + aw, \\ Q &= -hx \quad . \quad + fz + bw, \\ R &= \quad gx - fy \quad . \quad + cw, \\ S &= -ax - by - cz \quad . \end{aligned}$$

the equation is

$$PX + QY + RZ + SW = 0.$$

The plane in question is a tangent plane to the cone touched by the 6 lines. Now when 6 planes touch a quadric cone, their traces on any plane whatever touch a conic the intersection of the cone by that plane. Hence taking the plane $W=0$, the equation of the trace is

$$PX + QY + RZ = 0,$$

and forming in like manner tne equations belonging to each of the given lines, the condition that the 6 traces may touch a conic is

$$(P^2,\ Q^2,\ R^2,\ QR,\ RP,\ PQ) = 0,$$

where the left-hand side represents a determinant of 6 lines, the several lines being respectively P_1^2, Q_1^2, R_1^2, Q_1R_1, R_1P_1, P_1Q_1, P_2^2, &c.... Or more simply we may denote the equation by

$$[(P,\ Q,\ R)^2] = 0.$$

To ascertain the form of this, write for a moment $y=0$, $z=0$; the equation is

$$[(aw,\ -hx + bw,\ gx + cw)^2] = 0,$$

or attending only to the highest and lowest powers of w, this is

$$w^{12}\,[(a,\ b,\ c)^2] \ldots + w^4x^8\,[(a,\ -h,\ g)^2] = 0\,;$$

and it is thence easy to infer that the whole equation divides by w^4; so that, omitting this factor, the form of the equation is

$$((a,\ b,\ c,\ f,\ g,\ h)^2 \between x,\ y,\ z,\ w)^8 = 0\,;$$

viz., the equation is of the order 8 in the coordinates (x, y, z, w), and of the degree 2 in the coordinates (a, b, c, f, g, h) of each of the lines. It would not be very difficult to actually develope the equation; in fact, starting from the term $w^8[(a, b, c)^2]$ the other terms are obtained therefrom by changing a, b, c into $a+\frac{1}{w}(hy-gz)$, $b+\frac{1}{w}(-hx+fz)$, $c+\frac{1}{w}(gx-fy)$ respectively; the equation may therefore be written in the symbolic form

$$w^8 \,.\, \exp.\frac{1}{w}\{(hy-gz)\,\delta_a+(-hx+fz)\,\delta_b+(gx-fy)\,\delta_c\}\,.\,[(a,\ b,\ c)^2]=0,$$

or, what is the same thing,

$$w^8 \,.\, \exp.\frac{1}{w}\{x\,(g\delta_c-h\delta_b)+y\,(h\delta_a-f\delta_c)+z\,(f\delta_b-g\delta_a)\}\,.\,[(a,\ b,\ c)^2]=0,$$

where exp. θ (read exponential) denotes e^θ, and $[(a,\ b,\ c)^2]$ represents a determinant as above explained. The equation contains, it is clear, the four terms

$$x^8[(a,\ -h,\ g)^2]+y^8[(-h,\ b,\ -f)^2]+z^8[(-g,\ f,\ c)^2]+w^8[(a,\ b,\ c)^2].$$

I am not sure whether this surface of the eighth order has been anywhere considered.