444.

ON THE CENTRO-SURFACE OF AN ELLIPSOID.

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THE President [Prof. Cayley] gave an account of his investigations on the centro-surface of an ellipsoid (locus of the centres of curvature of the ellipsoid). The surface has been studied by Dr Salmon, and also by Prof. Clebsch, but in particular the theory of the nodal curve on the surface admits of further development. The position of a point on the ellipsoid is determined by means of the parameters, or elliptic coordinates, h, k; viz., if as usual a, b, c are the semi-axes, and if X, Y, Z are the coordinates of the point in question, then

$$\frac{X^2}{a^2+h}+\frac{Y^2}{b^2+h}+\frac{Z^2}{c^2+h}=1,$$

$$\frac{X^2}{a^2+k}+\frac{Y^2}{b^2+k}+\frac{Z^2}{c^2+k}=1;$$

and hence

$$\begin{aligned}-\beta\gamma X^2 &= a^2(a^2+h)(a^2+k),\\ -\gamma\alpha Y^2 &= b^2\,(b^2+h)(b^2+k),\\ -\alpha\beta Z^2 &= c^2\,(c^2+h)(c^2+k),\end{aligned}$$

if for shortness

$$\alpha=b^2-c^2,\quad \beta=c^2-a^2,\quad \gamma=a^2-b^2,\quad (\alpha+\beta+\gamma=0).$$

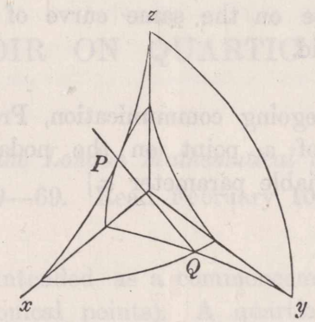
This being so, the coordinates of the point of intersection of the normal at (X, Y, Z) by the normal at the consecutive point of the curve of curvature

$$\frac{X^2}{a^2+k}+\frac{Y^2}{b^2+k}+\frac{Z^2}{c^2+k}=1$$

are given by the formulæ

$$-\beta\gamma a^2x^2 = (a^2+h)^3(a^2+k),$$
$$-\gamma\alpha b^2y^2 = (b^2+h)^3(b^2+k),$$
$$-\alpha\beta c^2z^2 = (c^2+h)^3(c^2+k);$$

viz., these equations, considering therein (h, k) as arbitrary parameters, determine the coordinates (x, y, z) of a point on the centro-surface. The principal sections (as is known) consist each of them of an ellipse counting three times, and of an evolute of an ellipse; the evolute and ellipse have four contacts (two-fold intersections) and four simple intersections, but the contacts and intersections respectively are in the different sections real and imaginary; and if (as we may without loss of generality assume) $a^2+c^2>2b^2$, then the form of the principal sections is as shown in the figure (which



represents only an octant of the surface); viz., there is a real contact at P in the plane of xz, and a real intersection at Q in the plane of xy. The surface has thus an exterior and an interior sheet, but (instead of meeting in a conical point, as in the wave surface) these intersect in a nodal curve QP. The curve has a cusp at Q, and a node at P; viz., the curve extends beyond P, but from that point is acnodal, or without any real sheet of the surface passing through it. For the nodal curve there must be two values (h, k), (h_1, k_1), giving the same values of (x, y, z); viz., there must exist the relations

$$(a^2+h)^3(a^2+k) = (a^2+h_1)^3(a^2+k_1),$$
$$(b^2+h)^3(b^2+k) = (b^2+h_1)^3(b^2+k_1),$$
$$(c^2+h)^3(c^2+k) = (c^2+h_1)^3(c^2+k_1);$$

from which equations eliminating h_1 and k_1, we should have between h, k a relation which, combined with the expressions of x, y, z in terms of (h, k), determines the nodal curve. But the better course is to eliminate k, k_1, thus obtaining a relation between h and h_1, in virtue whereof h_1 may be regarded as a known function of h; k and k_1 can then be readily expressed in terms of h, h_1; that is, we have k as a function of h, h_1, or in effect as a function of h. The relation between h, h_1 (after a reduction of some complexity) assumes ultimately a form which is very simple and remarkable; viz., writing

$$P = a^2 + b^2 + c^2, \quad Q = b^2c^2 + c^2a^2 + a^2b^2, \quad R = a^2b^2c^2,$$

the relation is

$$\begin{aligned} &(6R + 3Qh + Ph^2) \\ &+ h_1 (3Q + 4Ph + 3h^2) \\ &+ h_1^2 (P + 3h) = 0; \end{aligned}$$

this is a (2, 2) correspondence between the two parameters h, h_1; the united values $h_1 = h$, are given by the equation $6(R + Qh + Ph^2 + h^3) = 0$, that is

$$(a^2 + h)(b^2 + h)(c^2 + h) = 0;$$

viz., the two points on the ellipsoid which have their common centre of curvature on the nodal curve are only situate on the same curve of curvature when this curve is a principal section of the ellipsoid.

{Since the date of the foregoing communication, Prof. Cayley has found that the squared coordinates x^2, y^2, z^2 of a point on the nodal curve can be expressed as rational functions of a single variable parameter σ.}