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ON A CERTAIN SEXTIC TORSE.

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THE torse (developable surface) intended to be considered is that which has for its edge of regression an excubo-quartic curve, or say a *unicursal* quartic curve. I call to mind that (excluding the plane quartic) a quartic curve is either a quadriquadric, viz. it is the complete intersection of two quadric surfaces; or else it is an excubo-quartic, viz. there is through the curve only one quadric surface, and the curve is the partial intersection of this quadric surface with a cubic surface through two generating lines (of the same kind) of the quadric surface. Returning to the quadriquadric curve, this may be general, nodal, or cuspidal; viz. if the two quadric surfaces have an ordinary contact, the curve of intersection is a nodal quadriquadric; if they have a stationary contact, the curve is a cuspidal quadriquadric.

The unicursal quartic is a curve such that the coordinates (x, y, z, w) of any point thereof are proportional to rational and integral quartic functions $(*\between\theta, 1)^4$ of a variable parameter θ; and the general unicursal quartic is in fact the excubo-quartic; but included as particular cases of the unicursal curve (although not as cases of the excubo-quartic as above defined) we have the nodal quadriquadric and the cuspidal quadriquadric. The torse having for its edge of regression a unicursal curve is a sextic torse; and this is in fact the order of the torse derived from the excubo-quartic, and from the nodal quadriquadric; but for the cuspidal quadriquadric, there is a depression of *one*, and the torse becomes a quintic torse. The equations have been obtained of (1) the sextic torse derived from the nodal quadriquadric, (2) the quintic torse derived from the cuspidal quadriquadric, (3) the sextic torse derived from a certain special excubo-quartic; but the equation of the torse derived from the general unicursal quartic has not yet been found. To show at the outset what the analytical problem is, I anticipate the remark that the coordinates (x, y, z, w) of a point on the curve may by an obvious reduction be rendered proportional to the fourth powers $(\theta+\alpha)^4$, $(\theta+\beta)^4$, $(\theta+\gamma)^4$, $(\theta+\delta)^4$ in the parameter θ; this leads to an equation

$$\frac{x}{(\theta+\alpha)^2}+\frac{y}{(\theta+\beta)^2}+\frac{z}{(\theta+\gamma)^2}+\frac{w}{(\theta+\delta)^2}=0,$$

for the osculating plane at the point (x, y, z, w); or observing that this equation, when integralised, is of the form $(x, y, z, w \between \theta, 1)^6 = 0$, we see that the equation is obtained by equating to zero the discriminant of a certain sextic function in θ; the discriminant is of the order 10 in the coordinates (x, y, z, w), but it obviously contains the factor $xyzw$, or throwing this out we have an equation of the order 6, so that the torse is (as above stated) a sextic torse.

Theorem relating to Four Binary Quartics.

1. Consider the four quartics:

$$(a_1, b_1, c_1, d_1, e_1 \between x, y)^4,$$

$$(a_2, b_2, c_2, d_2, e_2 \between x, y)^4,$$

$$(a_3, b_3, c_3, d_3, e_3 \between x, y)^4,$$

$$(a_4, b_4, c_4, d_4, e_4 \between x, y)^4;$$

then if λ_1, λ_2, λ_3, λ_4 are any four quantities, these may be determined, and that in four different ways, so that

$$\lambda_1(a_1, \ldots \between x, y)^4 + \lambda_2(a_2, \ldots \between x, y)^4 + \lambda_3(a_3, \ldots \between x, y)^4 + \lambda_4(a_4, \ldots \between x, y)^4 = (\beta x + \alpha y)^4,$$

a perfect fourth power; in fact, equating the coefficients of the different powers of $(x, y)^4$, we have five equations, which determine the ratios of the unknown quantities $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; α, β: eliminating $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we find the equation

$$\begin{vmatrix} \beta^4, & \beta^3\alpha, & \beta^2\alpha^2, & \beta\alpha^3, & \alpha^4 \\ a_1, & b_1, & c_1, & d_1, & e_1 \\ a_2, & b_2, & c_2, & d_2, & e_2 \\ a_3, & b_3, & c_3, & d_3, & e_3 \\ a_4, & b_4, & c_4, & d_4, & e_4 \end{vmatrix} = 0,$$

giving four different values of the ratio $\alpha : \beta$; or, assigning at pleasure a value to α or β (say $\beta = 1$), then to each of the four sets of values of (α, β) there correspond a determinate set of values of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$; that is, we have as stated four sets of values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; α, β.

Standard Equation of the Unicursal Quartic.

2. The coordinates (x, y, z, w) being originally taken to be proportional to any four given quartic functions $(*\between\theta, 1)^4$ of the parameter θ, then forming a linear function of the coordinates, we have four sets of values of the multipliers, each reducing the function of θ to a perfect fourth power; that is, writing (X, Y, Z, W) for the linear functions of the original coordinates, and taking (X, Y, Z, W) as coordinates, it appears that the unicursal quartic may be represented by the equations

$$X : Y : Z : W = (\theta+\alpha)^4 : (\theta+\beta)^4 : (\theta+\gamma)^4 : (\theta+\delta)^4.$$

Tangent Line, and Osculating Plane of the Unicursal Quartic.

3. The equations of the tangent line at the point (θ) (that is, the point the coordinates whereof are as $(\theta+\alpha)^4 : (\theta+\beta)^4 : (\theta+\gamma)^4 : (\theta+\delta)^4$) are at once seen to be

$$\left\| \begin{matrix} X, & Y, & Z, & W \\ (\theta+\alpha)^4, & (\theta+\beta)^4, & (\theta+\gamma)^4, & (\theta+\delta)^4 \\ (\theta+\alpha)^3, & (\theta+\beta)^3, & (\theta+\gamma)^3, & (\theta+\delta)^3 \end{matrix} \right\| = 0,$$

and that of the osculating plane to be

$$\left| \begin{matrix} X, & Y, & Z, & W \\ (\theta+\alpha)^4, & (\theta+\beta)^4, & (\theta+\gamma)^4, & (\theta+\delta)^4 \\ (\theta+\alpha)^3, & (\theta+\beta)^3, & (\theta+\gamma)^3, & (\theta+\delta)^3 \\ (\theta+\alpha)^2, & (\theta+\beta)^2, & (\theta+\gamma)^2, & (\theta+\delta)^2 \end{matrix} \right| = 0.$$

Writing as in the sequel

$$\begin{aligned} a &= \beta-\gamma, & f &= \alpha-\delta, \\ b &= \gamma-\alpha, & g &= \beta-\delta, \\ c &= \alpha-\beta, & h &= \gamma-\delta, \end{aligned}$$

the equations of the tangent line become

$$\begin{matrix} \cdot & \dfrac{hY}{(\theta+\beta)^3} - \dfrac{gZ}{(\theta+\gamma)^3} + \dfrac{aZ}{(\theta+\delta)^3} = 0, \\ -\dfrac{hX}{(\theta+\alpha)^3} & \cdot \quad + \dfrac{fZ}{(\theta+\gamma)^3} + \dfrac{bZ}{(\theta+\delta)^3} = 0, \\ \dfrac{gX}{(\theta+\alpha)^3} - \dfrac{fY}{(\theta+\beta)^3} & \cdot \quad + \dfrac{cZ}{(\theta+\delta)^3} = 0, \\ -\dfrac{aX}{(\theta+\alpha)^3} - \dfrac{bY}{(\theta+\beta)^3} - \dfrac{cZ}{(\theta+\gamma)^3} & \cdot \quad = 0, \end{matrix}$$

(equivalent of course to two equations), and the equation of the osculating plane becomes

$$\frac{ahgX}{(\theta+\alpha)^2} + \frac{hbfY}{(\theta+\beta)^2} + \frac{gfcZ}{(\theta+\gamma)^2} + \frac{abcW}{(\theta+\delta)^2} = 0.$$

Modification of the foregoing notation, and final form for the Unicursal Quartic.

4. If instead of the coordinates (X, Y, Z, W) we introduce the coordinates (x, y, z, w) connected therewith by the relations

$$ahgX : hbfY : gfcZ : abcW = x : y : z : w,$$

or, what is the same thing,

$$X : Y : Z : W = bcfx : cagy : abhz : fghw,$$

then the curve is given by the equations

$$x : y : z : w = ahg(\theta+\alpha)^4 : hbf(\theta+\beta)^4 : cfg(\theta+\gamma)^4 : abc(\theta+\delta)^4.$$

The equations of the tangent line are

$$\begin{aligned}
\frac{cy}{(\theta+\beta)^3} - \frac{bz}{(\theta+\gamma)^3} + \frac{fw}{(\theta+\delta)^3} &= 0,\\
-\frac{cx}{(\theta+\alpha)^3} \quad\cdot\quad + \frac{az}{(\theta+\gamma)^3} + \frac{gw}{(\theta+\delta)^3} &= 0,\\
\frac{bx}{(\theta+\alpha)^3} - \frac{ay}{(\theta+\beta)^3} \quad\cdot\quad + \frac{hw}{(\theta+\delta)^3} &= 0,\\
-\frac{fx}{(\theta+\alpha)^3} - \frac{gy}{(\theta+\beta)^3} - \frac{hz}{(\theta+\gamma)^3} \quad\cdot\quad &= 0,
\end{aligned}$$

and the equation of the osculating plane is

$$\frac{x}{(\theta+\alpha)^2} + \frac{y}{(\theta+\beta)^2} + \frac{z}{(\theta+\gamma)^2} + \frac{w}{(\theta+\delta)^2} = 0.$$

Determination of the Sextic Torse.

5. Starting from the equation of the osculating plane written under the form

$$\begin{aligned}
&x(\theta+\beta)^2(\theta+\gamma)^2(\theta+\delta)^2\\
&+y(\theta+\gamma)^2(\theta+\delta)^2(\theta+\alpha)^2\\
&+z(\theta+\delta)^2(\theta+\alpha)^2(\theta+\beta)^2\\
&+w(\theta+\alpha)^2(\theta+\beta)^2(\theta+\gamma)^2 = 0,
\end{aligned}$$

the equation of the torse is obtained by equating to zero the discriminant of the sextic function. Writing as before

$$\begin{aligned}
a &= \beta-\gamma, & f &= \alpha-\delta,\\
b &= \gamma-\alpha, & g &= \beta-\delta,\\
c &= \alpha-\beta, & h &= \gamma-\delta,
\end{aligned}$$

equations which give

$$\begin{array}{rrrrl} & h & -g & +a & =0, \\ -h & . & +f & +b & =0, \\ g & -f & . & +c & =0, \\ -a & -b & -c & . & =0, \\ . & h\beta & -g\gamma & +a\delta & =0, \\ -h\alpha & . & +f\gamma & +b\delta & =0, \\ g\alpha & -f\beta & . & +c\delta & =0, \\ -a\alpha & -b\beta & -c\gamma & . & =0, \end{array}$$

and also

$$af+bg+ch=0,$$

then the discriminant is a function of (x, y, z, w), (a, b, c, f, g, h) of the degree 10 in (x, y, z, w) and the degree 30 in (a, b, c, f, g, h). But the equation in θ has two equal roots, or the discriminant vanishes, if any one of the quantities (x, y, z, w) is $=0$; and again, if any one of the differences $\alpha-\beta$, &c. (that is any one of the quantities a, b, c, f, g, h) is $=0$: the discriminant thus contains the factors $xyzw$ and $(abcfgh)^2$, and throwing these out, we have an equation of the form

$$\Delta=(a, b, c, f, g, h)^{18}(x, y, z, w)^6=0,$$

which is the equation of the sextic torse.

Principal Sections of the Torse.

6. Consider for instance the section by the plane $w=0$. Writing $w=0$, the equation of the osculating plane is

$$(\theta+\delta)^2[x(\theta+\beta)^2(\theta+\gamma)^2+y(\theta+\gamma)^2(\theta+\alpha)^2+z(\theta+\alpha)^2(\theta+\beta)^2]=0.$$

The discriminant of the sextic function vanishes identically in virtue of the double factor $(\theta+\delta)^2$. But omitting this factor, the equation becomes

$$x(\theta+\beta)^2(\theta+\gamma)^2+y(\theta+\gamma)^2(\theta+\alpha)^2+z(\theta+\alpha)^2(\theta+\beta)^2=0.$$

The discriminant of this quartic function of θ is a function of x, y, z, a, b, c of the degree 6 in (x, y, z) and 12 in (a, b, c); it contains however the factors xyz, $a^2b^2c^2$, and the remaining factor is of the degree 3 in (x, y, z) and 6 in (a, b, c); this remaining factor is as will presently be seen

$$=(a^2x+b^2y+c^2z)^3-27a^2b^2c^2\,xyz.$$

The last mentioned sextic equation in θ will have a triple root $\theta=-\delta$, if only the value $\theta=-\delta$ makes to vanish the factor in [ ], that is if we have

$$0=g^2h^2x+h^2f^2y+f^2g^2z.$$

The foregoing results lead to the conclusion that for $w=0$, we have

$$\Delta = (g^2h^2x + h^2f^2y + f^2g^2z)^3 [(a^2x + b^2y + c^2z)^3 - 27a^2b^2c^2\, xyz];$$

but this will appear more distinctly as follows.

7. First, as to the factor $(a^2x + b^2y + c^2z)^3 - 27a^2b^2c^2\, xyz$: writing in the equation of the osculating plane $w=0$, the equation becomes

$$\frac{x}{(\theta+\alpha)^2} + \frac{y}{(\theta+\beta)^2} + \frac{z}{(\theta+\gamma)^2} = 0,$$

which equation is therefore that of the trace of the osculating plane on the plane $w=0$; the envelope of the trace in question is *a part* of the section of the torse by the plane $w=0$. To find the equation of this envelope we must eliminate θ from the foregoing, and its derived equation

$$\frac{x}{(\theta+\alpha)^3} + \frac{y}{(\theta+\beta)^3} + \frac{z}{(\theta+\gamma)^3} = 0;$$

the two equations give

$$x : y : z = a(\theta+\alpha)^3 : b(\theta+\beta)^3 : c(\theta+\gamma)^3,$$

and thence

$$(a^2x)^{\frac{1}{3}} + (b^2y)^{\frac{1}{3}} + (c^2z)^{\frac{1}{3}} = a(\theta+\alpha) + b(\theta+\beta) + c(\theta+\gamma) = 0,$$

that is, we have

$$(a^2x)^{\frac{1}{3}} + (b^2y)^{\frac{1}{3}} + (c^2z)^{\frac{1}{3}} = 0,$$

or, what is the same thing,

$$(a^2x + b^2y + c^2z)^3 - 27a^2b^2c^2\, xyz = 0$$

for a part of the section in question.

8. I have said that the foregoing cubic is *a part* of the section; the equations

$$x : y : z : w = ahg(\theta+\alpha)^4 : bhf(\theta+\beta)^4 : cfg(\theta+\gamma)^4 : abc(\theta+\delta)^4,$$

which for $w=0$ give $\theta = -\delta$, and thence $x : y : z = af^3 : bg^3 : ch^3$, show that the last mentioned point is a four-pointic intersection of the curve with the plane $w=0$. But the curve, having four consecutive points, will have three consecutive tangents in the plane $w=0$; that is, the tangent at the point in question will present itself as a threefold factor in the equation of the torse. Writing in the equations of the tangent $w=0$, $\theta = -\delta$, we find for the equation of the tangent in question

$$\frac{x}{f^2} + \frac{y}{g^2} + \frac{z}{h^2} = 0,$$

or, what is the same thing,

$$g^2h^2x + h^2f^2y + f^2g^2z = 0.$$

Hence the section by the plane $w=0$ is made up of this line taken three times, and of the last mentioned cubic curve.

9. By symmetry, we conclude that the sections by the principal planes $x=0$, $y=0$, $z=0$, $w=0$, are each made up of a line taken three times, and of a cubic curve: viz. these are

$$\begin{array}{llll|l}
x=0, & . & b^2f^2y + c^2f^2z + b^2c^2w = 0, & & . \quad (h^2y)^{\frac{1}{3}} + (g^2z)^{\frac{1}{3}} + (a^2w)^{\frac{1}{3}} = 0, \\
y=0, & a^2g^2x & . \quad c^2g^2z + c^2a^2w = 0, & & (h^2x)^{\frac{1}{3}} \quad . \quad + (f^2z)^{\frac{1}{3}} + (b^2w)^{\frac{1}{3}} = 0, \\
z=0, & a^2h^2x + b^2h^2y & . \quad + a^2b^2w = 0, & & (g^2x)^{\frac{1}{3}} + (f^2y)^{\frac{1}{3}} \quad . \quad + (c^2w)^{\frac{1}{3}} = 0, \\
w=0, & g^2h^2x + h^2f^2y + f^2g^2z & . \quad = 0, & & (a^2x)^{\frac{1}{3}} + (b^2y)^{\frac{1}{3}} + (c^2z)^{\frac{1}{3}} \quad = 0,
\end{array}$$

where for shortness I have written the equations of the four cubics in their irrational forms respectively.

Partial Determination of the Equation.

10. As the value of Δ is known when any one of the coordinates x, y, z, w is put $=0$, we in fact know all the terms of Δ, except those which contain the factor $xyzw$, which unknown terms, as Δ is of the degree 6, are of the form $(*\between x, y, z, w)^2$.

I remark that if $(xyzw)$ is any homogeneous function $(*\between x, y, z, w)^2$, and (xyz), (xy), (x) are what $(xyzw)$ become on putting therein $(w=0)$, $(z=0,\ w=0)$, $(y=0,\ z=0,\ w=0)$ respectively, and the like for the other similar symbols, then that

$$\begin{aligned}
(xyzw) = \;& (x)+(y)+(z)+(w) \\
& -(xy)-(xz)-(xw)-(yz)-(yw)-(zw) \\
& +(xyz)+(xyw)+(xzw)+(yzw) \\
& + \text{terms multiplied by } xyzw;
\end{aligned}$$

in fact, omitting the last line, this equation on writing therein $x=0$ or $y=0$ or $z=0$ or $w=0$, becomes an identity, that is, the difference of the two sides vanishes when any one of these equations is satisfied, and such difference contains therefore the factor $xyzw$; which proves the theorem. It hence appears that the equation $\Delta=0$ of the torse is

$$\begin{aligned}
\Delta = \;& a^6g^6h^6x^6 + b^6h^6f^6y^6 + c^6f^6g^6z^6 + a^6b^6c^6w^6 \\
& -(g^2h^2x + h^2f^2y)^3(a^2x + b^2y)^3 \\
& -(h^2f^2y + f^2g^2z)^3(b^2y + c^2z)^3 \\
& -(g^2h^2x + f^2g^2z)^3(a^2x + c^2z)^3 \\
& -(a^2h^2x + a^2b^2w)^3(g^2x + c^2w)^3 \\
& -(b^2h^2y + a^2b^2w)^3(f^2y + c^2w)^3 \\
& -(c^2g^2z + c^2a^2w)^3(f^2z + b^2w)^3 \\
& +(b^2f^2y + c^2f^2z + b^2c^2w)^3[(h^2y + g^2z + a^2w)^3 - 27a^2g^2h^2yzw] \\
& +(a^2g^2x + c^2g^2z + c^2a^2w)^3[(h^2x + f^2z + b^2w)^3 - 27b^2h^2f^2zxw] \\
& +(a^2h^2x + b^2h^2y + a^2b^2w)^3[(g^2x + f^2y + c^2w)^3 - 27c^2f^2g^2xyw] \\
& +(g^2h^2x + h^2f^2y + f^2g^2z)^3[(a^2x + b^2y + c^2z)^3 - 27a^2b^2c^2xyz] \\
& + xyzw)(*\between x, y, z, w)^2,
\end{aligned}$$

where the ten coefficients of $(*\between x, y, z, w)^2$ remain to be found.

Process for the Determination of the Unknown Coefficients.

11. At a point of the cubic curve in the plane $w=0$, we have

$$x : y : z = a(\theta+\alpha)^3 : b(\theta+\beta)^3 : c(\theta+\gamma)^3;$$

and the tangent plane at this point is the osculating plane of the curve; that is, it is the plane

$$\frac{x'}{(\theta+\alpha)^2}+\frac{y'}{(\theta+\beta)^2}+\frac{z'}{(\theta+\gamma)^2}+\frac{w'}{(\theta+\delta)^2}=0$$

if for a moment (x', y', z', w') are the current coordinates of a point in the tangent plane. But the equation of the tangent plane as deduced from the equation $\Delta=0$ is

$$x'\frac{d\Delta}{dx}+y'\frac{d\Delta}{dy}+z'\frac{d\Delta}{dz}+w'\frac{d\Delta}{dw}=0,$$

where in the differential coefficients of Δ, the coordinates (x, y, z, w) are considered as having the values

$$x : y : z : w = a(\theta+\alpha)^3 : b(\theta+\beta)^3 : c(\theta+\gamma)^3 : 0.$$

Hence, with these values of (x, y, z, w), we have

$$\frac{d\Delta}{dx} : \frac{d\Delta}{dy} : \frac{d\Delta}{dz} : \frac{d\Delta}{dw} = \frac{1}{(\theta+\alpha)^2} : \frac{1}{(\theta+\beta)^2} : \frac{1}{(\theta+\gamma)^2} : \frac{1}{(\theta+\delta)^2};$$

conditions which determine the values of certain of the coefficients of $(*\between x, y, z, w)^2$, viz. the six coefficients of the terms independent of w; and when these are known the values of the remaining four coefficients are at once obtained by symmetry.

12. To develope this process, disregarding the higher powers of w, we may write

$$\Delta = \Theta + 3w\Phi + xyzw(*\between x, y, z)^2,$$

where Θ denotes the terms independent of w, $3w\Phi$ the known terms which contain the factor w, and $xyzw\,(*\between x, y, z)^2$ the unknown terms which contain this same factor; the value of $(*\between x, y, z)^2$ being clearly $=(*\between x, y, z, 0)^2$.

We have, moreover,

$$\Theta = (g^2h^2x+h^2f^2y+f^2g^2z)^3[(a^2x+b^2y+c^2z)^3-27a^2b^2c^2\,xyz],$$

and

$$\begin{aligned}\Phi = \;& f^4(b^2y+c^2z)^2[a^2f^2(h^4y^2-7h^2g^2yz+g^4z^2)(b^2y+c^2z)+b^2c^2(h^2y+g^2z)^3]\\ &+g^4(a^2x+c^2z)^2[b^2g^2(h^4x^2-7h^2f^2xz+f^4z^2)(a^2x+c^2z)+c^2a^2(h^2x+f^2z)^3]\\ &+h^4(a^2x+b^2y)^2[c^2h^2(g^4x^2-7g^2h^2xy+f^4y^2)(a^2x+b^2y)+a^2b^2(g^2x+f^2y)^3]\\ &-a^6h^4g^4(b^2g^2+c^2h^2)x^5\\ &-b^6h^4f^4(c^2h^2+a^2f^2)y^5\\ &-c^6f^4g^4(a^2f^2+b^2g^2)z^5.\end{aligned}$$

13. The equations, putting after the differentiations $w=0$, and writing for shortness $(*)$ in place of $(*\between x, y, z)^2$, become

$$\frac{d\Theta}{dx} : \frac{d\Theta}{dy} : \frac{d\Theta}{dz} : 3\Phi + xyz\,(*) = \frac{1}{(\theta+\alpha)^2} : \frac{1}{(\theta+\beta)^2} : \frac{1}{(\theta+\gamma)^2} : \frac{1}{(\theta+\delta)^2}.$$

Now, observing that the second factor of Θ vanishes for the values

$$a(\theta+\alpha)^3,\quad b(\theta+\beta)^3,\quad c(\theta+\gamma)^3 \text{ of } (x, y, z),$$

we have simply

$$\frac{d\Theta}{dx} = (g^2h^2x + h^2f^2y + f^2g^2z)^3 \,.\, 3a^2\,[(a^2x+b^2y+c^2z)^2 - 9b^2c^2yz].$$

But

$$a^2x+b^2y+c^2z = a^3(\theta+\alpha)^3 + b^3(\theta+\beta)^3 + c^3(\theta+\gamma)^3,$$
$$= 3abc\,(\theta+\alpha)(\theta+\beta)(\theta+\gamma),$$

in virtue of the relation $a(\theta+\alpha)+b(\theta+\beta)+c(\theta+\gamma)=0$ and hence

$$[(a^2x+b^2y+c^2z)^2 - 9b^2c^2yz] = 9b^2c^2(\theta+\beta)^2(\theta+\gamma)^2 \,.\, [a^2(\theta+\alpha)^2 - bc(\theta+\beta)(\theta+\gamma)],$$
$$= 9b^2c^2(\theta+\beta)^2(\theta+\gamma)^2\,Q,$$

where

$$Q = a^2(\theta+\alpha)^2 - bc\,(\theta+\beta)(\theta+\gamma),$$
$$= b^2(\theta+\beta)^2 - ca\,(\theta+\gamma)(\theta+\alpha),$$
$$= c^2(\theta+\gamma)^2 - ab\,(\theta+\alpha)(\theta+\beta).$$

Hence

$$\frac{d\Theta}{dx} = 27a^2b^2c^2\,(g^2h^2x + h^2f^2y + f^2g^2z)^3\,(\theta+\beta)^2(\theta+\gamma)^2\,Q,$$

and similarly

$$\frac{d\Theta}{dy} = 27a^2b^2c^2\,(g^2h^2x + h^2f^2y + f^2g^2z)^3\,(\theta+\gamma)^2(\theta+\alpha)^2\,Q,$$

$$\frac{d\Theta}{dz} = 27a^2b^2c^2\,(g^2h^2x + h^2f^2y + f^2g^2z)^3\,(\theta+\alpha)^2(\theta+\beta)^2\,Q;$$

whence the above-mentioned conditions reduce themselves to the single condition

$$(\theta+\delta)^2\,\{3\Phi + xyz\,(*)\} = 27a^2b^2c^2\,(g^2h^2x + h^2f^2y + f^2g^2z)^3\,(\theta+\alpha)^2(\theta+\beta)^2(\theta+\gamma)^2\,Q.$$

14. But we have

$$g^2h^2x + h^2f^2y + f^2g^2z$$
$$= g^2h^2a\,(\theta+\delta+f)^3 + h^2f^2b\,(\theta+\delta+g)^3 + f^2g^2c\,(\theta+\delta+h)^3,$$
$$= (\theta+\delta)^2\,[(g^2h^2a + h^2f^2b + f^2g^2c)(\theta+\delta) + 3\,(gha + hfb + fgc)\,fgh],$$
$$= -\,abc\,(\theta+\delta)^2\,[(gh + hf + fg)(\theta+\delta) + 3fgh],$$
$$= -\,abc\,(\theta+\delta)^2\,[gh(\theta+\alpha) + hf(\theta+\beta) + fg\,(\theta+\gamma)],$$
$$= -\,abc\,(\theta+\delta)^2\,P,$$

if for shortness

$$P = gh(\theta+\alpha) + hf(\theta+\beta) + fg(\theta+\gamma).$$

Hence, substituting,

$$3\Phi + abc(\theta+\alpha)^3(\theta+\beta)^3(\theta+\gamma)^3(*) = -27(abc)^5(\theta+\delta)^4(\theta+\alpha)^2(\theta+\beta)^2(\theta+\gamma)^2 P^3 Q;$$

which when the values $a(\theta+\alpha)^3$, $b(\theta+\beta)^3$, $c(\theta+\gamma)^3$ for (x, y, z) are substituted in the functions Φ and $(*)$, will be an identical equation in θ.

15. It is right to remark that what we require is the expression of $(*)$, $= (*\between x, y, z)^2$; the foregoing equation leads to the value of $(*)$ expressed in terms of θ; and it is necessary to show that this leads back to the expression for $(*)$ as a function of (x, y, z); in fact, that the function of θ is transformable in a *definite* manner into a function of (x, y, z). Suppose that the function of θ could be expressed in two different manners as a function of (x, y, z); then we should have two different functions $(x, y, z)^2$ each equivalent to the same function of θ; and the difference of these functions would be identically $=0$; that is, we should have a function $(x, y, z)^2$ vanishing identically by the substitution

$$x : y : z = a(\theta+\alpha)^3 : b(\theta+\beta)^3 : c(\theta+\gamma)^3;$$

but these relations are equivalent to the single relation

$$(a^2x + b^2y + c^2z)^3 - 27a^2b^2c^2\,xyz = 0,$$

which, *quà* cubic equation, is not equivalent to any equation whatever of the form

$$(x, y, z)^2 = 0;$$

that is, the function of θ is equivalent to a definite function $(x, y, z)^2$.

16. To proceed with the reduction, I remark that we have

$$\Phi = (a^2x + b^2y + c^2z)^2 \left(\begin{array}{l} f^4[a^2f^2(h^4y^2 - 7h^2g^2yz + g^4z^2)(b^2y + c^2z) + b^2c^2(h^2y + g^2z)^3] \\ + g^4[b^2g^2(h^4x^2 - 7h^2f^2xz + f^4z^2)(a^2x + c^2z) + c^2a^2(h^2x + f^2z)^3] \\ + h^4[c^2h^2(g^4x^2 - 7g^2f^2xy + f^4y^2)(a^2x + b^2y) + a^2b^2(g^2x + f^2y)^3] \\ - a^2h^4g^4(b^2g^2 + c^2h^2)x^3 \\ - b^2h^4f^4(c^2h^2 + a^2f^2)y^3 \\ - c^2f^4g^4(a^2f^2 + b^2g^2)z^3 \end{array}\right)$$

$$+ xyz\Omega$$

where

$$\begin{aligned} -\Omega = {} & 2(b^2c^2Ax^2 + c^2a^2By^2 + a^2b^2Cz^2) \\ & + (My + Nz)(a^4x + 2a^2b^2y + 2a^2c^2z) \\ & + (Oz + Px)(2a^2b^2x + b^4y + 2b^2c^2z) \\ & + (Qx + Ry)(2a^2c^2x + 2b^2c^2y + cz), \end{aligned}$$

if for shortness

$$A = a^2g^4h^4\,(b^2g^2 + c^2h^2),$$
$$B = b^2h^4f^4\,(c^2h^2 + a^2f^2),$$
$$C = c^2f^4g^4\,(a^2f^2 + b^2g^2),$$

$$M = f^4h^2\,(a^2f^2c^2h^2 - 7a^2f^2b^2g^2 + 3b^2g^2c^2h^2), \quad N = f^4g^2\,(-7a^2f^2c^2h^2 + a^2f^2b^2g^2 + 3b^2g^2c^2h^2),$$
$$O = g^4f^2\,(b^2g^2a^2f^2 - 7b^2g^2c^2h^2 + 3c^2h^2a^2f^2), \quad P = g^4h^2\,(-7b^2g^2a^2f^2 + b^2g^2c^2h^2 + 3c^2h^2a^2f^2),$$
$$Q = h^4g^2\,(c^2h^2b^2g^2 - 7c^2h^2a^2b^2 + 3a^2f^2b^2g^2), \quad R = h^4f^2\,(-7c^2h^2b^2g^2 + c^2h^2a^2f^2 + 3a^2f^2b^2g^2);$$

and I represent the foregoing equation by

$$\Phi = (a^2x + b^2y + c^2z)^2\,U + xyz\Omega.$$

Hence, writing for x, y, z the foregoing values, we have

$$\Phi = 9a^2b^2c^2\,(\theta+\alpha)^2\,(\theta+\beta)^2\,(\theta+\gamma)^2\,U + abc\,(\theta+\alpha)^3\,(\theta+\beta)^3\,(\theta+\gamma)^3\,\Omega;$$

and thence

$$27U + \frac{1}{abc}\,(\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)\,\big(3\Omega + (*)\big) = -27\,(abc)^3\,(\theta+\delta)^4\,P^3Q;$$

that is

$$27\,[U + (abc)^3\,(\theta+\delta)^4\,P^3Q] + \frac{1}{abc}\,(\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)\,\big(3\Omega + (*)\big) = 0.$$

In order that this may be the case, it is clear that we must have

$$U + (abc)^3\,(\theta+\delta)^4\,P^3Q = (\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)\,M,$$

viz. the left-hand side expressed as a function of θ must be divisible by the product $(\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)$. Assuming for a moment that this is so, the quotient M will be a function $(\theta, 1)^6$ expressible in a unique manner in the form $(x, y, z)^2$, and assuming it to be so expressed, we have

$$27Mabc + 3\Omega + (*) = 0;$$

which equation, without any further substitution of the θ-values of (x, y, z), gives $(*)$ in its proper form as a function of (x, y, z).

Reduction of the Equation, $U + (abc)^3\,(\theta+\delta)^4\,P^3\,Q = (\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)\,M$.

17. We have by an easy transformation

$$U = (a^2x + b^2y + c^2z)\left\{\begin{array}{l} a^2f^6\,(h^4y^2 - 7h^2g^2yz + g^4z^2) \\ + b^2g^6\,(f^4z^2 - 7f^2h^2zx + h^4x^2) \\ + c^2h^6\,(g^4x^2 - 7g^2f^2xy + f^4y^2) \end{array}\right\}$$
$$+ 7\,(a^4f^4 + b^4g^4 + c^4h^4)\,f^2g^2h^2\,xyz$$
$$+ U',$$

if for shortness

$$\begin{aligned} U' = \ & y^2z \,.\, c^2h^4f^4\,(3b^2g^2 - c^2h^2) \\ & + yz^2 \,.\, b^2g^4f^4\,(3c^2h^2 - b^2g^2) \\ & + z^2x \,.\, a^2f^4g^4\,(3c^2h^2 - a^2f^2) \\ & + zx^2 \,.\, c^2h^4g^4\,(3a^2f^2 - c^2h^2) \\ & + x^2y \,.\, b^2g^4h^4\,(3a^2f^2 - b^2g^2) \\ & + xy^2 \,.\, a^2f^4h^4\,(3b^2g^2 - a^2f^2). \end{aligned}$$

Substituting the θ-values, the terms of U, other than U', are at once seen to contain the factor $(\theta+\alpha)(\theta+\beta)(\theta+\gamma)$, and we have

$$\begin{aligned} M = \ & 3abc\left\{\begin{array}{l} a^2f^6\,(h^4y^2 - 7h^2g^2yz + g^4z^2) \\ + b^2g^6\,(f^4z^2 - 7f^2h^2zx + h^4x^2) \\ + c^2h^6\,(g^4x^2 - 7g^2f^2xy + f^4y^2) \end{array}\right\} \\ & + 7\,(a^4f^4 + b^4g^4 + c^4h^4)\,f^2g^2h^2\,abc\,(\theta+\alpha)^3(\theta+\beta)^2(\theta+\gamma)^2 \\ & + M', \end{aligned}$$

where

$$U' + (abc)^3(\theta+\delta)^4P^3Q = (\theta+\alpha)(\theta+\beta)(\theta+\gamma)\,M'.$$

18. Write for shortness $p, q, r = (af, bg, ch)$; after a complicated reduction, I obtain

$$\begin{aligned} 3abc\,M' = \ & a^2g^2h^2\,(r-p)(p-q)(-2p^4 + 5p^2qr - 6q^2r^2)\,x^2 \\ & + b^2h^2f^2\,(p-q)(q-r)(-2q^4 + 5q^2rp - 6q^2p^2)\,y^2 \\ & + c^2f^2g^2\,(q-r)(r-p)(-2r^4 + 5r^2pq - 6p^2q^2)\,z^2 \\ & + 2f^2g^2h^2b^2c^2\,(7p^4 - 20p^2qr + 4q^2r^2)\,yz \\ & + 2f^2g^2h^2c^2a^2\,(7q^4 - 20pq^2r + 4r^2p^4)\,zx \\ & + 2f^2g^2h^2a^2b^2\,(7r^4 - 20pqr^2 + 4p^2q^2)\,xy \\ & - 2f^2g^2h^2\,(p^4 + q^4 + r^4)(a^2x + b^2y + c^2z)^2. \end{aligned}$$

We then have

$$9abc\,M = \text{terms }(x, y, z)^2 + 9abc\,M', \quad \Omega = \text{terms }(x, y, z)$$

as above; and

$$27M\,abc + 3\Omega + (*) = 0,$$

which gives $(*)$.

19. After all reductions we find:

$$\begin{aligned} -\tfrac{1}{3}(*) = \ & a^2g^2h^2\,(28p^6 - 84p^4qr + 62p^2q^2r^2 - 28q^3r^3)\,x^2 \\ & + b^2h^2f^2\,(28q^6 - 84pq^4r + 62p^2q^2r^2 - 28r^3p^3)\,y^2 \\ & + c^2f^2g^2\,(28r^6 - 84pqr + 62p^2q^2r - 28p^3q^3)\,z^2 \\ & + f^2\,(-3p^8 + 14p^6qr - 130p^4q^2r^2 - 136p^2q^3r^3 + 42q^4r^4)\,yz \\ & + g^2\,(-3q^8 + 14pq^6r - 130p^2q^4r^2 - 136p^3q^2r^3 + 42r^4p^4)\,zx \\ & + h^2\,(-3r^8 + 14pqr^6 - 130p^2q^2r^4 - 136p^3q^3r^2 + 42p^4q^4)\,xy\,; \end{aligned}$$

or observing that the coefficients of $a^2g^2h^2x^2$, $b^2h^2f^2y^2$ and $c^2f^2g^2z^2$ are equal to each other and to

$$62p^2q^2r^2 - 28(q^3r^3 + r^3p^3 + p^3q^3),$$

the equation becomes

$$\begin{aligned}(*) = &-3(62p^2q^2r^2 - 28(q^3r^3 + r^3p^3 + p^3q^3)(a^2g^2h^2x^2 + b^2h^2f^2y^2 + c^2f^2g^2z^2)\\ &+3(3p^8 - 14p^6qr + 130p^4q^2r^2 + 136p^2q^3r^3 - 42q^4r^4)f^2yz\\ &+3(3q^8 - 14q^6pr + 130q^4p^2r^2 + 136q^2p^3r^3 - 42r^4p^4)g^2zx\\ &+3(3r^8 - 14r^6pq + 130r^4p^2q^2 + 136r^2p^3q^3 - 42p^4q^4)h^2xy;\end{aligned}$$

and we thence obtain by symmetry the complete value of $(*\between x, y, z, w)^2$, viz. we have only to complete the literal parts of the foregoing expression into the forms

$$\begin{aligned}&a^2g^2h^2x^2 + b^2h^2f^2y^2 + c^2f^2g^2z^2 + a^2b^2c^2w^2,\\ &f^2yz + a^2xw,\\ &g^2zx + b^2yw,\\ &c^2xy + a^2zw,\end{aligned}$$

respectively.

20. The equation of the torse thus is

$$\begin{aligned}\Delta = &a^6g^6h^6x^6 + b^6h^6f^6y^6 + c^6f^6g^6z^6 + a^6b^6c^6w^6\\ &-f^6(h^2y + g^2z)^3(b^2y + c^2z)^3\\ &-g^6(f^2z + h^2x)^3(c^2z + a^2x)^3\\ &-h^6(g^2x + f^2y)^3(a^2x + b^2y)^3\\ &-a^6(g^2x + c^2w)^3(h^2x + b^2w)^3\\ &-b^6(h^2y + a^2w)^3(f^2y + c^2w)^3\\ &-c^6(f^2z + b^2w)^3(g^2z + a^2w)^3\\ &+(b^2f^2y + c^2f^2z + b^2c^2w)^3[(h^2y + g^2z + a^2w)^3 - 27a^2g^2h^2yzw]\\ &+(a^2g^2x + c^2g^2z + c^2a^2w)^3[(f^2z + h^2x + b^2w)^3 - 27b^2h^2f^2zxw]\\ &+(a^2h^2x + b^2h^2y + a^2b^2w)^3[(g^2x + f^2y + c^2w)^3 - 27c^2f^2g^2xyw]\\ &+(g^2h^2x + h^2f^2y + f^2g^2z)^3[(a^2x + b^2y + c^2z)^3 - 27a^2b^2c^2xyz]\\ &+xyzw(*\between x, y, z, w)^2 = 0.\end{aligned}$$

I recall that

$$\begin{aligned}a &= \beta - \gamma, & f &= \alpha - \delta, & p &= af = (\alpha - \delta)(\beta - \gamma),\\ b &= \gamma - \alpha, & g &= \beta - \delta, & q &= bg = (\beta - \delta)(\gamma - \alpha),\\ c &= \alpha - \beta, & h &= \gamma - \delta, & r &= ch = (\gamma - \delta)(\alpha - \beta).\end{aligned}$$

Developing, we have finally the equation of the torse in the form following.

Equation of the Sextic Torse.

21. The equation is

$$
\begin{aligned}
0 = \; & (a^6g^6h^6,\ b^6h^6f^6,\ c^6f^6g^6,\ a^6b^6c^6 \between x^6,\ y^6,\ z^6,\ w^6) \\
& + 3\,(q^2+r^2)\ (b^4h^4f^6,\ c^4g^4f^6,\ g^4h^4a^6,\ b^4c^4a^6 \between y^5z,\ yz^5,\ x^5w,\ xw^5) \\
& + 3\,(r^2+p^2)\ (c^4f^4g^6,\ a^4h^4g^6,\ h^4f^4b^6,\ c^4a^4b^6 \between z^5x,\ zx^5,\ y^5w,\ yw^5) \\
& + 3\,(p^2+q^2)\ (a^4g^4h^6,\ b^4f^4h^6,\ f^4g^4c^6,\ a^4b^4c^6 \between x^5y,\ xy^5,\ z^5w,\ zx^5) \\
& + 3\,(q^4+3q^2r^2+r^4)\,(b^2h^2f^6,\ c^2g^2f^6,\ g^2h^2a^6,\ b^2c^2a^6 \between y^4z^2,\ y^2z^4,\ x^4w^2,\ x^2w^4) \\
& + 3\,(r^4+3r^2p^2+p^4)\,(c^2f^2g^6,\ a^2h^2g^6,\ h^2f^2b^6,\ c^2a^2b^6 \between z^4x^2,\ x^2z^4,\ y^4w^2,\ y^2w^4) \\
& + 3\,(p^4+3p^2q^2+q^4)\,(a^2g^2h^6,\ b^2f^2h^6,\ f^2g^2c^6,\ a^2b^2c^6 \between x^4y^2,\ x^2y^4,\ z^4w^2,\ z^2w^4) \\
& + (6p^4+9p^2q^2+9p^2r^2-21q^2r^2)\,(a^2g^4h^4,\ a^2b^4c^4,\ f^2h^4b^4,\ f^2g^4c^4 \between x^4yz,\ w^4yz,\ y^4wx,\ z^4wy) \\
& + (6q^4+9q^2r^2+9q^2p^2-21r^2p^2)\,(b^2h^4f^4,\ b^2c^4a^4,\ g^2f^4c^4,\ g^2h^4a^4 \between y^4zx,\ w^4zx,\ z^4wy,\ x^4wz) \\
& + (6r^4+9r^2p^2+9r^2q^2-21p^2q^2)\,(c^2f^4g^4,\ c^2a^4b^4,\ h^2g^4a^4,\ h^2f^4b^4 \between z^4xy,\ w^4xy,\ x^4wz,\ y^4wx) \\
& + (q^6+9q^4r^2+9q^2r^4+r^6)\ (f^6,\ a^6 \between y^3z^3,\ x^3w^3) \\
& + (r^6+9r^4p^2+9r^2p^4+p^6)\ (g^6,\ b^6 \between z^3x^3,\ y^3w^3) \\
& + (p^6+9p^4q^2+9p^2q^4+q^6)\ (h^6,\ c^6 \between x^3y^3,\ z^3w^3) \\
& + 9\,(q^4r^2+q^2r^4+r^4p^2+r^2p^4+p^4q^2+p^2q^4-14p^2q^2r^2) \\
& \qquad \times\ (f^2g^2h^2,\ f^2b^2c^2,\ g^2c^2a^2,\ h^2a^2b^2 \between x^2y^2z^2,\ y^2z^2w^2,\ z^2x^2w^2,\ x^2y^2w^2) \\
& + 3\,\{p^6+3p^4(2q^2+r^2)+3p^2(q^4-7q^2r^2)+q^4r^2\} \\
& \qquad \times\ (g^2h^4,\ h^4b^2,\ g^2c^4,\ c^4b^2 \between x^3y^2z,\ y^3wx^2,\ z^3w^2x,\ w^3yz^2) \\
& + 3\,\{q^6+3q^4(2r^2+p^2)+3q^2(r^4-7r^2p^2)+r^4p^2\} \\
& \qquad \times\ (h^2f^4,\ f^4c^2,\ h^2a^4,\ a^4c^2 \between xy^3z^2,\ z^3wy^2,\ x^3w^2y,\ w^3zx^2) \\
& + 3\,\{r^6+3r^4(2p^2+q^2)+3r^2(p^4-7p^2q^2)+p^4q^2\} \\
& \qquad \times\ (f^2g^4,\ g^4a^2,\ f^2b,\ b^4a^2 \between x^2yz^3,\ x^3wz^2,\ y^3w^2z,\ w^3xy^2) \\
& + 3\,\{p^6+3p^4(2r^2+q^2)+3p^2(r^4-7q^2r^2)+q^2r^4\} \\
& \qquad \times\ (g^4h^2,\ h^2b^4,\ g^4c^2,\ c^2b^4 \between x^3yz^2,\ y^3w^2x,\ z^3wx^2,\ w^3y^2z) \\
& + 3\,\{q^6+3q^4(2p^2+r^2)+3q^2(p^4-7r^2p^2)+r^2p^4\} \\
& \qquad \times\ (h^4f^2,\ f^2c^4,\ h^4a^2,\ a^2c^4 \between x^2y^3z,\ z^3wy^2,\ x^3wy^2,\ w^3z^2x) \\
& + 3\,\{r^6+3r^4(2q^3+p^2)+3r^2(q^4-7p^2q^2)+p^2q^4\} \\
& \qquad \times\ (f^4g^2,\ g^2a^4,\ f^4b^2,\ b^2a^4 \between xy^2z^3,\ x^3w^2z,\ y^3wz^2,\ w^3x^2y) \\
& + xyzw \left\{ \begin{array}{l}
-3\,\{62p^2q^2r^2-28(q^3r^3+r^3p^3+p^3q^3)\}\,(a^2g^2h^2,\ b^2h^2f^2,\ c^2f^2g^2,\ a^2b^2c^2 \between x^2,\ y^2,\ z^2,\ w^2) \\
+3\,(3p^8-14p^6qr+130p^4q^2r^3+136p^2q^3r^3-42q^4r^4)\ (f^2,\ a^2 \between yz,\ xw) \\
+3\,(3q^8-14q^6rp+130q^4r^2p^2+136q^2r^3p^3-42r^4p^4)\ (g^2,\ b^2 \between zx,\ yw) \\
+3\,(3r^8-14r^6pq+130r^4p^2q^2+136r^2p^3q^3-42p^4q^4)\ (h^2,\ c^2 \between xy,\ zw)
\end{array} \right\}.
\end{aligned}
$$

Comparison with the Equation of the Centro-surface of an Ellipsoid.

22. In the Equation

$$\frac{x}{(\theta+\alpha)^2}+\frac{y}{(\theta+\beta)^2}+\frac{z}{(\theta+\gamma)^2}+\frac{w}{(\theta+\delta)^2}=0,$$

for x, y, z, w write $\xi^2, \eta^2, \zeta^2, \omega^2$, and then $\delta=\infty$, the equation is converted into

$$\frac{\xi^2}{(\theta+\alpha)^2}+\frac{\eta^2}{(\theta+\beta)^2}+\frac{\zeta^2}{(\theta+\gamma)^2}+\omega^2=0\,;$$

or writing a^2, b^2, c^2 for α, β, γ, and understanding $\xi^2, \eta^2, \zeta^2, \omega^2$ to mean a^2x^2, b^2y^2, c^2z^2, -1, this is

$$\frac{a^2x^2}{(\theta+a^2)^2}+\frac{b^2y^2}{(\theta+b^2)^2}+\frac{c^2z^2}{(\theta+c^2)^2}-1=0.$$

This is an equation, the envelope of which in regard to the variable parameter θ, gives the surface which is the locus of the centres of curvature of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, or say the Centro-surface of the Ellipsoid. (Salmon's *Solid Geometry*, Ed. 2, p. 400, [Ed. 4, p. 465].)

Making the same substitution in the foregoing equation $(*\between x, y, z, w)^6=0$, the quantities f, g, h become equal to $-\delta$, and p, q, r to $-a\delta, -b\delta, -c\delta$ respectively, and the whole equation divides by δ^{12}; throwing out this factor, we have a result which is obtained more simply by changing

$$x,\ y,\ z,\ w,\ a,\ b,\ c,\ f,\ g,\ h,\ p,\ q,\ r,$$

into

$$\xi^2,\ \eta^2,\ \zeta^2,\ \omega^2,\ \alpha,\ \beta,\ \gamma,\ 1,\ 1,\ 1,\ \alpha,\ \beta,\ \gamma,$$

where α, β, γ now signify b^2-c^2, c^2-a^2, a^2-b^2 respectively, and $\xi^2, \eta^2, \zeta^2, \omega^2$ are retained as standing for a^2x^2, b^2y^2, c^2z^2, -1 respectively; viz. the equation of the centro-surface is found to be

$$\begin{aligned}
0 = \quad & (\alpha^6,\ \beta^6,\ \gamma^6,\ \alpha^6\beta^6\gamma^6\between\xi^{12},\ \eta^{12},\ \zeta^{12},\ \omega^{12}) \\
+3(\beta^2+\gamma^2) \quad & (\beta^4,\ \gamma^2,\ \alpha^6,\ \alpha^6\beta^4\gamma^4\between\eta^{10}\zeta^2,\ \eta^2\zeta^{10},\ \xi^{10}\omega^2,\ \xi^2\omega^{10}) \\
+3(\gamma^2+\alpha^2) \quad & (\gamma^4,\ \alpha^2,\ \beta^6,\ \beta^6\gamma^4\alpha^4\between\zeta^{10}\xi^2,\ \zeta^2\xi^{10},\ \eta^{10}\omega^2,\ \eta^2\omega^{10}) \\
+3(\alpha^2+\beta^2) \quad & (\alpha^4,\ \beta^2,\ \gamma^6,\ \gamma^6\alpha^4\beta^4\between\xi^{10}\eta^2,\ \xi^2\eta^{10},\ \zeta^{10}\omega^2,\ \zeta^2\omega^{10}) \\
+3(\beta^4+3\beta^2\gamma^2+\gamma^4) \quad & (\beta^2,\ \gamma^2,\ \alpha^6,\ \alpha^6\beta^2\gamma^2\between\eta^8\zeta^4,\ \eta^4\zeta^8,\ \xi^8\omega^4,\ \xi^4\omega^8) \\
+3(\gamma^4+3\gamma^2\alpha^2+\alpha^4) \quad & (\gamma^2,\ \alpha^2,\ \beta^6,\ \beta^6\gamma^2\alpha^2\between\zeta^8\xi^4,\ \zeta^4\xi^8,\ \eta^8\omega^4,\ \eta^4\omega^8) \\
+3(\alpha^4+3\alpha^2\beta+\beta^4) \quad & (\alpha^2,\ \beta^2,\ \gamma^6,\ \gamma^6\alpha^2\beta^2\between\xi^8\eta^4,\ \xi^4\eta^8,\ \zeta^8\omega^4,\ \zeta^4\omega^8) \\
+3(2\alpha^4+3\alpha^2\beta^2+3\alpha^2\gamma^2-7\beta^2\gamma^2) \quad & (\alpha^2,\ \alpha^2\beta^4\gamma^4,\ \beta^4,\ \gamma^4\between\xi^8\eta^2\zeta^2,\ \omega^8\eta^2\zeta^2,\ \eta^8\omega^2\xi^2,\ \zeta^8\omega^2\eta^2) \\
+3(2\beta^4+3\beta^2\gamma^2+3\beta^2\alpha^2-7\gamma^2\alpha^2) \quad & (\beta^2,\ \beta^2\gamma^4\alpha^4,\ \gamma^4,\ \alpha^4\between\eta^8\zeta^2\xi^2,\ \omega^8\zeta^2\xi^2,\ \zeta^8\omega^2\eta^2,\ \xi^8\omega^2\zeta^2) \\
+3(2\gamma^4+3\gamma^2\alpha^2+3\gamma^2\beta^2-7\alpha^2\beta^2) \quad & (\gamma^2,\ \gamma^2\alpha^4\beta^4,\ \alpha^4,\ \beta^4\between\zeta^8\xi^2\eta^2,\ \omega^8\xi^2\eta^2,\ \xi^8\omega^2\zeta^2,\ \eta^8\omega^2\xi^2)
\end{aligned}$$

$$+ (\beta^6 + 9\beta^4\gamma^2 + 9\beta^2\gamma^4 + \gamma^6) \quad (1,\ \alpha^6 \big)\!\big(\eta^3\zeta^3,\ \xi^3\omega^3)$$
$$+ (\gamma^6 + 9\gamma^4\alpha^2 + 9\gamma^2\alpha^4 + \alpha^6) \quad (1,\ \beta^6 \big)\!\big(\zeta^3\xi^3,\ \eta^3\omega^3)$$
$$+ (\alpha^6 + 9\alpha^4\beta^2 + 9\alpha^2\beta^4 + \beta^6) \quad (1,\ \gamma^6 \big)\!\big(\xi^3\eta^3,\ \zeta^3\omega^3)$$

$$+ 3\{\alpha^6 + 3\alpha^4(2\beta^2+\gamma^2) + 3\alpha^2(\beta^4 - 7\beta^2\gamma^2) + \beta^4\gamma^2\}(1,\ \beta^2,\ \gamma^4,\ \beta^2\gamma^4 \big)\!\big(\xi^6\eta^4\zeta^2,\ \eta^6\omega^2\xi^4,\ \zeta^6\omega^4\xi^2,\ \omega^6\eta^2\zeta^4)$$
$$+ 3\{\beta^6 + 3\beta^4(2\gamma^2+\alpha^2) + 3\beta^2(\gamma^4 - 7\gamma^2\alpha^2) + \gamma^4\alpha^2\}(1,\ \gamma^2,\ \alpha^4,\ \gamma^2\alpha^4 \big)\!\big(\eta^6\zeta^4\xi^2,\ \zeta^6\omega^2\eta^4,\ \xi^6\omega^4\eta^2,\ \omega^6\zeta^2\xi^4)$$
$$+ 3\{\gamma^6 + 3\gamma^4(2\alpha^2+\beta^2) + 3\gamma^2(\alpha^4 - 7\alpha^2\beta^2) + \alpha^4\beta^2\}(1,\ \alpha^2,\ \beta^4,\ \alpha^2\beta^4 \big)\!\big(\zeta^6\xi^4\eta^2,\ \xi^6\omega^2\zeta^4,\ \eta^6\omega^4\zeta^2,\ \omega^6\xi^2\eta^4)$$
$$+ 3\{\alpha^6 + 3\alpha^4(2\gamma^2+\beta^2) + 3\alpha^2(\gamma^4 - 7\beta^2\gamma^2) + \beta^2\gamma^4\}(1,\ \gamma^2,\ \beta^4,\ \beta^4\gamma^2 \big)\!\big(\xi^6\eta^2\zeta^4,\ \zeta^6\omega^2\xi^4,\ \eta^6\omega^4\xi^2,\ \omega^6\eta^4\zeta^2)$$
$$+ 3\{\beta^6 + 3\beta^4(2\alpha^2+\gamma^2) + 3\beta^2(\alpha^4 - 7\gamma^2\alpha^2) + \gamma^2\alpha^4\}(1,\ \alpha^2,\ \gamma^4,\ \gamma^4\alpha^2 \big)\!\big(\eta^6\zeta^2\xi^4,\ \xi^6\omega^2\eta^4,\ \zeta^6\omega^4\eta^2,\ \omega^6\zeta^4\xi^2)$$
$$+ 3\{\gamma^6 + 3\gamma^4(2\beta^2+\alpha^2) + 3\gamma^2(\beta^4 - 7\alpha^2\beta^2) + \alpha^2\beta^4\}(1,\ \beta^2,\ \alpha^4,\ \alpha^4\beta^2 \big)\!\big(\zeta^6\xi^2\eta^4,\ \eta^6\omega^2\zeta^4,\ \xi^6\omega^4\zeta^2,\ \omega^6\xi^4\eta^2)$$
$$+ 9(\beta^4\gamma^2 + \beta^2\gamma^4 + \gamma^4\alpha^2 + \gamma^2\alpha^4 + \alpha^4\beta^2 + \alpha^2\beta^4 - 14\alpha^2\beta^2\gamma^2)$$
$$(1,\ \beta^2\gamma^2,\ \gamma^2\alpha^2,\ \alpha^2\beta^2 \big)\!\big(\xi^4\eta^4\zeta^4,\ \eta^4\zeta^4\omega^4,\ \zeta^4\xi^4\omega^4,\ \xi^4\eta^4\omega^4)$$

$$+ \xi^2\eta^2\zeta^2\omega^2 \left\{ \begin{array}{l} -3\{62\alpha^2\beta^2\gamma^2 - 28(\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3)\}(\alpha^2,\ \beta^2,\ \gamma^2,\ \alpha^2\beta^2\gamma^2 \big)\!\big(\xi^4,\ \eta^4,\ \zeta^4,\ \omega^4) \\ +3(3\alpha^8 - 14\alpha^6\beta\gamma + 130\alpha^4\beta^2\gamma^2 + 136\alpha^2\beta^3\gamma^3 - 42\beta^4\gamma^4)\ (1,\ \alpha^2 \big)\!\big(\eta^2\zeta^2,\ \xi^2\omega^2) \\ +3(3\beta^8 - 14\beta^6\gamma\alpha + 130\beta^4\gamma^2\alpha^2 + 136\beta^2\gamma^3\alpha^3 - 42\gamma^4\alpha^4)\ (1,\ \beta^2 \big)\!\big(\zeta^2\xi^2,\ \eta^2\omega^2) \\ +3(3\gamma^8 - 14\gamma^6\alpha\beta + 130\gamma^4\alpha^2\beta^2 + 136\gamma^2\alpha^3\beta^3 - 42\alpha^4\beta^4)\ (1,\ \gamma^2 \big)\!\big(\xi^2\eta^2,\ \zeta^2\omega^2) \end{array} \right\}.$$

This agrees with the result given in Salmon's *Solid Geometry*, Ed. 2, p. 151, [Ed. 4, p. 178], and *Quarterly Mathematical Journal*, vol. II. p. 220 (1858); in the latter place, however, the term

$$\beta^2\eta^8\zeta^4 + \gamma^2\eta^4\zeta^8 + \alpha^6\xi^8\omega^4 + \alpha^6\beta^2\gamma^2\xi^4\omega^8$$

is by mistake written

$$\beta^2\eta^8\zeta^4 + \gamma^2\eta^4\zeta^8 + \alpha^6\xi^8\omega^4 + \quad \beta^2\gamma^2\xi^4\omega^8;$$

viz. a factor α^6 is omitted in one of the coefficients.

Some of the coefficients are presented under slightly different forms; viz. instead of

$$62\alpha^2\beta^2\gamma^2 - 28(\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3)$$

Salmon has

$$14(\beta^4\gamma^2 + \beta^2\gamma^4 + \gamma^4\alpha^2 + \gamma^2\alpha^4 + \alpha^4\beta^2 + \alpha^2\beta^4) + 20\alpha^2\beta^2\gamma^2;$$

and instead of

$$3\alpha^8 - 14\alpha^6\beta\gamma + 130\alpha^4\beta^2\gamma^2 + 136\alpha^2\beta^3\gamma^3 - 42\beta^4\gamma^4,$$

he has

$$-4\alpha^8 + 7\alpha^6(\beta^2+\gamma^2) + 196\alpha^4\beta^2\gamma^2 - 68\alpha^2\beta^2\gamma^2(\beta^2+\gamma^2) - 42\beta^4\gamma^4,$$

but these different forms are respectively equivalent in virtue of the relation

$$\alpha + \beta + \gamma = 0.$$