424.

ON THE ATTRACTION OF A TERMINATED STRAIGHT LINE.

[From the *Philosophical Magazine*, vol. XLI. (1871), pp. 358—360.]

WRITE for shortness $(a, b, c; \epsilon)$ to denote the shell included between the ellipsoids

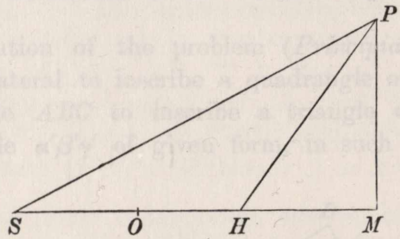
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=(1+\epsilon)^2$$

(where ϵ is indefinitely small); then, if the ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ and } \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1$$

are confocal, the attractions of the shells $(a, b, c; \epsilon)$ and $(a', b', c'; \epsilon)$ upon any exterior point P are proportional to their masses. Hence, considering a prolate spheroid of revolution, $c=b$, the attractions of the shell $(a, b, b; \epsilon)$ will be proportional to those of the shell $(\sqrt{a^2-h}, \sqrt{b^2-h}, \sqrt{b^2-h}; \epsilon)$; or if, as usual, $b^2=a^2(1-e^2)$, then, if h increases and becomes ultimately equal to b^2, to those of the shell $(ae, 0, 0; \epsilon)$; viz. this last is the portion of the axis of x included between the limits $x=-ae$, $x=+ae$; or say it is the terminated line $x=\pm ae$; and I say that the mass is distributed over this line *uniformly*.

To see that this is so, observe in general that, in the spheroid $\frac{x^2}{a'^2}+\frac{y^2+z^2}{b'^2}=1$, the volume included between the planes $x=\alpha$, $x=\alpha+d\alpha$, is $=(y^2+z^2)\,d\alpha$, $=\pi\left(b'^2-\frac{b'^2}{a'^2}\alpha^2\right)d\alpha$; and thence, writing $a'(1+\epsilon)$, $b'(1+\epsilon)$ for a', b', in the shell $(a', b', b'; \epsilon)$ the volume included between the planes $x=\alpha$, $x=\alpha+d\alpha$ is $=\pi b'^2 . 2\epsilon d\alpha$; viz. this is independent of α, and simply proportional to $d\alpha$. Hence, writing $b'=0$, when the shell shrinks up into a line, the mass must be disturbed uniformly over the line. It follows that for a line of uniform density the equipotential surfaces are each of them a prolate spheroid of revolution having the extremities of the line for its foci, and that, if we have a shell bounded by any such surface and the consecutive *similar* surface, with its mass equal to that of the line, then such shell and the line will exert the same



attractions upon any point P exterior to the shell. The attractions of the line are obtained most easily by means of its potential; viz. taking S, H for the extremities of the line, and, as above, the origin at the middle point, and the axis of x in the direction of the line, and writing $2ae$ for the length of the line, x, y, z for the coordinates of P, and r, s for the values of HP, SP (that is, $r=\sqrt{(x-ae)^2+y^2+z^2}$, $s=\sqrt{(x+ae)^2+y^2+z^2}$), then the potential is at once found to be

$$V=\log\frac{x+ae+s}{x-ae+r};$$

and we can hereby verify that the equipotential surface is in fact a spheroid of revolution having the foci S, H; for, taking the equation of such a spheroid to be

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{a^2(1-e^2)}=1,$$

(a is an arbitrary parameter, since only the value of ae has been defined), we have

$$s=a+ex, \quad r=a-ex$$

and thence

$$x+ae+s=(1+e)(x+a),$$

$$x-ae+r=(1-e)(x+a),$$

and the quotient is $=\dfrac{1+e}{1-e}$, a constant value, as it should be. The equation $V=$ const. may in fact be written

$$\frac{1+e}{1-e}=\frac{x+ae+s}{x-ae+r};$$

viz. this equation, apparently of the fourth order, breaks up into the twofold plane $y^2=0$, and the spheroid $\dfrac{x^2}{a^2}+\dfrac{y^2+z^2}{a^2(1-e^2)}=1$.

The foregoing results in regard to the attraction of a line are not new. See Green's *Essay on Electricity*, 1828, and *Collected Works*, Cambridge, 1871, p. 68; also

Joachimsthal, "On the Attraction of a Straight Line," with Sir W. Thomson's Note, *Camb. and Dubl. Math. Journ.*, vol. III. (1848), p. 93; but it does not appear to have been noticed that they are, in fact, included in the theory of the attraction of ellipsoids.

The like considerations show that the attractions of the ellipsoidal shell $(a, b, c; \epsilon)$ upon an exterior point are equal to those of an elliptic disk $z=0$, $\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1$, the mass of which is equal to that of the shell, and which has the density at the point (x, y) proportional to $\left(1-\frac{x^2}{a^2-c^2}-\frac{y^2}{b^2-c^2}\right)^{-\frac{1}{2}}$.

Sir W. Thomson informs me that the foregoing results have long been familiar to him.