

The like considerations show that the attractions of the ellipsoidal shell  $(a, b, c; \epsilon)$  upon an exterior point are equal to those of an elliptic disk  $x=0, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ , the mass of which is equal to that of the shell, and which has the density at any point  $(x, y)$  proportional to  $(\frac{x}{a^2} - \frac{y}{b^2})$ .

424.

### ON THE ATTRACTION OF A TERMINATED STRAIGHT LINE.

[From the *Philosophical Magazine*, vol. XLI, (1871), pp. 358—360.]

WRITE for shortness  $(a, b, c; \epsilon)$  to denote the shell included between the ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = (1 + \epsilon)^2$$

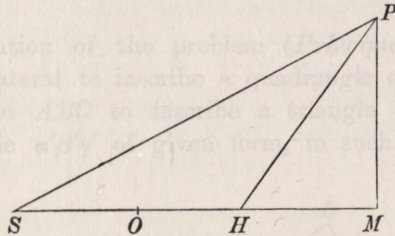
(where  $\epsilon$  is indefinitely small); then, if the ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1$$

are confocal, the attractions of the shells  $(a, b, c; \epsilon)$  and  $(a', b', c'; \epsilon)$  upon any exterior point  $P$  are proportional to their masses. Hence, considering a prolate spheroid of revolution,  $c=b$ , the attractions of the shell  $(a, b, b; \epsilon)$  will be proportional to those of the shell  $(\sqrt{a^2-h}, \sqrt{b^2-h}, \sqrt{b^2-h}; \epsilon)$ ; or if, as usual,  $b^2 = a^2(1-e^2)$ , then, if  $h$  increases and becomes ultimately equal to  $b^2$ , to those of the shell  $(ae, 0, 0; \epsilon)$ ; viz. this last is the portion of the axis of  $x$  included between the limits  $x = -ae, x = +ae$ ; or say it is the terminated line  $x = \pm ae$ ; and I say that the mass is distributed over this line *uniformly*.

To see that this is so, observe in general that, in the spheroid  $\frac{x^2}{a'^2} + \frac{y^2+z^2}{b'^2} = 1$ , the volume included between the planes  $x = \alpha, x = \alpha + d\alpha$ , is  $= (y^2 + z^2) d\alpha = \pi \left( b'^2 - \frac{b'^2}{a'^2} \alpha^2 \right) d\alpha$ ; and thence, writing  $a'(1+\epsilon), b'(1+\epsilon)$  for  $a', b'$ , in the shell  $(a', b', b'; \epsilon)$  the volume included between the planes  $x = \alpha, x = \alpha + d\alpha$  is  $= \pi b'^2 \cdot 2\epsilon' d\alpha$ ; viz. this is independent of  $\alpha$ , and simply proportional to  $d\alpha$ . Hence, writing  $b' = 0$ , when the shell shrinks up into a line, the mass must be distributed uniformly over the line. It follows that for a line of uniform density the equipotential surfaces are each of them a prolate

spheroid of revolution having the extremities of the line for its foci, and that, if we have a shell bounded by any such surface and the consecutive *similar* surface, with its mass equal to that of the line, then such shell and the line will exert the same



attractions upon any point  $P$  exterior to the shell. The attractions of the line are obtained most easily by means of its potential; viz. taking  $S, H$  for the extremities of the line, and, as above, the origin at the middle point, and the axis of  $x$  in the direction of the line, and writing  $2ae$  for the length of the line,  $x, y, z$  for the coordinates of  $P$ , and  $r, s$  for the values of  $HP, SP$  (that is,  $r = \sqrt{(x - ae)^2 + y^2 + z^2}$ ,  $s = \sqrt{(x + ae)^2 + y^2 + z^2}$ ), then the potential is at once found to be

$$V = \log \frac{x + ae + s}{x - ae + r};$$

and we can hereby verify that the equipotential surface is in fact a spheroid of revolution having the foci  $S, H$ ; for, taking the equation of such a spheroid to be

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{a^2(1 - e^2)} = 1,$$

( $a$  is an arbitrary parameter, since only the value of  $ae$  has been defined), we have

$$s = a + ex, \quad r = a - ex$$

and thence

$$x + ae + s = (1 + e)(x + a),$$

$$x - ae + r = (1 - e)(x + a),$$

and the quotient is  $= \frac{1 + e}{1 - e}$ , a constant value, as it should be. The equation  $V = \text{const.}$  may in fact be written

$$\frac{1 + e}{1 - e} = \frac{x + ae + s}{x - ae + r};$$

viz. this equation, apparently of the fourth order, breaks up into the twofold plane  $y^2 = 0$ , and the spheroid  $\frac{x^2}{a^2} + \frac{y^2 + z^2}{a^2(1 - e^2)} = 1$ .

The foregoing results in regard to the attraction of a line are not new. See Green's *Essay on Electricity*, 1828, and *Collected Works*, Cambridge, 1871, p. 68; also



Joachimsthal, "On the Attraction of a Straight Line," with Sir W. Thomson's Note, *Camb. and Dubl. Math. Journ.*, vol. III. (1848), p. 93; but it does not appear to have been noticed that they are, in fact, included in the theory of the attraction of ellipsoids.

The like considerations show that the attractions of the ellipsoidal shell ( $a, b, c; \epsilon$ ) upon an exterior point are equal to those of an elliptic disk  $z=0, \frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1$ , the mass of which is equal to that of the shell, and which has the density at the point  $(x, y)$  proportional to  $\left(1 - \frac{x^2}{a^2-c^2} - \frac{y^2}{b^2-c^2}\right)^{-\frac{1}{2}}$ .

Sir W. Thomson informs me that the foregoing results have long been familiar to him.

NOTE ON THE GEODESIC LINES ON AN ELLIPSOID.

U. U. (1870) the Ellipsoidal Attraction, vol. XII. (1878), pp. 231-255; and in the same work to which I have referred a note on the attraction of a straight line, vol. XIV. (1880), pp. 1-10. The general computation of the geodesic lines on an ellipsoid is established by means of the known theorem on the immediate properties of Jacobian fundamental forms, but which was first given by Mr. Michael Roberts, *Geometriae Axiomata*, vol. XXI. p. 1476 (Dec. 1875); that every geodesic line touches a curve of curvature; that, in attending to the two opposite ovals which constitute the curve of curvature, the geodesic line is in general an infinite curve including between these opposite ovals, and so touching each of them an infinite number of times (but possibly in particular cases it is a recurrent curve touching each oval a finite number of times). The geodesic lines thus divide themselves into two kinds according as they touch a curve of curvature of the one or the other kind; and there is besides a third limiting kind, the lines which pass through an umbilic; any such geodesic line passes through the opposite umbilics, and is in general an infinite curve passing an infinite number of times, alternately through the two umbilics; but possibly it is in particular cases a recurrent curve passing a finite number of times through the two umbilics. I annex a figure giving a general idea of the configuration of the geodesic lines drawn in different directions from a given point P on the surface of the ellipsoid: this is drawn (as it were) on the plane of the greatest and least axes; but it is not a perspective or geometrical representation of any kind, but a mere diagram for the purpose in question. We have A, A', B, C, O the extremities of the axes; U, U', U'', the umbilics; P the point on the surface; U'P and U''P the curves of curvature through P; six lines are drawn