419.

A THEOREM ON DIFFERENTIAL OPERATORS.

[From a *paper by* PROF. SYLVESTER, "*Note on the Test Operators which occur in the Calculus of Invariants, &c.*," *Philosophical Magazine*, vol. XXXII. (1866), pp. 461—472, see p. 471.]

THE paper concludes with an Observation from Professor Cayley as follows:

"In the case of two variables, if

$$P_1 = (ax+by)\frac{d}{dx} + (cx+dy)\frac{d}{dy},$$

then in the notation of matrices,

$$P_1 = \begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix} (x,\ y)\left(\frac{d}{dx},\ \frac{d}{dy}\right),$$

$$P_2 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^2 (x,\ y)\left(\frac{d}{dx},\ \frac{d}{dy}\right),$$

$$P_3 = \tfrac{1}{6}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x,\ y)\left(\frac{d}{dx},\ \frac{d}{dy}\right);$$

whence also

$$P * P_2 = P_2 * P_1 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x,\ y)\left(\frac{d}{dx},\ \frac{d}{dy}\right) = 3P_3,$$

which accords with your theorem,

$$E_1 * E_2 * = E_2 * E_1 * = E_1 E_2 * + 3E_3 *."$$

I have taken the liberty of writing in the above $\frac{d}{dx}$, $\frac{d}{dy}$ for δ_x, δ_y, and P for δ in the original. It will be useful to bear in mind that in any operator such as E_1* or E_2*, *the asterisk forms an integral part* of the symbol. Thus $E_1 * E_2 *$, if we choose, may be written under the form of E_1* multiplied by E_2*, i.e. $(E_1*)\times(E_2*)$, where the cross is the sign of ordinary algebraical multiplication.