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ON THE LOCUS OF THE FOCI OF THE CONICS WHICH PASS THROUGH FOUR GIVEN POINTS.

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THE curve which is the locus of the foci of the conics which pass through four given points is, as appears from a general theorem of M. Chasles, a sextic curve having a double point at each of the circular points at infinity; and Professor Sylvester, in his "Supplemental Note on the Analogues in Space to the Cartesian Ovals *in plano*" (*Phil. Mag.*, May 1866), has further remarked that the lines (eight in all) joining the circular points at infinity with any one of the four points are all of them double tangents of the curve; whence each of these points is a focus (more accurately a quadruple focus) of the curve. It is to be added that, besides the circular points at infinity, the curve has 6 double points (3 of these are the centres of the quadrangles formed by the 4 points), in all 8 double points; the class is therefore $=14$. Hence also the number of tangents to the curve from a circular point at infinity is $=10$; viz. these are the 4 double tangents each reckoned twice, and 2 single tangents; and the theoretical number of foci is $=100$; viz. we have

16 quadruple foci, or intersections of a double tangent by a double tangent . .	$16 \times 4 = 64$
16 double foci, or intersections of a double tangent by a single tangent . .	$16 \times 2 = 32$
4 single foci, or intersections of a single tangent by a single tangent . . .	$4 \times 1 = 4$
	$\overline{100}$

To verify the foregoing results, consider any two given points I, J, and the series of conics which pass through four given points A, B, C, D; we have thus a curve the locus of the intersections of the tangents from I and the tangents from J to any conic of the series; which curve, if I, J are the circular points at infinity, is the required curve of foci. Taking $U+\lambda V=0$ for the equation of a conic of the series, the pair of tangents from I is given by an equation of the form

$$(\lambda, 1)^2 (x, y, z)^2 = 0,$$

and the pair of tangents from J by an equation of the like form

$$(\lambda, 1)^2 (x, y, z)^2 = 0;$$

and by eliminating λ from these equations, we obtain the equation of the required curve. This in the first instance presents itself as an equation of the eighth order; but it is to be observed that in the series of conics there are two conics each of them touching the line IJ, and that, considering the tangents drawn to either of these conics, the line IJ presents itself as part of the locus; that is, the line IJ twice repeated is part of the locus; and the residual curve is thus of the order $8-2, =6$; that is, the required curve is of the order 6. The consideration of the same two conics shows that each of the points I, J is a double point on the curve. Moreover, by taking for the conic any one of the line-pairs through the four points, it appears that each of the points $(AB\,.\,CD)$, $(AC\,.\,BD)$, $(AD\,.\,BC)$ is a double point on the curve: this establishes the existence of five double points. The two conics of the series which touch the line IA are a single conic taken twice, and the consideration of this conic shows that the line IA is a double tangent to the curve; similarly each of the eight lines $I(A, B, C, D)$ and $J(A, B, C, D)$ is a double tangent to the curve. Instead of seeking to establish directly the existence of the remaining three double points, the easier course is to show that, besides the four double tangents from I, the number of tangents from I to the curve is $=2$; for, this being so, the total number of tangents from I to the curve will be $(2\times 4+2=)\,10$; that is, I being a double point, the class of the curve is $=14$; and assuming that the depression $(6\times 5-14=)\,16$ in the class of the curve is caused by double points, the number of double points will be $=8$. But observing that in the series of conics there is one conic which passes through J, so that the tangents from J to this conic are the tangent at J twice repeated, then it is easy to see that the tangents from I to this conic, at the points where they meet the tangent at J, *touch* the required curve, and that these two tangents are in fact (besides the double tangents) the only tangents from I to the curve; that is, the number of tangents from I to the curve is $=2$.

Considering I, J as the circular points at infinity, and writing A, B, C, D to denote the squared distances of a point P from the four points A, B, C, D respectively, then, as remarked by Professor Sylvester, the equation

$$\lambda\sqrt{A}+\mu\sqrt{B}+\nu\sqrt{C}+\pi\sqrt{D}=0$$

(where λ, μ, ν, π are constants) is in general a curve of the order 8; but the ratios $\lambda:\mu:\nu:\pi$ may be so determined that the order of the curve in question shall be

$= 6$; the resulting curve of the order 6 is (not one of a group of curves, but the very curve) the locus of the foci of the conics through the four points. And the determination of the ratios $\lambda : \mu : \nu : \pi$ is in fact quite simple; for writing

$$A = (x-a)^2 + (y-a_1)^2$$
$$= \rho^2 - 2(ax + a_1y) + \&\text{c.}$$
$$(\text{if } \rho^2 = x^2 + y^2),$$

and therefore

$$\sqrt{A} = \rho - \frac{ax + a_1y}{\rho} + \&\text{c.},$$

with similar values for $\sqrt{B}$, $\sqrt{C}$, $\sqrt{D}$, it is easy to see that, considering λ, μ, ν, π as standing for $\pm\lambda$, $\pm\mu$, $\pm\nu$, $\pm\pi$ respectively, the conditions for the reduction to the order 6 are

$$\lambda + \mu + \nu + \pi = 0,$$
$$\lambda a + \mu b + \nu c + \pi d = 0,$$
$$\lambda a_1 + \mu b_1 + \nu c_1 + \pi d_1 = 0,$$

and hence that the required equation of the curve of foci is

$$\Sigma \left\{ \begin{vmatrix} 1, & 1, & 1 \\ b, & c, & d \\ b_1, & c_1, & d_1 \end{vmatrix} \sqrt{(x-a)^2 + (y-a_1)^2} \right\} = 0,$$

or, as this may also be written,

$$\Sigma \pm (B,\quad C,\quad D)\sqrt{A} = 0,$$

where (B, C, D), &c. are the areas of the triangles B, C, D, &c.

I remark, in conclusion, that the number of conditions to be satisfied in order that a curve may have for double points two given points I, J, may have besides six double points, and may have for double tangents eight given lines, is $(3 + 3 + 6 + 16 =) 28$; the number of constants contained in the general equation of the order 6 is $= 27$. The conditions that a curve of the order 6 shall have for double points two given points I, J, shall besides have six double points, and shall have for double tangents four given lines through I and four given lines through J, are more than sufficient for the determination of the sextic curve; and the existence of a sextic curve satisfying these conditions is therefore a theorem.

In the case where the points I, J lie on a conic of the series, the consideration of this conic shows that the curve has a ninth double point, the pole of the line IJ in regard to the conic in question: in this case the sextic curve, as is known, breaks up into two cubic curves. [It need not do so, for a proper sextic curve may have nine (or indeed ten) double points.]

P.S. In general the curve $\lambda\sqrt{A}+\mu\sqrt{B}+\nu\sqrt{C}+\pi\sqrt{D}=0$ has (exclusively of multiple points at infinity) six double points; viz. these are situate at the intersections of the pairs of circles,

$$(\lambda\sqrt{A}+\mu\sqrt{B}=0,\quad \nu\sqrt{C}+\pi\sqrt{D}=0),$$

$$(\lambda\sqrt{A}+\nu\sqrt{C}=0,\quad \mu\sqrt{B}+\pi\sqrt{D}=0),$$

$$(\lambda\sqrt{A}+\pi\sqrt{D}=0,\quad \mu\sqrt{B}+\nu\sqrt{C}=0).$$

In the case of the curve of foci, the first, second, and third pairs of circles intersect respectively in the points $(AB\,.\,CD)$, $(AC\,.\,BD)$, $(AD\,.\,BC)$, which, as mentioned above, are double points on the curve; and they besides intersect in three other points, which are the other three double points mentioned above.

Professor Sylvester reminds me that he mentioned to me in conversation that he had himself obtained the foregoing equation $\Sigma\pm(B,\,C,\,D)\sqrt{A}=0$, for the locus of the foci of the conics which pass through the four points A, B, C, D.

Cambridge, October 10, 1866.