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BRIEF NOTES

On the existence and uniqueness of solutions in linear theory of Cosserat elasticity. I

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The basic equations of motion of linear theory of Cosserat elasticity have been derived in [6, 7]. In the present paper is given a theorem of existence and uniqueness of a generalized solution (in the sense of VISHIK and LADYZENSKAYA [4]) to these equations. Sec. 1 is devoted to preliminaries and general notations. In Sec. 2, we define the classical solution of the initial boundary-value problem for Cosserat elasticity. In Sec. 3, we construct a Hilbert space associated with governing equations and prove that a unique generalized solution exists in this space.

1. Introduction

Let Ω *BE* a bounded domain and properly regular in the sense of FICHERA [2] in the Euclidean space E^3 with orthogonal coordinates $x \equiv (x_1, x_2, x_3)$. Under this assumption, Ω has the segment property and the cone property, so that integration on the boundary 2Ω is meaningful and integration by parts over Ω is permissible.

Let $(0, T)$ be a time-interval with $0 < T < +\infty$ and Q the right-hand cylinder $Q = \Omega \times (0, T)$.

We shall consider spaces C^m, $L_2(\Omega)$, $\mathbf{C}^m(\overline{\Omega})$, $\mathbf{L}_2(\Omega)$ of scalar and vector functions, defined in the usual way. We denote by $W_2^m(\Omega)$ and $\mathbf{W}_2^m(\Omega)$ the completions of the spaces $C^m(\overline{\Omega})$ and $\mathbf{C}^m(\Omega)$ in the norms induced by the inner products (1)

$$(1.1)\qquad (\varphi, \psi)_{W_2^m(\Omega)} \equiv \sum_{K=0}^{m} \int_{\Omega} \varphi_{,i_1 \cdots i_k} \psi_{,i_1 \ldots i_k} dx$$

and

$$(1.2)\qquad (\mathbf{u}, \mathbf{v})_{\mathbf{W}_2^m(\Omega)} \equiv \sum_{j=1}^{3} (u_j, v_j)_{W_2^m(\Omega)},$$

respectively.

(1) Here and further the summation convention is adopted.

Since Ω has the segment property, the Beppo-Levi spaces $W_2^m(\Omega)$ coincide with the Sobolev spaces of functions possesing L_2 — strong generalized derivatives up to the order m in Ω [see AGMON [1]].

Let H be a Banach space. We denote by $C^m([0, T]; H)$ the space of mappings from $[0, T]$ to H, which possess on $(0, T)$ time derivatives in H up to the m-th order, continuous on $[0, T]$. In an analogous manner we introduce the spaces $L_1([0, T]; H)$, $L_2([0, T]; H)$, $L_p([0, T]; H)$ $(p > 2)$ [see [5]].

2. Formulation of the initial-boundary value problem

The basic equations in the linear theory of nonhomogeneous and anisotropic Cosserat elastic solids are [7]:

the equations of motion

$$\begin{aligned} \tau_{ji,j}+F_i &= \varrho\ddot{u}_i, \\ \mu_{ji,j}+\varepsilon_{ijk}\tau_{jk}+M_i &= \varrho J_{ik}\ddot{\varphi}_k, \end{aligned} \tag{2.1}$$

the constitutive law

$$\begin{aligned} \tau_{ij} &= E_{ijkl}\gamma_{kl}+K_{ijkl}\varkappa_{kl}, \\ \mu_{ij} &= K_{klij}\gamma_{kl}+M_{ijkl}\varkappa_{kl}, \end{aligned} \tag{2.2}$$

the kinematic relations

$$\gamma_{ij} = u_{j,i}-\varepsilon_{ijk}\varphi_k, \quad \varkappa_{ij} = \varphi_{j,i}. \tag{2.3}$$

In these equations, $\tau_{ij}(x, t)$ and $\mu_{ij}(x, t)$ represent the stress tensor and the couple-stress tensor, respectively; $u_i(x, t)$ — the displacement vector; $\varphi_i(x, t)$ — the micro-rotation vector; $F_i(x, t)$ — the body force vector; $M_i(x, t)$ — the body couple vector; $\gamma_{ij}(x, t)$ — the strain tensor; $\varkappa_{ij}(x, t)$ — the micro-strain tensor; $\varrho(x)$ — the mass density; $J_{ik}(x)$ the micro-inertia coefficients; $E_{ijkl}(x)$, $K_{ijkl}(x)$, $M_{ijkl}(x)$ — the characteristic constants of the material; ε_{ijk} — the unit antisymmetric tensor. The tensors $E_{ijkl}(x)$, $M_{ijkl}(x)$, $J_{ik}(x)$ are assumed to meet in Ω the following conditions of symmetry:

$$E_{ijkl}(x) = E_{klij}(x), \quad M_{ijkl}(x) = M_{klij}(x), \quad J_{ik}(x) = J_{ki}(x). \tag{2.4}$$

Now, we give the definition of a classical solution to the initial-boundary value problem which is to be studied in the present paper.

Let $\Omega \subset E^3$ be a bounded domain and C^1 — smooth. By $\partial\Omega$, we denote the boundary of Ω.

DEFINITION 1. *By a classical solution to the initial-boundary value problem of the linear theory of Cosserat elasticity in the cylinder* $Q = \Omega\times(0, T)$, *we mean a pair* $(\mathbf{u}, \boldsymbol{\varphi}) \in [\mathbf{C}^2(Q)\cap \mathbf{C}^1(\overline{Q})]\times[\mathbf{C}^2(Q)\cap\mathbf{C}^1(\overline{Q})]$ *satisfying the system* (2.1)–(2.3) *for* $(x, t)\in Q$, *together with the boundary conditions*:

$$\mathbf{u} = 0, \quad \boldsymbol{\varphi} = 0 \quad \text{on} \quad \partial\Omega\times(0, T), \tag{2.5}$$

and initial conditions:

$$\big(\mathbf{u}(x, 0), \dot{\mathbf{u}}(x, 0), \boldsymbol{\varphi}(x, 0), \dot{\boldsymbol{\varphi}}(x, 0)\big) = \big(\mathbf{u}_0(x), \dot{\mathbf{u}}_0(x), \boldsymbol{\varphi}_0(x), \dot{\boldsymbol{\varphi}}_0(x)\big), \tag{2.6}$$

where $\mathbf{u}_0(x)$, $\dot{\mathbf{u}}_0(x)$, $\boldsymbol{\varphi}_0(x)$, $\dot{\boldsymbol{\varphi}}_0(x)$ *are prescribed functions on* Ω.

3. The existence and uniqueness of a generalized solution

In the present section, we establish the existence of a unique generalized solution to the equations of linear Cosserat elasticity.

We make the following assumptions: $E_{ijkl}(x)$, $K_{ijkl}(x)$, $M_{ijkl}(x)$, $\varrho(x)$, $J_{ik}(x)$ are given (Lebesgue) measurable functions, essentially bounded on Ω, and satisfying (2.4) on Ω.

We introduce the following spaces of vector fields: $\hat{\mathbf{C}}^1(\Omega) \equiv \{\mathbf{V} \in \mathbf{C}^1(\overline{\Omega}) : \mathbf{V} = 0$ on $\partial\Omega\}$, $\mathbf{W}_2^1(\Omega)$ = the completion of $\hat{\mathbf{C}}^1(\Omega)$ in $\mathbf{W}_2^1(\Omega)$.

Let $\mathbf{u}, \mathbf{v}, \boldsymbol{\varphi}, \boldsymbol{\psi} \in \hat{\mathbf{C}}^1(\Omega)$ and $y = (\mathbf{u}, \boldsymbol{\varphi})$, $z = (\mathbf{v}, \boldsymbol{\psi})$. We set

$$A(y, z) = \int_{\Omega} \{E_{ijkl}\gamma_{ij}(y)\gamma_{kl}(z) + K_{ijkl}[\gamma_{ij}(y)\varkappa_{kl}(z) + \gamma_{ij}(z)\varkappa_{kl}(y)] + M_{ijkl}\varkappa_{ij}(y)\varkappa_{kl}(z)\}\, dx. \tag{3.1}$$

Obviously, the bilinear form $A(y, z)$ may be extended by continuity onto $(\hat{W}_2^1(\Omega))^4$.

In order to establish the existence of solutions, we make the following additional assumptions:

1. The mass-density and the micro-inertia coefficients satisfy the conditions:

$$\operatorname*{essinf}_{\Omega} \varrho(x) > 0, \quad J_{ik}(x)\xi_i\xi_k \geqslant \lambda\xi_e\xi_e \quad (\lambda > 0). \tag{3.2}$$

2. The energy of deformation denoted by $\mathscr{A}$ is uniform positive definite for $x \in \overline{\Omega}$ and $t \in (0, T)$ — i.e., there exists a positive constant c such that

$$\mathscr{A}(\gamma_{ij}, \varkappa_{ij}) \equiv \frac{1}{2}E_{ijkl}\gamma_{ij}(z)\gamma_{kl}(z) + K_{ijkl}\gamma_{ij}(z)\varkappa_{kl}(z) + \frac{1}{2}M_{ijkl}\varkappa_{ij}(z)\varkappa_{kl}(z) \geqslant c\sum_{i,j=1}^{3}(\gamma_{ij}^2(z) + \varkappa_{ij}^2(z)), \tag{3.3}$$

for every second-order tensor $\gamma_{ij}(z)$ and $\varkappa_{ij}(z)$ with $z \in \hat{\mathbf{C}}^1(\Omega) \times \hat{\mathbf{C}}^1(\Omega)$.

Using the Schwarz inequality and elementary inequalities, we deduce from (3.1) and (3.3) that

$$A(z, z) \geqslant \tilde{c} \int_{\Omega} (v_{i,j}v_{i,j} + \psi_{i,j}\psi_{i,j} + \psi_i\psi_i)\, dx, \tag{3.4}$$

where $\tilde{c}$ is a positive constant depending only on Ω.

For $z = (\mathbf{v}, \boldsymbol{\psi}) \in \hat{\mathbf{C}}^1(\Omega) \times \mathbf{C}^1(\Omega)$, we have the Poincaré inequality

$$kA(z, z) \geqslant \int_{\Omega} (v_iv_i + \psi_i\psi_i)\, dx, \tag{3.5}$$

where k is a positive constant.

From (3.4) and (3.5) it easily results that:

$$A(z, z) \geqslant \alpha \int_{\Omega} (v_iv_i + \psi_i\psi_i + v_{i,j}v_{i,j} + \psi_{i,j}\psi_{i,j})\, dx, \tag{3.6}$$

with $\alpha > 0$ — i.e., $A(z, z)$ is coercive on $\|z\|_{\mathbf{W}_2^1(\Omega) \times \mathbf{W}_2^1(\Omega)}$ in $\hat{\mathbf{W}}_2^1(\Omega) \times \hat{\mathbf{W}}_2^1(\Omega)$.

Before proceeding to the definition of a generalized solution, we consider the sets: $\mathscr{C}(\Omega) \equiv \mathbf{C}(\Omega)\times\mathbf{C}(\Omega)$, $\mathscr{C}^1(\Omega) \equiv \hat{\mathbf{C}}^1(\Omega)\times\hat{\mathbf{C}}^1(\Omega)$, $\mathscr{L}_2(\Omega) \equiv L_2(\Omega)\times L_2(\Omega)$, $\mathscr{F}(Q) \equiv C^\infty((0, T); \hat{\mathscr{C}}^1(\Omega))$, $\mathring{\mathscr{F}}(Q) \equiv \{(\mathbf{v}, \boldsymbol{\psi}) : (\mathbf{v}, \boldsymbol{\psi}) \in \mathscr{F}(Q)$ and $\mathbf{v}(x, 0) = \boldsymbol{\psi}(x, 0) = 0$ on $\Omega\}$. For $y = (\mathbf{u}, \boldsymbol{\varphi}) \in \mathscr{F}(Q)$, $z = (\mathbf{v}, \boldsymbol{\psi}) \in \mathring{\mathscr{F}}(Q)$, $f = (\mathbf{F}, \mathbf{M}) \in C^\infty([0, T]; \mathscr{C}(\Omega))$ and $\dot{y}_0 \in \mathscr{C}^1(\Omega)$ we define:

$$\mathscr{L}(y, z) \equiv \int_0^T\int_\Omega \{(t-T)[\varrho(\dot{u}_i\ddot{v}_i + J_{ik}\dot{\varphi}_k\ddot{\psi}_i) - (\tau_{ij}(y)\dot{\gamma}_{ij}(z) + \mu_{ij}(y)\dot{\varkappa}_{ij}(z))] + \varrho(\dot{u}_i\dot{v}_i + J_{ik}\dot{\varphi}_k\dot{\psi}_i)\}\,dx\,dt,$$

$$\mathscr{D}(f, z) \equiv \int_0^T\int_\Omega (t-T)[-F_i\dot{v}_i - M_i\dot{\psi}_i]\,dx\,dt,$$

$$\mathscr{E}(\dot{y}_0, z) \equiv T\int_\Omega \varrho[\dot{u}_{0i}\dot{v}_{i/t=0} + J_{ik}\dot{\varphi}_{0k}\dot{\psi}_{i/t=0}]\,dx.$$

It is easy to verify the identity

$$\mathscr{L}(z, z) = \frac{1}{2}\int_0^T\int_\Omega [\varrho(\dot{v}_i\dot{v}_i + J_{ik}\dot{\psi}_i\dot{\psi}_k) + E_{ijkl}\gamma_{ij}\gamma_{kl} + 2K_{ijkl}\gamma_{ij}\varkappa_{kl} + M_{ijkl}\varkappa_{ij}\varkappa_{kl}]\,dx\,dt. \tag{3.7}$$

We denote by $\mathscr{H}^1(Q)$ the Hilbert space obtained as the completion of $\mathscr{F}(Q)$ by means of the norm $|\cdot|$ induced by the inner product

$$\langle y, z\rangle = \langle(\mathbf{u}, \boldsymbol{\varphi}), (\mathbf{v}, \boldsymbol{\psi})\rangle \equiv \int_0^T\int_\Omega [u_iv_i + \varphi_i\psi_i + u_{i,j}v_{i,j} + \varphi_{i,j}\psi_{i,j} + \dot{u}_i\dot{v}_i + \dot{\varphi}_i\dot{\psi}_i]\,dx\,dt$$

$\mathring{\mathscr{H}}^1(Q)$ the closed linear subspace of $\mathscr{H}^1(Q)$ obtained as the completion of $\mathring{\mathscr{F}}(Q)$ by means of $|\cdot|$. From (3.2), (3.6), (3.7) we deduce that there is a constant $c_1 > 0$ depending only on α, λ ess. inf., ϱ such that

$$\mathscr{L}(z, z) \geqslant c_1|z|^2, \quad \forall\, z \in \mathring{\mathscr{F}}(Q). \tag{3.8}$$

By $\mathring{\mathscr{H}}^1_{\cdot}(Q)$ we denote the Hilbert space obtained as the completion of $\mathscr{F}(Q)$ in the norm induced by the inner product

$$[y, z] = [(\mathbf{u}, \boldsymbol{\varphi}), (\mathbf{v}, \boldsymbol{\psi})] \equiv \langle(\mathbf{u}, \boldsymbol{\varphi}), (\mathbf{v}, \boldsymbol{\psi})\rangle + \langle(\dot{\mathbf{u}}, \dot{\boldsymbol{\varphi}}), (\dot{\mathbf{v}}, \dot{\boldsymbol{\psi}})\rangle.$$

Using Schwarz's inequality and Sobolev's embedding theorem, it is seen that the bilinear forms $\mathscr{L}$ and $\mathscr{D}$ can be extended by continuity onto $\mathscr{H}^1(Q)\times\mathring{\mathscr{H}}^1_{\cdot}(Q)$ and $L_1([0, T]; \mathscr{L}_2(\Omega))\times\mathring{\mathscr{H}}^1(Q))$, respectively.

The inequality (3.8) remains valid on $\mathring{\mathscr{H}}^1(Q)$, and $\mathscr{E}(\dot{y}_0, z)$ makes sense for $z \in \mathring{\mathscr{H}}^1(Q)$.

Let us define now a generalized (weak) solution to the linear equations of Cosserat elastodynamics in the sense of VISHIK and LADYZENSKAYA [4].

DEFINITION 2. *The pair* $y = (\mathbf{u}, \boldsymbol{\varphi}) \in \mathscr{H}^1(Q)$ *will be called a generalized solution with finite energy for the system* (2.1)–(2.3) *with the boundary conditions* $(y_0, \dot{y}_0) \in \mathscr{L}_2(\Omega)\times$