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Uniqueness, existence and estimate of the solution in the dynamical problem of thermodiffusion in an elastic solid

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THE PAPER is devoted to the theory of a system of five second-order partial differential equations describing the dynamic problem of thermodiffusion in a deforming elastic body. The system is hyperbolic with respect to some of the unknown functions, parabolic with respect to the rest of them. A particular case is the coupled thermoelasticity.

Rozpatruje się teorię układu pięciu równań różniczkowych cząstkowych drugiego rzędu, opisującego zagadnienia dynamiczne termodyfuzji w odkształcającym się ciele sprężystym. Układ hiperboliczny ze względu na część nieznanych funkcji, paraboliczny ze względu na pozostałe. Przypadkiem szczególnym jest sprzężona termosprężystość.

Рассматривается теория системы пяти дифференциальных уравнений второго порядка в частных производных, описывающая динамическую проблему термодиффузии в деформирующемся упругом теле. Система является гиперболической относительно части неизвестных функций и параболической относительно остальных. Частным случаем является сопряжённая термоупругость.

THE ANALYTICAL problem which will be discussed in this paper was proposed to me by Prof. Witold NOWACKI, during the Polish Conference on Mechanics of Solids, held in Zakopane last September. This problem is originated by the study of the dynamical processes of thermodiffusion in an elastic solid(1). It consists in investigating a system of five 2nd order partial differential equations which is hyperbolic with respect to some of the unknown functions and parabolic with respect to the others. This system contains as particular case (assuming equal to zero some of the coefficients) the well known system of partial differential equations of coupled thermoelasticity (see [2], p. 41). The existence problem in the case of thermoelasticity is studied in [3]. More general "coupled equations" are considered in [4].

Suitable boundary and initial conditions must be associated with the system of partial differential equations.

I have restricted myself to consider here only Dirichlet boundary conditions. However the method used in this paper should work even with more general or different boundary conditions.

It must also be remarked that the method developed in this paper applies to non isotropic, inhomogeneous bodies as well. However this extension will not be discussed here.

(1) For more information on the physical aspects of the problem we refer the reader to the forthcoming paper [1] of Prof. NOWACKI and to the bibliography quoted in it.

I am very pleased indeed to submit for publication in a Polish journal this paper, originated by stimulating discussions with Polish Colleagues and, in particular, with Prof. NOWACKI, during the unforgettable meeting in Zakopane.

1. Statement of the problem

Let us denote by $u \equiv (u_1, u_2, u_3)$ a 3-vector valued function with real components and by θ and μ real valued functions. In this paper by the term *function* we refer to vector valued functions with any number of real or complex components, and, in particular, to scalar functions. If v is a function, by $v_{/h}$ and $v_{/hk}$ we denote the partial derivatives

$$\frac{\partial v}{\partial x_h}, \frac{\partial^2 v}{\partial x_h \partial x_k}$$

with respect to space variables. Differentiation with respect to the time variable t will be denoted either in the usual way or by a dot, i.e.,

$$\dot{v} = \frac{\partial v}{\partial t}, \quad \ddot{v} = \frac{\partial^2 v}{\partial t^2}.$$

We shall consider the following system of partial differential equations

$$(1.1)\qquad \begin{gathered} Gu_{h/jj} + (\lambda + G)u_{j/hj} - \varrho \ddot{u}_h - p_\theta \theta_{/h} - p_\mu \mu_{/h} = F_h(x, t), \quad h = 1, 2, 3, \\ K\theta_{/jj} - c\dot{\theta} - d\dot{\mu} - p_\theta \dot{u}_{j/j} = f(x, t), \\ D\mu_{/jj} - n\dot{\mu} - d\dot{\theta} - p_\mu \dot{u}_{j/j} = g(x, t). \end{gathered}$$

$G, \lambda, \varrho, p_\theta, p_\mu, K, c, d, D, n$ are given real constants; $F(x, t) \equiv F_1(x, t), F_2(x, t), F_3(x, t)$, $f(x, t), g(x, t)$ are given functions with real components. The point $x \equiv (x_1, x_2, x_3)$ varies in a bounded domain (open set) A of the 3-dimensional space and the variable t is such that $0 \leqslant t < +\infty$.

We assume that the domain A has a piece-wise smooth boundary and that the Gauss-Green formulas which transform volume integrals into surface integrals hold for the domain A.

With the system (1.1) we associate the following *boundary conditions*, for $x \in \partial A$, $0 \leqslant t < +\infty$,

$$(1.2)\qquad \begin{gathered} u(x, t) = \bar{u}(x, t), \\ \theta(x, t) = \bar{\theta}(x, t), \quad \mu(x, t) = \bar{\mu}(x, t), \end{gathered}$$

where $\bar{u}, \bar{\theta}, \bar{\mu}$ are given functions with real components and the following *initial conditions* for $x \in A$

$$(1.3)\qquad \begin{gathered} u(x, 0) = u^\circ(x), \quad \dot{u}(x, 0) = u'(x), \\ \theta(x, 0) = \theta^\circ(x), \quad \mu(x, 0) = \mu^\circ(x), \end{gathered}$$

where $u^\circ, u', \theta^\circ, \mu^\circ$ are given functions with real components.

Let us define the function class where we shall study the problem (1.1), (1.2), (1.3). Let $v(x, t)$ be a function defined in $\bar{A} \times [0, +\infty)$. We consider the following conditions

i) $v \in \mathscr{C}^1\{\bar{A} \times [0, +\infty)\} \cap \mathscr{C}^2\{A \times [0, +\infty)\}$.

ii) Two positive constants c_0 and s_0 exist which depend only on v and are such that, if $\mathbf{D}^m$ denotes any partial derivative of $v(x, t)$ of order m, we have

$$|\mathbf{D}^m v(x, t)| \leqslant c_0 e^{s_0 t}, \quad x \in A, \quad 0 \leqslant t < +\infty, \quad m = 0, 1, 2.$$

We denote by $\mathscr{F}$ the function class, formed by all the 3-vector valued functions u and by all the scalar functions θ and μ, such that each of the functions u, θ, μ satisfies conditions i) and ii).

If (u, θ, μ) belongs to $\mathscr{F}$ we may assume that the positive constants c_0 and s_0 are the same for each function u, θ, μ.

2. Uniqueness theorem

Let us assume from now on that the following conditions are satisfied

$$\begin{gathered} G > 0, \quad K > 0, \quad D > 0, \quad \varrho > 0, \quad c > 0, \quad n > 0, \\ \lambda + 2G > 0, \quad d^2 \leqslant cn. \end{gathered} \tag{2.1}$$

Suppose that the "data" of the problem are identically zero, i.e.

$$\begin{gathered} F(x, t) \equiv 0, \quad f(x, t) \equiv 0, \quad g(x, t) \equiv 0, \\ \bar{u}(x, t) \equiv 0, \quad \bar{\theta}(x, t) \equiv 0, \quad \bar{\mu}(x, t) \equiv 0, \\ u^\circ(x) \equiv 0, \quad \theta^\circ(x) \equiv 0, \quad \mu^\circ(x) \equiv 0, \quad u'(x) \equiv 0. \end{gathered} \tag{2.2}$$

Let (u, θ, μ) be a solution of the problem (1.1), (1.2), (1.3), belonging to $\mathscr{F}$, and assume that the conditions (2.1), (2.2) are satisfied. For any complex number s such that $\mathscr{R}s > s_0$(²), we may consider the Laplace transforms

$$\hat{u}(x, s) = \int_0^\infty u(x, t) e^{-st} dt,$$

$$\hat{\theta}(x, s) = \int_0^\infty \theta(x, t) e^{-st} dt,$$

$$\hat{\mu}(x, s) = \int_0^\infty \mu(x, t) e^{-st} dt.$$

From (1.1), (1.3) and (2.2), we get

$$\begin{gathered} G\hat{u}_{h/jj} + (\lambda + G)\hat{u}_{j/hj} - \varrho s^2 \hat{u}_h - p_\theta \hat{\theta}_{/h} - p_\mu \hat{\mu}_h = 0, \quad h = 1, 2, 3, \\ K\hat{\theta}_{/jj} - cs\hat{\theta} - ds\hat{\mu} - p_\theta s\hat{u}_{j/j} = 0, \\ D\hat{\mu}_{/jj} - ns\hat{\mu} - ds\hat{\theta} - p_\mu s\hat{u}_{j/j} = 0. \end{gathered} \tag{2.3}$$

The functions $\hat{u}$, $\hat{\theta}$, $\hat{\mu}$ belong to $\mathscr{C}^1(\bar{A}) \cap \mathscr{C}^2(A)$ and satisfy the boundary conditions

$$\hat{u} = 0, \quad \hat{\theta} = 0, \quad \hat{\mu} = 0 \quad \text{on } \partial A. \tag{2.4}$$

(²) If a is a complex number, we set $a = \mathscr{R}a + i\,\mathscr{I}a$ with $\mathscr{R}a$ and $\mathscr{I}a$ real.

If a is a complex number, we denote by a^* its complex conjugate. By multiplying both sides of the h-th equation ($h = 1, 2, 3$) of system (2.3) by $-\hat{u}_h^*$, both sides of the fourth equations by $-\frac{1}{s}\hat{\theta}^*$ and both sides of the fifth equation by $-\frac{1}{s}\hat{\mu}^*$, by summing and by integrating over A, we get, after some integrations by parts and recalling (2.4),

$$\int_A \Big\{ G\hat{u}_{h/j}\hat{u}^*_{h/j} + (\lambda+G)\hat{u}_{j/j}\hat{u}^*_{h/h} + \varrho s^2 \hat{u}_h \hat{u}^*_h + 2ip\,\mathscr{I}(\hat{\theta}_{/h}\hat{u}^*_h) + 2ip_\mu \mathscr{I}(\hat{\mu}_{/h}\hat{u}^*_h) + \frac{K}{s}\hat{\theta}_{/j}\hat{\theta}^*_{/j} + \frac{D}{s}\hat{\mu}_{/j}\hat{\mu}^*_{/j} + c\hat{\theta}\hat{\theta}^* + 2d\mathscr{R}(\hat{\mu}\hat{\theta}^*) + n\hat{\mu}\hat{\mu}^* \Big\} dx = 0. \tag{2.5}$$

Set $s = s_1 + is_2$ (s_1, s_2 real). Considering the real part of (2.5) we have

$$\int_A \Big\{ G\hat{u}_{h/j}\hat{u}^*_{h/j} + (\lambda+G)\hat{u}_{j/j}\hat{u}^*_{h/h} + \varrho(s_1^2 - s_2^2)|\hat{u}|^2 + \frac{Ks_1}{|s|^2}\hat{\theta}_{/j}\hat{\theta}^*_{/j} + \frac{Ds_1}{|s|^2}\hat{\mu}_{/j}\hat{\mu}^*_{/j} + c|\hat{\theta}|^2 + 2d\mathscr{R}(\mu\hat{\theta}^*) + n|\hat{\mu}|^2 \Big\} dx = 0. \tag{2.6}$$

Set

$$\sigma = \frac{1}{2}\Big\{ \min\Big[G, 2G + \frac{2}{3}\lambda \Big] - G \Big\}. \tag{2.7}$$

The following Hermitian quadratic form in the 9 variables $\hat{u}_{h/j}$ ($h, j = 1, 2, 3$)

$$Q = G\hat{u}_{h/j}\hat{u}^*_{h/j} + (\lambda+G)\hat{u}_{j/j}\hat{u}^*_{h/h} + 2\sigma\mathscr{R}(\hat{u}_{1/2}\hat{u}^*_{2/1} - \hat{u}_{1/1}\hat{u}^*_{2/2} + \hat{u}_{2/3}\hat{u}^*_{3/2} - \hat{u}_{2/2}\hat{u}^*_{3/3} + \hat{u}_{3/1}\hat{u}^*_{1/3} - \hat{u}_{3/3}\hat{u}^*_{1/1}) \tag{2.8}$$

is positive definite([3]).

Because of the boundary condition $u = 0$ on ∂A, we can write (2.6) as follows:

$$\int_A \Big\{ G\hat{u}_{h/j}\hat{u}^*_{h/j} + (\lambda+G)\hat{u}_{j/j}\hat{u}^*_{h/h} + 2\sigma\mathscr{R}(\hat{u}_{1/2}\hat{u}^*_{2/1} - \hat{u}_{1/1}\hat{u}^*_{2/2} + \hat{u}_{2/3}\hat{u}^*_{3/2} - \hat{u}_{2/2}\hat{u}_{3/3} + \hat{u}_{3/1}\hat{u}^*_{1/3} - \hat{u}_{3/3}\hat{u}^*_{1/1}) + \varrho(s_1^2 - s_2^2)|\hat{u}|^2 + \frac{Ks_1}{|s|^2}\hat{\theta}_{/j}\hat{\theta}^*_{/j} + \frac{Ds_1}{|s|^2}\hat{\mu}_{/j}\hat{\mu}^*_{/j} + c|\hat{\theta}|^2 + 2d\mathscr{R}(\hat{\mu}\hat{\theta}^*) + n|\hat{\mu}|^2 \Big\} dx = 0. \tag{2.9}$$

From (2.9) we deduce that, if $s = s_1 + is_2$ satisfies the condition

$$s_1 \geqslant 0, \quad s_1 - s_2 \geqslant 0, \quad s_2 + s_1 \geqslant 0, \tag{2.10}$$

the problem (2.3), (2.4) has only the trivial solution $\hat{u} \equiv 0$, $\hat{\theta} \equiv 0$, $\hat{\mu} \equiv 0$ in the class $\mathscr{C}^1(\bar{A}) \cap \mathscr{C}^2(A)$.

I. *The problem* (1.1), (1.2), (1.3) *has at most one solution in the class* $\mathscr{F}$.

Suppose that conditions (2.2) are satisfied. Then the Laplace transforms $\hat{u}(x, s)$, $\hat{\theta}(x, s)$, $\hat{\mu}(x, s)$ vanish identically for $x \in \bar{A}$ and for s real and such that $s = s_1 > s_0$. Hence, for a classical theorem on Laplace transforms, we have $u(x, t) \equiv 0$, $\theta(x, t) \equiv 0$, $\mu(x, t) \equiv 0$ ([4]).

([3]) See [5], Lemma I.

([4]) See [6], p. 62: Corollary 6.2b.

For getting the existence theorem we need to prove that the uniqueness theorem for the problem (2.3), (2.4) holds in a larger subset of the complex plane than that determined by conditions (2.10). Actually we have:

II. *If s is such that $s_1 > 0$, the problem* (2.3), (2.4) *has only the trivial solution* $\hat{u} \equiv 0$, $\hat{\theta} \equiv 0$, $\hat{\mu} \equiv 0$ *in the function class* $\mathscr{C}^1(\bar{A}) \cap \mathscr{C}^2(A)$.

Suppose $s_1 > 0$. Let us multiply the h-th equation ($h = 1, 2, 3$) of (2.3) by $-\hat{u}_h^*$, the fourth by $\frac{1}{s}\hat{\theta}^*$, the fifth by $\frac{1}{s}\hat{\mu}^*$. By summing and integrating we obtain

$$(2.11) \qquad \int_A \Big\{ G\hat{u}_{h/j}\hat{u}^*_{h/j} + (\lambda+G)\hat{u}_{j/j}\hat{u}^*_{h/h} + \varrho^2 s^2 \hat{u}_h \hat{u}_h^* + 2p_j \mathscr{R}(\hat{\theta}_{/h}\hat{u}_h^*) + 2p_\mu \mathscr{R}(\hat{\mu}_{/h}\hat{u}_h^*) - \frac{K}{s}\hat{\theta}_{/j}\hat{\theta}^*_{/j} - \frac{D}{s}\hat{\mu}_{/j}\hat{\mu}^*_{/j} - c\hat{\theta}\hat{\theta}^* - 2d\mathscr{R}(\hat{\mu}\hat{\theta}^*) - n\hat{\mu}\hat{\mu}^* \Big\} dx = 0.$$

Considering the imaginary part of (2.11) we have

$$(2.12) \qquad \int_A \Big\{ 2\varrho s_1 s_2 |\hat{u}|^2 + \frac{Ks_2}{|s|^2}\hat{\theta}_{/j}\hat{\theta}^*_{/j} + \frac{Ds_2}{|s|^2}\hat{\mu}_{/j}\mu^*_{/j} \Big\} dx = 0,$$

which for $s_2 \neq 0$ proves the theorem. For $s_2 = 0$ the uniqueness was already known.

3. "A priori" estimates for the Laplace transforms

Let us now assume that the functions $F(x, t), f(x, t), g(x, t)$ have Laplace transforms $\hat{F}(x, s), \hat{f}(x, s), \hat{g}(x, s)$ which for any s, such that $\mathscr{R}s > s_0 > 0$, belong to the space $L^2(A)$. Suppose that the functions $\bar{u}(x, t)$, $\bar{\theta}(x, t)$, $\bar{\mu}(x, t)$, $u^\circ(x)$, $u'(x)$, $\theta^\circ(x)$, $\mu^\circ(x)$ vanish identically. Then the Laplace transforms $\hat{u}$, $\hat{\theta}$, $\hat{\mu}$ of u, θ, μ satisfy in A the differential system

$$(3.1) \qquad \begin{aligned} &G\hat{u}_{h/jj} + (\lambda+G)\hat{u}_{j/hj} - \varrho s^2 \hat{u}_h - p_\theta \hat{\theta}_{/h} - p_\mu \hat{\mu}_{/h} = \hat{F}_h, \quad h = 1, 2, 3,\\ &K\hat{\theta}_{/jj} - cs\hat{\theta} - ds\hat{\mu} - p\ s\hat{u}_{j/j} = \hat{f},\\ &D\hat{\mu}_{/jj} - ns\hat{\mu} - ds\hat{\theta} - p_\mu s\hat{u}_{j/j} = \hat{g} \end{aligned}$$

and the boundary conditions (2.4).

Assume $s_1 = \mathscr{R}s > s_0 > 0$. By the same procedure used in Sec. 2 we get the analogous equations of (2.6), (2.12)

$$(3.2) \qquad \int_A \Big\{ G\hat{u}_{h/j}\hat{u}^*_{h/j} + (\lambda+G)\hat{u}_{j/j}\hat{u}^*_{h/h} + \varrho(s_1^2 - s_2^2)|\hat{u}|^2 + \frac{Ks_1}{|s|^2}\hat{\theta}_{/j}\hat{\theta}^*_{/j} + \frac{Ds_1}{|s|^2}\hat{\mu}_{/j}\hat{\mu}^*_{/j} + c|\hat{\theta}|^2 + 2d\mathscr{R}(\hat{\mu}, \hat{\theta}^*) + n|\hat{\mu}|^2 \Big\} dx = -\mathscr{R}\int_A \Big(\hat{F}_h u_h^* + \frac{1}{s}\hat{f}\hat{\theta}^* + \frac{1}{s}\hat{g}\hat{\mu}^* \Big) dx,$$

$$(3.3) \qquad \int_A \Big\{ 2\varrho s_1 s_2 |\hat{u}|^2 + \frac{Ks_2}{|s|^2}\hat{\theta}_{/j}\hat{\theta}^*_{/j} + \frac{Ds_2}{|s|^2}\hat{\mu}_{/j}\hat{\mu}^*_{/j} \Big\} dx = -\mathscr{I}\int_A \Big(\hat{F}_h \hat{u}_h^* - \frac{1}{s}\hat{f}\hat{\theta}^* - \frac{1}{s}g\hat{\mu}^* \Big) dx.$$

If we multiply both sides of (3.3) by s_2/s_1 and sum to (3.2), we get

$$(3.4)\quad \int_A \Big\{ G\hat{u}_{h/j}\hat{u}^*_{h/j} + (\lambda+G)\hat{u}_{j/j}\hat{u}^*_{h/h} + \varrho|s|^2|\hat{u}|^2 + \frac{K}{s_1}\hat{\theta}_{/j}\hat{\theta}^*_{/j} + \frac{D}{s_1}\hat{\mu}_{/j}\hat{\mu}^*_{/j} + c|\hat{\theta}|^2 + 2d\mathcal{R}(\hat{\mu}, \hat{\theta}^*)$$

$$+n|\hat{\mu}|^2\Big\} dx = -\mathcal{R}\int_A \Big(\hat{F}_h\hat{u}^*_h + \frac{1}{s}\hat{f}\hat{\theta}^* + \frac{1}{s}\hat{g}\hat{\mu}^*\Big)dx - \frac{s_2}{s_1}\mathcal{I}\int_A \Big(\hat{F}_h\hat{u}^*_h - \frac{1}{s}\hat{f}\hat{\theta}^* - \frac{1}{s}g\hat{\mu}^*\Big)dx.$$

Let us denote by $\gamma_0(\lambda, G)$ the lowest eigenvalue of the quadratic form Q given by (2.8).

Set

$$\gamma_1(c, d, n) = \frac{1}{2}\{(c+n) - [(c-n)^2 + 4d^2]^{\frac{1}{2}}\}(^5).$$

From (3.4) we deduce(6)

$$\int_A \{s_0\gamma_0(\lambda, G)\hat{u}_{h/j}\hat{u}^*_{h/j} + \varrho s_0|s|^2|\hat{u}|^2 + K\hat{\theta}_{/j}\hat{\theta}^*_{/j} + D\hat{\mu}_{/j}\hat{\mu}^*_{/j} + s_0\gamma_1(c, d, n)[|\hat{\theta}|^2 + |\hat{\mu}|^2]\} dx$$

$$\leqslant 2\Big(\int_A (|\hat{F}|^2 + |\hat{f}|^2 + |\hat{g}|^2)dx\Big)^{\frac{1}{2}}\Big(\int_A (|s|^2|\hat{u}|^2 + |\hat{\theta}|^2 + |\hat{\mu}|^2)dx\Big)^{\frac{1}{2}}.$$

Set

$$(3.5)\qquad \gamma_2 = \min[\varrho, \gamma_1(c, d, n)].$$

We have

$$\int_A \{s_0\gamma_0(\lambda, G)\hat{u}_{h/j}\hat{u}^*_{h/j} + K\hat{\theta}_{/j}\hat{\theta}^*_{/j} + D\hat{\mu}_{/j}\hat{\mu}^*_{/j} + \gamma_2 s_0[|s|^2|\hat{u}|^2 + |\hat{\theta}|^2 + |\hat{\mu}|^2]\} dx$$

$$\leqslant \frac{4}{\gamma_2 s_0}\int_A (|\hat{F}|^2 + |\hat{f}|^2 + |\hat{g}|^2)dx.$$

Hence we have the following theorem:

III. *Let s_0 be a given positive real number. Suppose that $\mathcal{R}s > s_0$. If $\hat{F}, \hat{f}, \hat{g}$ belong to $L^2(A)$ and if $(\hat{u}, \hat{\theta}, \hat{\mu})$ is a solution of the problem* (3.1), (2.4) *in the function class $\mathscr{C}^1(\bar{A}) \cap \mathscr{C}^2(A)$, the following integral estimate holds*

$$(3.6)\quad \int_A (\hat{u}_{h/j}\hat{u}^*_{h/j} + \hat{\theta}_{/h}\hat{\theta}^*_{/h} + \hat{\mu}_{/h}\hat{\mu}^*_{/h} + |\hat{u}|^2 + |\hat{\theta}|^2 + |\hat{\mu}|^2)dx \leqslant \gamma^2\int_A (|\hat{F}|^2 + |\hat{f}|^2 + |\hat{g}|^2)dx,$$

where

$$(3.7)\qquad \gamma^2 = \frac{4}{\gamma_2 s_0}\min[s_0\gamma_0(\lambda, G), K, D, \gamma_2 s_0].$$

Let us denote by **v** the 5-vector valued function whose components are the functions

$$\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{\theta}, \hat{\mu}$$

(5) $\gamma_1(c, d, n)$ is the lowest eigenvalue of the positive Hermitian quadratic form $c|\hat{\theta}|^2 + 2d\mathcal{R}(\hat{\mu}\hat{\theta}^*) + n|\hat{\mu}|^2$.

(6) Like in Sec. 2 we use the identity

$$\int_A \{G\hat{u}_{h/j}\hat{u}^*_{h/j} + (\lambda+G)\hat{u}_{j/j}\hat{u}^*_{h/h}\} dx = \int_A Q dx.$$

and by $\hat{\boldsymbol{\varphi}}$ the 5-vector valued function whose components are the functions

$$\hat{F}_1, \hat{F}_2, \hat{F}_3, \frac{1}{s}\hat{f}, \frac{1}{s}\hat{g}, \quad (\mathcal{R}s > s_0 > 0).$$

Moreover, let us multiply both the fourth and the fifth equations of the system (3.1) by $1/s$ and write the resulting differential system in the abridged form

$$\mathbf{L}\hat{\mathbf{v}} = \hat{\boldsymbol{\varphi}}, \tag{3.8}$$

where the meaning of the 5×5 matrix differential operator $\mathbf{L}$ is self explanatory.

Let us consider the problem

$$\begin{aligned} \mathbf{L}\hat{\mathbf{w}} - \varrho s_2^2 \hat{\mathbf{w}} &= \hat{\boldsymbol{\psi}} \quad \text{in } A, \\ \hat{\mathbf{w}} &= 0 \quad \text{on } \partial A, \end{aligned} \tag{3.9}$$

where $\boldsymbol{\psi}$ is a given 5-vector valued function and $s_2 = \mathcal{I}s$.

We shall denote by $H_m(A)$ the space of 5-vector valued functions with L^2 generalized derivative up to the order m endowed by the usual norm $\| \ \|_m$(7). Let $\mathring{H}_1(A)$ be the subspace of $H_1(A)$ formed by the functions vanishing on ∂A (in the sense of the functions of $H_1(A)$).

The quadratic form associated to the boundary value problem (3.9), assuming $\hat{\mathbf{z}} \equiv (\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{\tau}, \hat{\nu})$, is the following

$$B(\hat{\mathbf{z}}, \hat{\mathbf{z}}) = \int_A \Big\{ G\hat{z}_{h/j}\hat{z}^*_{h/j} + (\lambda+G)\hat{z}_{j/j}\hat{z}^*_{h/h} + \varrho(s^2+s_2^2)\hat{z}_h\hat{z}^*_h + p_\circ\hat{\tau}_{/h}\hat{z}^*_h + p_\mu\hat{\nu}_{/h}\hat{z}^*_h$$
$$-p\ \hat{z}_h\hat{\tau}^*_{/h} - p_\mu\hat{z}_h\hat{\nu}^*_{/h} + \frac{K}{s}\hat{\tau}_{/j}\hat{\tau}^*_{/j} + \frac{D}{s}\hat{\nu}_{/j}\hat{\nu}^*_{/j} + \varrho s_2^2(\hat{\tau}\hat{\tau}^* + \hat{\nu}\hat{\nu}^*) + c\hat{\tau}\hat{\tau}^* + d\hat{\nu}\hat{\tau}^* + d\hat{\tau}\hat{\nu}^* + n\hat{\nu}\hat{\nu}^* \Big\} dx.$$

We have for every $\hat{\mathbf{z}} \in \mathring{H}_1(A)$ and for $\mathcal{R}s > s_0$

$$\mathcal{R}B(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \geqslant q|s|^{-2}\|\hat{\mathbf{z}}\|_1^2, \tag{3.10}$$

where q is a positive constant independent of $\hat{\mathbf{z}}$.

From now on we assume that ∂A is $\mathscr{C}^\infty$-smooth(8). The theory of strongly elliptic differential systems enables us to conclude that there exists one and only one solution $\hat{\mathbf{w}}$ of the problem (3.9) which belongs to $H_{m+2}(A)$ if $\hat{\boldsymbol{\psi}}$ belongs to $H_m(A)$. Moreover, we have the following estimate

$$\|\hat{\mathbf{w}}\|_{m+2} \leqslant K_m(s)\,\|\hat{\boldsymbol{\psi}}\|_m.$$

Since the Frobenius modulus of the inverse matrix of the characteristic matrix of the dominant part of the operator $\mathbf{L}$ is $O(|s|)$ and the coefficients of $\mathbf{L}$ are $O(|s|^2)$ and since (3.10) holds, we may assume

$$K_m(s) = k_m|s|^5,$$

where k_m is a constant independent of s(9).

(7) See [7], Sec. 2.

(8) See [7] p. 369.

(9) See [7], Secs. 3, 4, 5, 6.

Let us denote by $\hat{\mathbf{w}} = \mathbf{G}\hat{\boldsymbol{\psi}}$ the resolvent operator (Green's operator) of the problem (3.9), which we consider as a compact operator from $L^2(A)$ into $L^2(A)$.

The differential system (3.8), with the boundary condition

$$\hat{\mathbf{v}} = 0 \quad \text{on } \partial A$$

may be written

$$\text{(3.11)} \qquad \begin{cases} \mathbf{L}\hat{\mathbf{v}} - \varrho s_2^2 \hat{\mathbf{v}} + \varrho s_2^2 \mathbf{v} = \hat{\boldsymbol{\varphi}} & \text{on } A, \\ \hat{\mathbf{v}} = 0 & \text{on } \partial A. \end{cases}$$

If we set $\mathbf{L}\hat{\mathbf{v}} - \varrho s_2^2 \hat{\mathbf{v}} = \hat{\boldsymbol{\psi}}$, the problem (3.11) is equivalent to the problem

$$\text{(3.12)} \qquad \hat{\boldsymbol{\psi}} + \varrho s_2^2 \mathbf{G}\hat{\boldsymbol{\psi}} = \hat{\boldsymbol{\varphi}}.$$

Since for $\mathcal{R}s > s_0 > 0$ we have an uniqueness theorem for the problem (3.11), we have that there exists one and only one solution $\hat{\boldsymbol{\psi}}$ of (3.12) belonging to $L^2(A)$. Hence $\mathbf{G}\hat{\boldsymbol{\psi}}$ belongs to $H_2(A)$. By an induction argument we deduce from (3.12) that if $\hat{\boldsymbol{\varphi}}$ belongs to $H_m(A)$, $\hat{\boldsymbol{\psi}}$ belongs to $H_m(A)$ and, in consequence, $\hat{\mathbf{v}} = \mathbf{G}\hat{\boldsymbol{\psi}}$ belongs to $H_{m+2}(A)$; hence

IV. *If* $(\hat{F}, \hat{f}, \hat{g})$ *belongs to* $H_m(A)$, *the problem* (3.1), (2.4) *has a solution belonging to* $H_{m+2}(A)$.

From Theorem III we deduce ($q_0, q_1, q_2, \ldots$ denote positive constants)

$$\|\hat{\mathbf{v}}\|_0 = \|\mathbf{G}\hat{\boldsymbol{\psi}}\|_0 \leqslant q_0 \|\hat{\boldsymbol{\varphi}}\|_0 \qquad (q_0 > 0).$$

Hence from (3.12)

$$\|\hat{\boldsymbol{\psi}}\|_0 \leqslant q_1 |s|^2 \|\hat{\boldsymbol{\varphi}}\|_0$$

and in consequence

$$\text{(3.13)} \qquad \|\hat{\mathbf{v}}\|_2 \leqslant k_2 |s|^5 \|\hat{\boldsymbol{\psi}}\|_0 \leqslant k_2 q_1 |s|^7 \|\hat{\boldsymbol{\varphi}}\|_0.$$

From (3.12), (3.13) we have

$$\|\hat{\boldsymbol{\psi}}\|_2 \leqslant \varrho |s|^2 \|\hat{\mathbf{v}}\|_2 + \|\hat{\boldsymbol{\varphi}}\|_2 \leqslant \varrho k_2 q_1 |s|^9 \|\hat{\boldsymbol{\varphi}}\|_0 + \|\hat{\boldsymbol{\varphi}}\|_2.$$

Hence

$$\text{(3.14)} \qquad \|\hat{\mathbf{v}}\|_4 \leqslant k_2 |s|^5 \|\hat{\boldsymbol{\psi}}\|_2 \leqslant q_2 |s|^{14} \|\hat{\boldsymbol{\varphi}}\|_2.$$

By using the Sobolev lemma(10) from (3.14) we deduce the following theorem:

V. *For the solution of the problem* (3.1), (2.4), *the following estimate holds*:

$$\text{(3.15)} \qquad \sum_{k=1}^{3} \max_{\bar{A}} |\hat{u}_k| + \sum_{h,k}^{1,3} \max_{\bar{A}} |\hat{u}_{k/h}| + \sum_{j,h,k}^{1,3} \max_{\bar{A}} |\hat{u}_{k/jh}| + \max_{\bar{A}} |\hat{\theta}| + \sum_{h=1}^{3} \max_{\bar{A}} |\hat{\theta}_{/h}| + \sum_{j,h}^{1,3} \max_{\bar{A}} |\hat{\theta}_{/jh}|$$
$$+ \max_{\bar{A}} |\hat{\mu}| + \sum_{h=1}^{3} \max_{\bar{A}} |\hat{\mu}_{/h}| + \sum_{j,h}^{1,3} \max_{\bar{A}} |\hat{\mu}|_{/jh} \leqslant q_3 |s|^{14} \Big\{ \sum_{k=1}^{3} \max_{\bar{A}} |\hat{F}_k| + \sum_{h,k}^{1,3} \max_{\bar{A}} |\hat{F}_{k/h}|$$
$$+ \sum_{j,h,k}^{1,3} \max_{\bar{A}} \hat{F}_{k/jh} + \max_{\bar{A}} |\hat{f}| + \sum_{h=1}^{3} \max_{\bar{A}} |\hat{f}_{/h}| + \sum_{j,h}^{1,3} \max_{\bar{A}} |\hat{f}_{/jh}|$$
$$+ \max_{\bar{A}} |\hat{g}| + \sum_{h=1}^{3} \max_{\bar{A}} |\hat{g}_{/h}| + \sum_{j,h}^{1,3} \max_{\bar{A}} |\hat{g}_{/jh}| \Big\}.$$

(10) See [7], p. 354.

4. Existence theorem

We shall denote by $\mathscr{L}$ the matrix differential operator which operates on the 5-vector valued function $\mathbf{v} \equiv (u, \theta, \mu)$ in Eqs. (1.1).

In addition to the above specified hypotheses concerning the physical constants of the problem and the boundary ∂A of A, we assume the following ones:

α) There exists a 5-vector $\mathbf{w}$ belonging to $\mathscr{F}$ and satisfying the boundary conditions (1.2) and the initial conditions (1.3) ([11]).

β) Set $\mathbf{f} = (F, f, g)$. The vector $\mathbf{f} - \mathscr{L}\mathbf{w}$ is defined in the whole four-dimensional cylinder $\bar{A} \times (-\infty, +\infty)$, belongs to the class $\mathscr{C}^\infty[\bar{A} \times (-\infty, +\infty)]$ and has a bounded support contained in the half-space $t \geqslant 0$.

VI. *If hypotheses* (2.1), α) *and* β) *are satisfied and if* ∂A *is* $\mathscr{C}^\infty$*-smooth, the solution in the class* $\mathscr{F}$ *of the problem* (1.1), (1.2), (1.3) *exists.*

Hypothesis α) enables us to assume identically vanishing functions as boundary data and as initial conditions. Hence we may suppose that hypothesis β) is satisfied by the vector $\mathbf{f}$. Since $F(x, t), f(x, t), g(x, t)$ belong to $\mathscr{C}^\infty$ and have a bounded support contained in the half-space $t \geqslant 0$, if we denote by m any positive integer and by s_0 an arbitrary positive real number, we have for $\mathscr{R}s > s_0$

$$\hat{\omega}(x, s) = \int_0^\infty e^{-st}\omega(x, t)dt = \frac{1}{s^m}\int_0^\infty e^{-st}\frac{\partial^m}{\partial t^m}\omega(x, t)dt,$$

$$\hat{\omega}_{/h}(x, s) = \int_0^\infty e^{-st}\omega_{/h}(x, t)dt = \frac{1}{s^m}\int_0^\infty e^{-st}\frac{\partial^m}{\partial t^m}\omega_{/h}(x, t)dt,$$

$$\hat{\omega}_{/jh}(x, s) = \int_0^\infty e^{-st}\omega_{/jh}(x, t)dt = \frac{1}{s^m}\int_0^\infty e^{-st}\frac{\partial^m}{\partial t^m}\omega_{/jh}(x, t)dt,$$

where $\omega(x, t)$ is any of the functions $F(x, t), f(x, t), g(x, t)$ and $\hat{\omega}(x, s)$ the corresponding Laplace transform. Hence we have

$$|\hat{\omega}(x, s)| \leqslant \frac{a_m}{|s|^m}, \quad |\hat{\omega}_{/h}(x, s)| \leqslant \frac{a_m}{|s|^m}, \quad |\hat{\omega}_{/jh}(x, s)| \leqslant \frac{a_m}{|s|^m},$$

where a_m is a positive constant which we may assume depending only on m.

Let $(\hat{u}, \hat{\theta}, \hat{\mu})$ be the solution of the problem (3.1), (2.4). From the estimate (3.15) we get

$$\begin{aligned} &|\hat{v}(x, s)| \leqslant 65a_m q_3|s|^{14-m},\\ (4.1)\qquad &|\hat{v}_{/h}(x, s)| \leqslant 65a_m q_3|s|^{14-m},\\ &|\hat{v}_{/jh}(x, s)| \leqslant 65a_m q_3|s|^{14-m}, \end{aligned}$$

where $\hat{v}(x, s)$ is any of the functions $\hat{u}(x, s)$, $\hat{\theta}(x, s)$, $\hat{\mu}(x, s)$.

([11]) Hypothesis α) will be discussed in Sec. 6.

Consider the functions $u(x, t)$, $\theta(x, t)$, $\mu(x, t)$ defined by the *inversion integrals* $(s_1 > s_0)$

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(s_1+is_2)t} \hat{u}(x, s_1+is_2)\, ds_2,$$

$$\theta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(s_1+is_2)t} \hat{\theta}(x, s_1+is_2)\, ds_2,$$

$$\mu(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(s_1+is_2)t} \hat{\mu}(x, s_1+is_2)\, ds_2.$$

From (4.1), assuming $m > 17$, we see that these integrals exist (as integrals of absolutely integrable functions) and the functions defined by them have as Laplace transforms the functions $\hat{u}(x, s)$, $\hat{\theta}(x, s)$, $\hat{\mu}(x, s)$, respectively ([12]). Moreover,

$$v_{/h}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(s_1+is_2)t} \hat{v}_{/h}(x, s_1+is_2)\, ds_2,$$

$$v_{/jh}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(s_1+is_2)t} \hat{v}_{/jh}(x, s_1+is_2)\, ds_2,$$

$$\dot{v}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s e^{(s_1+is_2)t} \hat{v}(x, s_1+is_2)\, ds_2,$$

$$\ddot{v}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s^2 e^{(s_1+is_1)t} \hat{v}(x, s_1+is_2)\, ds_2,$$

where $v(x, t)$ is any of the functions $u(x, t)$, $\theta(x, t)$, $\mu(x, t)$ and $\hat{v}(x, s)$ the corresponding Laplace transform([13]). The vector (u, θ, μ) belongs to $\mathscr{F}$ and satisfies the initial conditions

(4.2) $$u(x, 0) \equiv 0, \quad u_t(x, 0) \equiv 0, \quad \theta(x, 0) \equiv 0, \quad \mu(x, 0) \equiv 0 \text{ ([14])}.$$

On the other hand, since $\hat{u}(x, s)$, $\hat{\theta}(x, s)$, $\hat{\mu}(x, s)$ satisfy (3.1), (2.4), the functions $u(x, t)$, $\theta(x, t)$, $\mu(x, t)$ satisfy the differential system (1.1) and the boundary conditions

(4.3) $$u(x, t) = 0, \quad \theta(x, t) = 0, \quad \mu(x, t) = 0, \quad \{x \in \partial A, t \in [0, +\infty)\}.$$

5. Continuous dependence of the solution on the data

From the analysis developed in the last Section it should be easy to prove, by assuming suitable norms in the function space $\mathscr{F}$ and in the function space of the "data", that

([12]) See [8], Theorem 5, p. 178.

([13]) See [8], Theorem 9, p. 185.

([14]) See [8], Theorem 8, p. 184.

the solution (u, θ, μ) of the problem (1.1), (4.2), (4.3) depends continuously on the "data" (F, f, g). However, we prefer to use a new approach for proving this continuous dependence, since we wish to have an *explicit* estimate of the solution, which could be used for bounding the approximation error in numerical computations.

Let $v(x,t)$, $\hat{v}(x,s)$ have the same meaning as in the last Section.

We have for $s_1 > s_0 > 0$ and recalling Theorem III

$$(5.1)\quad \int_A \Big\{|v(x,t)|^2 + \sum_{h=1}^{3} |v_{/h}(x,t)|^2 + |\dot{v}(x,t)|^2\Big\} dx$$
$$= \frac{e^{2s_1 t}}{4\pi^2} \int_A \Big\{\Big|\int_{-\infty}^{\infty} e^{is_2 t}\hat{v}(x,s)\,ds_2\Big|^2 + \sum_{h=1}^{3}\Big|\int_{-\infty}^{\infty} e^{is_2 t}\hat{v}_{/h}(x,s)\,ds_2\Big|^2 + \Big|\int_{-\infty}^{\infty} e^{is_2 t} s\hat{v}(x,s)\,ds_2\Big|^2\Big\} dx$$
$$\leqslant \frac{e^{2s_1 t}}{4\pi^2}\int_{-\infty}^{\infty}\frac{ds_2}{|s|^2}\Big\{\int_A dx\int_{-\infty}^{\infty}|s|^2|\hat{v}(x,s)|^2 ds_2 + \sum_{h=1}^{3}\int_A dx\int_{-\infty}^{\infty}|s|^2|\hat{v}_{/h}(x,s)|^2 ds_2$$
$$+\int_A dx\int_{-\infty}^{\infty}|s|^4|\hat{v}(x,s)|^2 ds_2\Big\} \leqslant \frac{\gamma^2}{4\pi}\,\frac{e^{2s_1 t}}{s_1}\Big(1+\frac{1}{s_0^2}\Big)\int_{-\infty}^{\infty}|s|^4 ds_2\int_A\{|\hat{F}(x,s)|^2$$
$$+|\hat{f}(x,s)|^2 + |\hat{g}(x,s)|^2\}\,dx.$$

If ω and $\hat{\omega}$ have the same meaning as in the previous Section, we have

$$(5.2)\quad \int_{-\infty}^{\infty}|s|^4 ds_2\int_A|\hat{\omega}(x,s)|^2 dx = \int_{-\infty}^{\infty}|s|^4 ds_2\int_A\Big|\frac{1}{s^3}\int_0^{\infty}e^{-st}\frac{\partial^3}{\partial t^3}\omega(x,t)\,dt\Big|^2 dx$$
$$\leqslant \int_{-\infty}^{\infty}\frac{ds_2}{|s|^2}\int_0^{\infty}e^{-2s_1 t}dt\int_A dx\int_0^{\infty}\Big|\frac{\partial^3}{\partial t^3}\omega(x,t)\Big|^2 dt = \frac{\pi}{2s_1^2}\int_A dx\int_0^{\infty}\Big|\frac{\partial^3}{\partial t^3}\omega(x,t)\Big|^2 dt.$$

Since we are permitted to assume in (5.1), (5.2) $s_1 = s_0$, we have the estimate

$$(5.3)\quad \int_A\{|u(x,t)|^2 + u_{/h}(x,t)u_{/h}(x,t) + |\dot{u}(x,t)|^2 + |\theta(x,t)|^2 + \theta_{/h}(x,t)\theta_{/h}(x,t)$$
$$+|\dot{\theta}(x,t)|^2 + |\mu(x,t)|^2 + \mu_{/h}(x,t)\mu_{/h}(x,t) + |\dot{\mu}(x,t)|^2\}\,dx$$
$$\leqslant \frac{\gamma^2}{8}\,\frac{e^{2s_0 t}}{s_0^3}\Big(1+\frac{1}{s_0^2}\Big)\int_A dx\int_0^{\infty}\Big\{\Big|\frac{\partial^3}{\partial t^3}F(x,t)\Big|^2 + \Big|\frac{\partial^3}{\partial t^3}f(x,t)\Big|^2 + \Big|\frac{\partial^3}{\partial t^3}g(x,t)\Big|^2\Big\}dt.$$

Let us denote by $\mathbf{v}$ the 5-vector valued function with components $u_1, u_2, u_3, \theta, \mu$ and by $\mathbf{f}$ the 5-vector function with components F_1, F_2, F_3, f, g. Define as $\|\mathbf{v}\|^2$ the integral on the left-hand side of (5.3) and by $|||\mathbf{f}|||^2$ the integral over $A\times[0, +\infty)$ which appears on the right-hand side of (5.3).

Inequality (5.3) gives for every $T > 0$

$$(5.4)\quad \max_{0\leqslant t\leqslant T}\|\mathbf{v}\|^2 \leqslant p(s_0)e^{2s_0 T}|||\mathbf{f}|||^2,$$

where

$$p(s_0) = \frac{\gamma^2}{8s_0^3}\left(1+\frac{1}{s_0^2}\right). \tag{5.5}$$

Inequality (5.4) proves the continuous dependence of the solution of (1.1), (4.2), (4.3) on the "data" in the above introduced norms.

6. Remarks on hypothesis α) and integral estimate of the solution in the general case

We intend to show in this Section how it is possible, under reasonably general assumptions, to satisfy hypothesis α) of Sec. 4.

Let us suppose that, in addition to (2.1) and to the hypothesis concerning the $\mathscr{C}^\infty$- smoothness of ∂A, we have

i) The functions $\bar{u}(x, t)$, $\bar{\theta}(x, t)$, $\bar{\mu}(x, t)$ belong to the class $\mathscr{C}^\infty\{\partial A \times [0, +\infty)\}$ and have a bounded support.

ii) The functions $u^\circ(x)$, $u'(x)$, $\theta^\circ(x)$, $\mu^\circ(x)$ belong to $\mathscr{C}^\infty(\bar{A})$.

iii) The following equations are satisfied for every $x \in \partial A$

$$\bar{u}(x, 0) = u^\circ(x), \quad \dot{\bar{u}}(x, 0) = u'(x),$$
$$\bar{\theta}(x, 0) = \theta^\circ(x), \quad \bar{\mu}(x, 0) = \mu^\circ(x).$$

VII. *Under the assumptions* i), ii), iii), *hypothesis* α) *is satisfied.*

Because of the hypotheses on ∂A, it is possible to determine $\varrho_0 > 0$ such that, if $\xi \in \partial A$ and $0 \leqslant \varrho \leqslant \varrho_0$, the mapping

$$x = \xi + \varrho\nu(\xi), \tag{6.1}$$

where $\nu(\xi)$ is the inward unit normal to ∂A in ξ, is a one-to-one mapping of the Cartesian product $\partial A \times [0, \varrho_0]$ onto the closed domain $\bar{A} - A_0$ (where A_0 is a domain such that A_0 is interior to A). If ξ is determined on ∂A by the local coordinates ξ_1, ξ_2, let us introduce in $\bar{A} - A_0$ the curvilinear coordinates ξ_1, ξ_2, ϱ.

Let $\varphi(\varrho)$ be a $\mathscr{C}^\infty$ real valued function such that $\varphi(\varrho) \equiv 1$ for $0 \leqslant \varrho \leqslant \frac{1}{3}\varrho_0$, $\varphi(\varrho) \equiv 0$ for $\varrho \geqslant \frac{2}{3}\varrho_0$. Set

$$u^{(1)}(x, t)\begin{cases} = \varphi(\varrho)\bar{u}(\xi, t) & \text{for} \quad x = \xi + \varrho\nu(\xi), \quad 0 \leqslant \varrho \leqslant \varrho_0, \\ = 0 & \text{for} \quad x \in \bar{A} - A_0; \end{cases} \tag{6.2}$$

$$\theta^{(1)}(x, t)\begin{cases} = \varphi(\varrho)\bar{\theta}(\xi, t) & \text{for} \quad x = \xi + \varrho\nu(\xi), \quad 0 \leqslant \varrho \leqslant \varrho_0, \\ = 0 & \text{for} \quad x \in \bar{A} - A_0; \end{cases} \tag{6.3}$$

$$\mu^{(1)}(x, t)\begin{cases} = \varphi(\varrho)\bar{\mu}(\xi, t) & \text{for} \quad x = \xi + \varrho\nu(\xi), \quad 0 \leqslant \varrho \leqslant \varrho_0, \\ = 0 & \text{for} \quad x \in \bar{A} - A_0. \end{cases} \tag{6.4}$$

It is evident that $u^{(1)}(x, t)$, $\theta^{(1)}(x, t)$, $\mu^{(1)}(x, t)$ belong to $\mathscr{C}^\infty\{\bar{A} \times [0, +\infty)\}$ and satisfy the boundary conditions (1.2).

Set $\mathbf{v}^{(1)} \equiv (u^{(1)}, \theta^{(1)}, \mu^{(1)})$. Let us now define $\mathbf{v}^{(2)} \equiv (u^{(2)}, \theta^{(2)}, \mu^{(2)})$ as follows

$$u^{(2)}(x,t) = u^{\circ}(x) - u^{(1)}(x,0) + t[u'(x) - \dot{u}^{(1)}(x,0)],$$
$$\theta^{(2)}(x,t) = \theta^{\circ}(x) - \theta^{(1)}(x,0),$$
$$\mu^{(2)}(x,t) = \mu^{\circ}(x) - \mu^{(1)}(x,0).$$

It is easy to see that the vector $\mathbf{w} = \mathbf{v}^{(1)} + \mathbf{v}^{(2)} \equiv (u^{(1)} + u^{(2)}, \theta^{(1)} + \theta^{(2)}, \mu^{(1)} + \mu^{(2)})$ belongs to $\mathscr{F}$ and satisfies hypothesis α).

Let us now suppose that

iv) $\mathbf{f} \equiv (F, f, g)$ is such that hypothesis β) of Sec. 4 is satisfied, if we assume as $\mathbf{w}$ the above defined vector.

Let $\mathbf{z}$ be the solution of the differential system

$$\mathscr{L}\mathbf{z} = \mathbf{f} - \mathscr{L}\mathbf{w}$$

satisfying the homogeneous boundary and initial conditions (4.2), (4.3). This solution is provided by Theorem VI. Hence $\mathbf{v} = \mathbf{z} + \mathbf{w}$ is the solution of problem (1.1), (1.2), (1.3) belonging to $\mathscr{F}$. From (5.3) we easily get

$$\|\mathbf{v}\| \leqslant \|\mathbf{w}\| + \sqrt{p(s_0)}\, e^{s_0 t}(|||\mathbf{f}||| + |||\mathscr{L}\mathbf{w}|||). \tag{6.5}$$

We have

$$\|\mathbf{w}\| \leqslant \|\mathbf{v}^{(1)}\| + \|\mathbf{v}^{(2)}\| \tag{6.6}$$

and

$$\|\mathbf{v}^{(1)}\| = \Big\{\int_A [|u^{(1)}(x,t)|^2 + u^{(1)}_{i/h}(x,t)u^{(1)}_{i/h}(x,t) + |\dot{u}^{(1)}(x,t)|^2 + |\theta^{(1)}(x,t)|^2$$
$$+ \theta^{(1)}_{/h}(x,t)\theta^{(1)}_{/h}(x,t) + |\dot{\theta}^{(1)}(x,t)|^2 + |\mu^{(1)}(x,t)|^2 + \mu^{(1)}_{i/h}(x,t)\mu^{(1)}_{i/h}(x,t) + |\dot{\mu}^{(1)}(x,t)|^2]\,dx\Big\}^{\frac{1}{2}},$$

$$\|\mathbf{v}^{(2)}\| \leqslant \Big\{\int_A [|u^{\circ}(x) + tu'(x)|^2 + \big(u^{\circ}_{/h}(x) + tu'_{/h}(x)\big)\big(u^{\circ}_{/h}(x) + tu'_{/h}(x)\big) + |u'(x)|^2 + |\theta^{\circ}(x)|^2$$
$$+ \theta^{\circ}_{/h}(x)\theta^{\circ}_{/h}(x) + |\mu^{\circ}(x)|^2 + \mu^{\circ}_{/h}(x)\mu^{\circ}_{/h}(x)]\,dx\Big\}^{\frac{1}{2}} + \Big\{\int_A [|u^{(1)}(x,0)|^2 + u^{(1)}_{i/h}(x,0)u^{(1)}_{i/h}(x,0)$$
$$+ |\theta^{(1)}(x,0)|^2 + \theta^{(1)}_{/h}(x,0)\theta^{(1)}_{/h}(x,0) + |\mu^{(1)}(x,0)|^2 + \mu^{(1)}_{i/h}(x,0)\,\mu^{(1)}_{i/h}(x,0)]\,dx\Big\}^{\frac{1}{2}}$$
$$+ t\Big\{\int_A [|\dot{u}^{(1)}(x,0)|^2 + \dot{u}^{(1)}_{i/h}(x,0)\dot{u}^{(1)}_{i/h}(x,0) + |\ddot{u}^{(1)}(x,0)|^2]\,dx\Big\}^{\frac{1}{2}},$$

$$|||\mathscr{L}\mathbf{w}||| = |||\mathscr{L}\mathbf{v}^{(1)}||| \tag{6.7}$$

$$\leqslant \Big[\int_0^\infty dt \int_A \Big\{9[(\lambda+G)^2 + G^2]\frac{\partial^3}{\partial t^3}u^{(1)}_{i/hk}\frac{\partial^3}{\partial t^3}u^{(1)}_{i/hk} + 9\Big|\frac{\partial^3}{\partial t^3}\ddot{u}^{(1)}\Big|^2 + 9(p_\theta^2 + p_0^2)\frac{\partial^3}{\partial t^3}\theta^{(1)}_{/h}\frac{\partial^3}{\partial t^3}\theta^{(1)}_{/h}$$
$$+ 9(p_\mu^2 + p_0^2)\frac{\partial^3}{\partial t^3}\mu^{(1)}_{i/h}\frac{\partial^3}{\partial t^3}\mu^{(1)}_{i/h} + 8K^2\frac{\partial^3}{\partial t^3}\theta^{(1)}_{/hk}\frac{\partial^3}{\partial t^3}\theta^{(1)}_{/hk} + 8(c^2 + d^2)\Big|\frac{\partial^3}{\partial t^3}\dot{\theta}^{(1)}\Big|^2$$
$$+ 8(d^2 + n^2)\Big|\frac{\partial^3}{\partial t^3}\dot{\mu}^{(1)}\Big|^2 + 8(p_\theta^2 + p_\mu^2 + p_0^2)\frac{\partial^3}{\partial t^3}u^{(1)}_{i/h}\frac{\partial^3}{\partial t^3}u^{(1)}_{i/h} + 8D^2\frac{\partial^3}{\partial t^3}\mu^{(1)}_{i/hk}\frac{\partial^3}{\partial t^3}\mu^{(1)}_{i/hk}\Big\}dx\Big]^{\frac{1}{2}};$$

p_0 is an arbitrarily chosen positive constant.

If we denote by $\varphi(t)$ a vector valued function belonging to $\mathscr{C}^\infty[0, +\infty)$ and with support contained in the bounded interval $[0, T]$, we have, for any positive integer n and any $t \geqslant 0$,

$$\varphi(t) = \int_T^t \frac{(t-\tau)^{n-1}}{(n-1)!} \varphi^{(n)}(\tau)\, d\tau.$$

Hence

$$|\varphi(t)|^2 \leqslant \frac{|T-t|^{2n-1}}{(2n-1)[(n-1)!]^2} \left| \int_t^T |\varphi^{(n)}(\tau)|^2 d\tau \right|.$$

For $0 \leqslant t \leqslant T$ we deduce

$$|\varphi(t)|^2 \leqslant \frac{T^{2n-1}}{(2n-1)[(n-1)!]^2} \int_0^T |\varphi^{(n)}(\tau)|^2 d\tau. \tag{6.8}$$

Suppose that the functions $\bar{u}(x, t)$, $\bar{\theta}(x, t)$, $\bar{\mu}(x, t)$ vanish identically for $t \geqslant T$.

From (6.1), (6.2), (6.3) we deduce that $u^{(1)}(x, t)$, $\theta^{(1)}(x, t)$, $\mu^{(1)}(x, t)$ vanish identically for $t \geqslant T$. Hence, if we denote by $P(\mathbf{v}^{(1)})$ the integral which bounds $|||\mathscr{L}\mathbf{v}^{(1)}|||^2$ by (6.7), using (6.8) we have

$$\begin{aligned} \|\mathbf{v}^{(1)}\| + \Big\{\int_A [|u^{(1)}(x, 0)|^2 + u^{(1)}_{i/h}(x, 0) u^{(1)}_{i/h}(x, 0) + |\theta^{(1)}(x, 0)|^2 \\ + \theta^{(1)}_{/h}(x, 0)\theta^{(1)}_{/h}(x, 0) + |\mu^{(1)}(x, 0)|^2 + \mu^{(1)}_{/h}(x, 0)\, \mu^{(1)}_{/h}(x, 0)] dx\Big\}^{\frac{1}{2}} \\ + t\Big\{\int_A [|\dot{u}^{(1)}(x, 0)|^2 + \dot{u}^{(1)}_{i/h}(x, 0) \dot{u}^{(1)}_{i/h}(x, 0) + |\ddot{u}^{(1)}(x, 0)|^2] dx\Big\}^{\frac{1}{2}} \leqslant (b_0 + tb_1)\{P[\mathbf{v}^{(1)}]\}^{\frac{1}{2}}, \end{aligned} \tag{6.9}$$

where

$$\begin{aligned} b_0 = \Big\{\frac{T^9}{2^6 3^6} + \frac{T^7}{2^2 3^3 7}\left(\frac{1}{3^2} + \frac{1}{2^3(c^2+d^2)} + \frac{1}{2^3(d^2+n^2)}\right) \\ + \frac{T^5}{2^2 5}\left(\frac{1}{2^3(p_\theta^2+p_\mu^2+p_0^2)} + \frac{1}{3^2(p_\theta^2+p_0^2)} + \frac{1}{3^2(p_\mu^2+p_0^2)} + \frac{1}{2^3(d^2+n^2)}\right)\Big\}^{\frac{1}{2}} \\ + \Big\{\frac{T^9}{2^6 3^6} + \frac{T^7}{2^2 3^3 7}\left(\frac{1}{2^3(c^2+d^2)} + \frac{1}{2^3(d^2+n^2)}\right) \\ + \frac{T^5}{2^2 5}\left(\frac{1}{2^3(p_\theta^2+p_\mu^2+p_0^2)} + \frac{1}{3^2(p_\theta^2+p_0^2)} + \frac{1}{3^2(p_\mu^2+p_0^2)}\right)\Big\}^{\frac{1}{2}}; \end{aligned} \tag{6.10}$$

$$b_1 = \left\{\frac{T^7}{2^2 3^4 7} + \frac{T^5}{2^2 3^2 5} + \frac{T^3}{3 \cdot 2^3(p_\theta^2+p_\mu^2+p_0^2)}\right\}^{\frac{1}{2}}. \tag{6.11}$$

Let us decompose ∂A by a triangulation into a finite set of non overlapping open surfaces $\Sigma_1, \ldots, \Sigma_l$, such that Σ_i $(i = 1, 2, \ldots, l)$ admits the parametric representation

$x = x(\xi_1, \xi_2)$ with $x(\xi_1, \xi_2)$ 3-vector valued $\mathscr{C}^\infty$ function in the triangular closed domain S_i of the (ξ_1, ξ_2)-plane; $x(\xi_1, \xi_2)$ maps one-to-one S_i onto Σ_i and the Jacobian matrix

$$\frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2)}$$

has rank 2 at each point of S_i.

If we consider for $(\xi_1, \xi_2) \in S_i$, $0 \leqslant \varrho \leqslant \varrho_0$, the above introduced curvilinear coordinates (ξ_1, ξ_2, ϱ), we have that the determinant $J(\xi, \varrho)$ of the Jacobian matrix

$$\frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \varrho)} \tag{6.12}$$

is given by

$$J(\xi, \varrho) = [1-2\varrho H(\xi)+\varrho^2 K(\xi)]\sqrt{EG-F^2},$$

where $H(\xi)$ and $K(\xi)$ are respectively the mean curvature and the total curvature at the point ξ of ∂A and E, F, G have the usual meaning, i.e., they are the coefficients of the first fundamental form on ∂A

$$ds^2 = Ed\xi_1^2+2Fd\xi_1 d\xi_2+Gd\xi_2^2.$$

Let M be a positive constant such that for $(\xi_1, \xi_2) \in S_i$ $(i = 1, \ldots, l)$ we have

$$\left|\frac{\partial x_h(\xi_1, \xi_2)}{\partial \xi_k}\right| \leqslant M, \quad |\nu_h(\xi_1, \xi_2)| \leqslant M,$$

$$\left|\frac{\partial \nu_h(\xi_1, \xi_2)}{\partial \xi_k}\right| \leqslant M, \quad \left|\frac{\partial^2 x_h(\xi_1, \xi_2)}{\partial \xi_j \partial \xi_k}\right| \leqslant M, \quad \left|\frac{\partial^2 \nu_h(\xi_1, \xi_2)}{\partial \xi_j \partial \xi_k}\right| \leqslant M, \tag{6.13}$$

$$h = 1, 2, 3; \quad k, j = 1, 2,$$

$$|H(\xi)| \leqslant M, \quad |K(\xi)| \leqslant M.$$

We are permitted to assume M independent of i. Moreover, we suppose $\varrho_0 < 1$. It is convenient to denote by ξ_3 the coordinate ϱ in some of the subsequent computations.

Let $J_{hk}(\xi_1, \xi_2, \xi_3)$ be the co-factor of the entry in the h-th row and in the k-th column of the Jacobian matrix (6.12). We have

$$J(\xi_1, \xi_2, \xi_3) \geqslant (1-3\varrho M)\sqrt{EG-F^2}.$$

We suppose that $\varrho_0 < (3M)^{-1}$.

Let ε be a positive constant, independent of i, such that $\sqrt{EG-F^2} \geqslant \varepsilon$ for $(\xi_1, \xi_2) \in S_i$. We have

$$\frac{\partial \xi_k}{\partial x_h} = \frac{J_{hk}(\xi_1, \xi_2, \xi_3)}{J(\xi_1, \xi_2, \xi_3)},$$

$$\frac{\partial}{\partial \xi_j}\frac{\partial \xi_k}{\partial x_h} = \frac{1}{J}\frac{\partial J_{hk}}{\partial \xi_i} - \frac{1}{J}\frac{\partial J}{\partial \xi_j}\frac{\partial \xi_k}{\partial x_h},$$

$$\left|\frac{\partial}{\partial \xi_j} J\right| \leqslant 18M^3(1+\varrho_0)^3, \quad \left|\frac{\partial}{\partial \xi_j} J_{hk}\right| \leqslant 4M^2(1+\varrho_0)^2.$$

Hence

$$\left|\frac{\partial \xi_k}{\partial x_h}\right| \leqslant \frac{2M^2(1+\varrho_0)^2}{(1-3\varrho_0 M)\varepsilon} = \alpha; \tag{6.14}$$

$$\left|\frac{\partial}{\partial \xi_j}\frac{\partial \xi_k}{\partial x_h}\right| \leqslant \frac{4M^2(1+\varrho_0)^2}{(1-3\varrho_0 M)\varepsilon}\left(1+9\frac{M^3(1+\varrho_0)^3}{(1-3\varrho_0 M)\varepsilon}\right) = \beta, \tag{6.15}$$
$$\left|\frac{\partial^2 \xi_k}{\partial x_j \partial x_h}\right| \leqslant 3\alpha\beta.$$

Denote by ψ a function which is $\mathscr{C}^\infty$ in $\overline{A}-A_0$. We have, after elementary computations,

$$\psi_{/h}\psi_{/h} \leqslant 9\alpha^2 \frac{\partial \psi}{\partial \xi_k}\frac{\partial \psi}{\partial \xi_k}; \tag{6.16}$$

$$\psi_{/hk}\psi_{/hk} \leqslant 2\cdot 3^4\alpha^4 \frac{\partial^2\psi}{\partial \xi_i \partial \xi_l}\frac{\partial^2\psi}{\partial \xi_j \partial \xi_l} + 2\cdot 3^5\alpha^2\beta^2 \frac{\partial \psi}{\partial \xi_j}\frac{\partial \psi}{\partial \xi_j}. \tag{6.17}$$

Suppose that we have for $x = \xi+\varrho\nu(\xi)$, $\xi \in \partial A$, $0 \leqslant \varrho \leqslant \varrho_0$:

$$\psi(x) = \varphi(\varrho)\overline{\psi}(\xi),$$

where $\overline{\psi}(\xi)$ is a $\mathscr{C}^\infty$ function defined on ∂A.

Setting for $(\xi_1, \xi_2) \in S_i$ $(i = 1, \dots, l)$

$$|\nabla'_\xi \overline{\psi}|^2 = |\overline{\psi}_{\xi_1}|^2 + |\overline{\psi}_{\xi_2}|^2,$$
$$|\nabla''_\xi \psi|^2 = |\overline{\psi}_{\xi_1\xi_1}|^2 + 2|\overline{\psi}_{\xi_1\xi_2}|^2 + |\overline{\psi}_{\xi_2\xi_2}|^2,$$

we have

$$\frac{\partial \psi}{\partial \xi_k}\frac{\partial \psi}{\partial \xi_k} = |\nabla'_\xi\overline{\psi}|^2|\varphi(\varrho)|^2 + |\overline{\psi}|^2|\varphi'(\varrho)|^2; \tag{6.18}$$

$$\frac{\partial^2\psi}{\partial \xi_j \partial \xi_l}\frac{\partial^2\psi}{\partial \xi_j \partial \xi_l} = |\nabla''_\xi \overline{\psi}|^2|\varphi(\varrho)|^2 + 2|\nabla'_\xi|^2|\varphi'(\varrho)|^2 + |\overline{\psi}|^2|\varphi''(\varrho)|^2. \tag{6.19}$$

If we use (6.16), (6.17), (6.18), (6.19) for estimating $P[\mathbf{v}^{(1)}]$ and estimate the right-hand side of (6.5) by the inequalities (6.6) and (6.9), we get the proof of the following theorem:

VIII. *Under the assumptions.* i), ii), iii), iv) *of this Section, the following estimate holds for the solution of the problem* (1.1), (1.2), (1.3$_l$), *in the function class* $\mathscr{F}$

$$\Big\{\int_A \Big[|u(x,t)|^2 + u_{/h}(x,t)u_{/h}(x,t) + |\dot{u}(x,t)|^2 + |\theta(x,t)|^2 + \theta_{/h}(x,t)\theta_{/h}(x,t)$$
$$+ |\dot{\theta}(x,t)|^2 + |\mu(x,t)|^2 + \mu_{/h}(x,t)\mu_{/h}(x,t) + |\dot{\mu}(x,t)|^2\Big]dx\Big\}^{\frac{1}{2}}$$
$$\leqslant \Big\{\int_A |u^\circ(x)+tu'(x)|^2 + \big(u^\circ_{/h}(x)+tu'_{/h}(x)\big)\big(u^\circ_{/h}(x)+tu'_{/h}(x)\big) + |u'(x)|^2$$
$$+ |\theta^\circ(x)|^2 + \theta^\circ_{/h}(x)\theta^\circ_{/h}(x) + |\mu^\circ(x)|^2 + \mu^\circ_{/h}(x)\mu^\circ_{/h}(x)\Big]dx\Big\}^{\frac{1}{2}}$$

$$+[p(s_0)]^{\frac{1}{2}} e^{s_0 t}\left\{\int_0^\infty dt \int_A \left(\left|\frac{\partial^3 F}{\partial t^3}\right|^2 + \left|\frac{\partial^3 f}{\partial t^3}\right|^2 + \left|\frac{\partial^3 g}{\partial t^3}\right|^2\right) dx\right\}^{\frac{1}{2}} + \{b_0 + b_1 t + [p(s_0)]^{\frac{1}{2}} e^{s_0 t}\} \times$$

$$\times \left\{c_1 \int_0^\infty dt \int_{\partial A} \left|\nabla''_\xi \frac{\partial^3 \bar{u}}{\partial t^3}\right|^2 d\sigma + c_2 \int_0^\infty dt \int_{\partial A} \left|\nabla'_\xi \frac{\partial^3 \bar{u}}{\partial t^3}\right|^2 d\sigma + c_3 \int_0^\infty dt \int_{\partial A} \left|\frac{\partial^5 \bar{u}}{\partial t^5}\right|^2 d\sigma\right.$$

$$+ c_4 \int_0^\infty dt \int_{\partial A} \left|\nabla''_\xi \frac{\partial^3 \bar{\theta}}{\partial t^3}\right|^2 d\sigma + c_5 \int_0^\infty dt \int_{\partial A} \left|\nabla'_\xi \frac{\partial^3 \bar{\theta}}{\partial t^3}\right|^2 d\sigma + c_6 \int_0^\infty dt \int_{\partial A} \left|\frac{\partial^4 \bar{\theta}}{\partial t^4}\right| d\sigma$$

$$\left. + c_7 \int_0^\infty dt \int_{\partial A} \left|\nabla''_\xi \frac{\partial^3 \bar{\mu}}{\partial t^3}\right| d\sigma + c_8 \int_0^\infty dt \int_{\partial A} \left|\nabla'_\xi \frac{\partial^3 \bar{\mu}}{\partial t^3}\right| d\sigma + c_9 \int_0^\infty dt \int_{\partial A} \left|\frac{\partial^4 \bar{\mu}}{\partial t^4}\right| d\sigma\right\}^{\frac{1}{2}}.$$

The constants $p(s_0)$, b_0 and b_1 are given by (5.5), (6.10), (6.11). The constants $c_1, \ldots, c_9$ are given by the following equations:

$$c_1 = 2 \cdot 9^3 \alpha^4 (\lambda^2 + 2G^2 + 2\lambda G) \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi(\varrho)|^2 d\varrho,$$

$$c_2 = 2^2 9^3 \alpha^4 (\lambda^2 + 2G^2 + 2\lambda G) \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi'(\varrho)|^2 d\varrho + [6 \cdot 9^3 \alpha^2 \beta^2 (\lambda^2 + 2G^2 + 2\lambda G)$$
$$+ 2^3 9 \alpha^2 (p_\theta^2 + p_\mu^2 + p_0^2)] \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi(\varrho)|^2 d\varrho,$$

$$c_3 = 6 \cdot 9^2 T^4 \alpha^4 (\lambda^2 + 2G^2 + 2\lambda G) \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi''(\varrho)|^2 d\varrho$$
$$+ T^4 [2 \cdot 9^3 \alpha^2 \beta^2 (\lambda^2 + 2G^2 + 2\lambda G) + 24 \alpha^2 (p_\theta^2 + p_\mu^2 + p_0^2)] \times$$
$$\times \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi'(\varrho)|^2 d\varrho + 9 \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi(\varrho)|^2 d\varrho,$$

$$c_4 = 2^4 9^2 K^2 \alpha^4 \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi(\varrho)|^2 d\varrho,$$

$$c_5 = 2^5 9^2 K^2 \alpha^4 \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi'(\varrho)|^2 d\varrho$$
$$+ [9^2 \alpha^2 (p_\theta^2 + p_0^2) + 2^4 3^5 \alpha^2 \beta^2 K^2] \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi(\varrho)|^2 d\varrho,$$

$$c_6 = T^2 [3^4 \alpha^2 (p_\theta^2 + p_0^2) + 2^4 3^5 K^2 \alpha^2 \beta^2] \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi'(\varrho)|^2 d\varrho$$
$$+ 2^4 3^4 T^2 K^2 \alpha^4 \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi''(\varrho)|^2 d\varrho + 8(c^2 + d^2) \int_0^{\varrho_0} (1 - 3M\varrho)\, |\varphi(\varrho)|^2 d\varrho,$$

$$c_7 = 2^4 3^4 D^2 \alpha^4 \int_0^{\varrho_0} (1-3M\varrho)\,|\varphi(\varrho)|^2 d\varrho,$$

$$c_8 = 2^5 3^4 D^2 \alpha^4 \int_0^{\varrho_0} (1-3M\varrho)\,|\varphi'(\varrho)|^2 d\varrho + [9^2\alpha^2(p_\mu^2+p_0^2)+2^4 3^4 \alpha^2\beta^2 D^2] \int_0^{\varrho_0} (1-3M\varrho)\,|\varphi(\varrho)|^2 d\varrho,$$

$$c_9 = T^2 [3^4\alpha^2(p_\mu^2+p_0^2)+2^4 3^5 K^2\alpha^2\beta^2] \int_0^{\varrho_0} (1-3M\varrho)\,|\varphi'(\varrho)|^2 d\varrho + 2^4 3^4 T^2 K^2 \alpha^4 \int_0^{\varrho_0} (1-3M\varrho)\,|\varphi''(\varrho)|^2 d\varrho + 8(d^2+n^2) \int_0^{\varrho_0} (1-3M\varrho)\,|\varphi(\varrho)|^2 d\varrho.$$

It must be remarked that all the constants considered in the above theorem are explicitly expressed in terms of the physical constants and of the geometry of the domain A. One could get better values for these constants by performing more refined (and much more tedious!) computations. We leave this task to the reader interested in concrete numerical computations.

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